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# A NOTE ON FUSION BANACH FRAMES

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Abstract. For a fusion Banach frame $(\{G_n, v_n\}, S)$ for a Banach space $E$, if $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for $E^*$, then $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is called a fusion bi-Banach frame for $E$. It is proved that if $E$ has an atomic decomposition, then $E$ also has a fusion bi-Banach frame. Also, a sufficient condition for the existence of a fusion bi-Banach frame is given. Finally, a characterization of fusion bi-Banach frames is given.

## 1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [5] in 1952 and re-introduced in 1986 by Daubechies, Grossmann and Meyer [4]. Casazza [2] and Benedetto and Fickus [1] have studied frames in finite dimensional spaces which attracted more attention due to their use in signal processing. Frames are now used as a tool in many areas like data compression, sampling theory, optics, filter banks, signal detection, time-frequency analysis etc.

The concept of frames in Hilbert spaces was extended to Banach spaces by Feichtinger and Gröchenig [6] who introduced the concept of atomic decompositions in Banach spaces. This concept was further generalized by Gröchenig [7] who introduced the notion of Banach frames for Banach spaces. Jain et al. [9], generalized Banach frames in Banach spaces and introduced frames of subspaces (Fusion Banach frames) for Banach spaces. They gave the following definition of a fusion Banach frame.

**Definition 1.1** ([9])**.** Let $E$ be a Banach space. Let $\{G_n\}$ be a sequence of non-trivial subspaces of $E$ and $\{v_n\}$ be a sequence of bounded linear projections such that $v_n(E) = G_n$, $n \in \mathbb{N}$. We associate a Banach space $\mathcal{A}$ and an operator $S : \mathcal{A} \to E$ with the space $E$. Then $(\{G_n, v_n\}, S)$ is called a *frame of subspaces* (*fusion Banach frame*) for $E$ with respect to $\mathcal{A}$ if

(i) $\{v_n(x)\} \in \mathcal{A}$, for all $x \in E$,

(ii) there exist constants $A$, $B$ $(0 < A \leq B < \infty)$ such that

$$A\|x\|_E \leq \|\{v_n(x)\}\|_{\mathcal{A}} \leq B\|x\|_E\,, \quad x \in E\,,$$

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(iii) $S$ is a bounded linear operator such that

$$S(\{v_n(x)\}) = x\,, \quad x \in E\,.$$

The following lemma, proved in [9], is used in the sequel

**Lemma 1.2.** *Let $\{G_n\}$ be a sequence of non-trivial subspaces of $E$ and $\{v_n\}$ be a sequence of bounded linear projections with $v_n(E) = G_n$, $n \in \mathbb{N}$. If $\{v_n\}$ is total over $E$, i.e., $\{x \in E : v_n(x) = 0, \text{ for all } n \in \mathbb{N}\} = \{0\}$, then $\mathcal{A} = \{\{v_n(x)\} : x \in E\}$ is a Banach space with norm $\|\{v_n(x)\}\|_{\mathcal{A}} = \|x\|_E$, $x \in E$.*

For other related notions on frames in Banach spaces one may refer to [3, 8, 10, 11].

In the present paper, we introduce fusion bi-Banach frames for a Banach space $E$. We prove that if $E$ has an atomic decomposition, then $E$ also has a fusion bi-Banach frame. Also, a sufficient condition for the existence of fusion bi-Banach frames is given. Finally, a characterization of fusion bi-Banach frames is obtained.

## 2. Main Results

One may observe that, if $(\{G_n, v_n\}, S)$ is a fusion Banach frame for $E$ with respect to some associated Banach space $\mathcal{A}$, then there may not exist a Banach space $\mathcal{A}_1$ associated with $E^*$ together with an operator $T\colon \mathcal{A}_1 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for $E^*$ with respect to $\mathcal{A}_1$.

In this regard, we have the following examples

**Example 2.1.** Consider the Banach space

$$E = \ell^\infty(X) = \big\{\{x_n\} : x_n \in X;\ \sup_{1 \le n < \infty} \|x_n\|_X < \infty\big\}$$

equipped with the norm $\|\{x_n\}\|_E = \sup\limits_{1 \le n < \infty} \|x_n\|_X$, $\{x_n\} \in E$, where $(X, \|\cdot\|)$ is a Banach space. For each $n \in \mathbb{N}$, define $G_n = \{\delta_n^x : x \in X\}$ and $v_n(x) = \delta_n^{x_n}$, $x = \{x_n\} \in E$, where $\delta_n^x = (0, 0, \ldots, 0, \underset{\substack{\downarrow \\ n\text{-th place}}}{x}, 0, \ldots)$ for all $n \in \mathbb{N}$ and $x \in X$. Then by Lemma 1.2, there exist an associated Banach space $\mathcal{A} = \{\{v_n(x)\} : x \in E\}$ with norm $\|\{v_n(x)\}\|_{\mathcal{A}} = \|x\|_E$, $x \in E$ together with an operator $S\colon \mathcal{A} \to E$ given by $S(\{v_n(x)\}) = x$, $x \in E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for $E$ with respect to $\mathcal{A}$. But, there does not exist a Banach space $\mathcal{A}_1$ associated with $E^*$ together with an operator $T\colon \mathcal{A}_1 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for $E^*$ with respect to $\mathcal{A}_1$. For otherwise, $\big[\bigcup\limits_{n=1}^{\infty} G_n\big] = E$, which is not true.

**Example 2.2.** Let $E$ be a Banach space defined as

$$E = c_0(X) = \big\{\{x_n\} : x_n \in X;\ \lim_{n \to \infty} \|x_n\|_X = 0\big\}$$

equipped with the norm given by

$$\|\{x_n\}\|_E = \sup_{1 \le n < \infty} \|x_n\|_X\,, \quad \text{where } (X, \|\cdot\|) \text{ is a Banach space.}$$

Define a sequence $\{G_n\}$ of subspaces of $E$ by

$$G_{2n-1} = \{\delta^x_{2n-1} - 2^{n-1}\delta^x_{2n} : x \in X\}$$
$$G_{2n} = \{\delta^x_{2n} : x \in X\}\,.$$

Also define operators $v_n$ on $E$ by

$$v_{2n-1}(x) = \delta^{x_{2n-1}}_{2n-1} - 2^{n-1}\delta^{x_{2n-1}}_{2n}$$
$$v_{2n}(x) = \delta^{2^{n-1}x_{2n-1}+x_{2n}}_{2n} \quad \text{for all } x = \{x_n\} \in E \text{ and } n \in \mathbb{N}\,.$$

Then by Lemma 1.2 there exist an associated Banach space $\mathcal{A}$ and an operator $S\colon \mathcal{A} \to E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for $E$ with respect to $\mathcal{A}$.

If $\big[\bigcup\limits_{n=1}^{\infty} G_n\big] \neq E$, then there exists $0 \neq f = \{f_i\} \in E^*$ such that $f(y) = 0$ for all $y \in G_n$, $n \in \mathbb{N}$. This would imply $f_n = 0$ for all $n \in \mathbb{N}$ and hence $f = 0$. Therefore, by Lemma 1.2 again, there exist a Banach space $\mathcal{A}_1$ associated to $E^*$ and an operator $T\colon \mathcal{A}_1 \to E^*$ such that $(\{v^*_n(E^*), v^*_n\}, T)$ is a fusion Banach frame for $E^*$ with respect to $\mathcal{A}_1$.

In view of the above discussion, we define the following

**Definition 2.3.** Let $E$ be a Banach space. Let $\{G_n\}$ be a sequence of non-trivial subspaces of $E$ and $\{v_n\}$ be a sequence of bounded linear projections such that $v_n(E) = G_n$, $n \in \mathbb{N}$. If there exist Banach spaces $\mathcal{A}$ and $\mathcal{A}_1$ associated with $E$ and $E^*$ respectively and operators $S\colon \mathcal{A} \to E$ and $T\colon \mathcal{A}_1 \to E^*$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for $E$ with respect to $\mathcal{A}$ and $(\{v^*_n(E^*), v^*_n\}, T)$ is a fusion Banach frame for $E^*$ with respect to $\mathcal{A}_1$, then we call the system $(\{G_n, v_n\}, S; \{v^*_n(E^*), v^*_n\}, T)$ a *fusion bi-Banach frame* for $E$

In view of Remark 3.2.1 in [9], we have

*Every reflexive Banach space has a fusion bi-Banach frame.*

Recall that if $E$ is a Banach space and $E_d$ is an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}, \{x_n\}$ is a sequence in $E$ and $\{f_n\}$ is a sequence in $E^*$, then the pair $(\{f_n\}, \{x_n\})$ is called an *atomic decomposition* for $E$ with respect to $E_d$ if

(i) $\{f_n(x)\} \in E_d$, $x \in E$;

(ii) there exist constants $A, B$ with $0 < A \leq B < \infty$ such that
$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E\,, \quad x \in E\,;$$

(iii) $x = \sum\limits_{n=1}^{\infty} f_n(x)x_n$, $x \in E$.

The next result is regarding the existence of fusion bi-Banach frames for a Banach space having an atomic decomposition.

**Theorem 2.4.** *Let $E$ be a Banach space. If $E$ has an atomic decomposition, then it also has a fusion bi-Banach frame.*

**Proof.** Let $(\{f_n\}, \{x_n\})$ be an atomic decomposition for $E$ with respect to $E_d$. Define $G_n = [x_n]$, $n \in \mathbb{N}$ and $v_n(x) = f_n(x)x_n$, $n \in \mathbb{N}$. Then there exist an associated Banach space $\mathcal{A} = \{\{v_n(x)\} : x \in E\}$ together with an operator $S\colon \mathcal{A} \to E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for $E$ with respect to $\mathcal{A}$. Further $\big[\bigcup_{n=1}^{\infty} G_n\big] = E$ (as $[x_n] = E$). So, $v_n^*(f) = 0$ for all $n \in \mathbb{N}$ imply $f = 0$, where $f \in E^*$. Thus, $\{v_n^*\}$ is total over $E^*$ and so by Lemma 1.2, there exist an associated Banach space $\mathcal{A}_1$ and an operator $T\colon \mathcal{A}_1 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for $E^*$ with respect to $\mathcal{A}_1$. Hence, $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is a fusion bi-Banach frame for $E$. $\square$

Next, we observe that if $E$ be a Banach space and $\{G_n\}$ be a sequence of non-trivial subspaces of $E$ with associated sequence of projections $\{v_n\}$ with $v_n(E) = G_n$, $n \in \mathbb{N}$, then it is possible that there exist a Banach space $\mathcal{A}_1$ associated with $E^*$ together with a bounded linear operator $T\colon \mathcal{A}_1 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for $E^*$ with respect to $\mathcal{A}_1$ and there may not exist any Banach space $\mathcal{A}$ associated with $E$ together with an operator $S\colon \mathcal{A} \to E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for $E$ with respect to $\mathcal{A}$. Indeed, let

$$E = \ell^2(X) = \left\{\{x_n\} : x_n \in X; \sum_{n=1}^{\infty} \|x_n\|_X^2 < \infty\right\},$$

where $(X, \|\cdot\|)$ is a Banach space, equipped with the norm given by

$$\|\{x_n\}\|_E = \left(\sum_{n=1}^{\infty} \|x_n\|_X^2\right)^{1/2}.$$

Define for $n \in \mathbb{N}$, $G_n = \{\delta_1^x + \delta_{n+1}^x : x \in X\}$ and $v_n(x) = \delta_1^{x_{n+1}} + \delta_{n+1}^{x_{n+1}}$, $x = \{x_n\} \in E$, where $\delta_n^x = (0, 0, \ldots, 0, \underset{\substack{\downarrow \\ n\text{th place}}}{x}, 0, \ldots)$, $x \in X$.

Then $\big[\bigcup_{n=1}^{\infty} G_n\big] = E$ and $v_i v_j = 0$ for all $i \neq j$.

But, since for any $0 \neq x \in X$, $\delta_1^x = (x, 0, 0, \ldots) \in E$ is such that $v_n(\delta_1^x) = 0$, for all $n \in N$, there exist no associated Banach space $\mathcal{A}$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for $E$ with respect to $\mathcal{A}$. However, there exist a Banach space $\mathcal{A}_0$ and an operator $T\colon \mathcal{A}_0 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for $E^*$ with respect to $\mathcal{A}_0$.

In view of the above discussion, we prove the following result

**Theorem 2.5.** *Let $E$ be a Banach space and $\{G_n\}$ be a sequence of subspaces of $E$ with $\big[\bigcup_{n=1}^{\infty} G_n\big] = E$. Let $\{v_n\}$ be a sequence of projections on $E$ satisfying $v_n(E) = G_n$, $n \in \mathbb{N}$ and $v_i v_j = 0$ for all $i \neq j$. Then there exist Banach spaces $\mathcal{A}$ and $\mathcal{A}_1$ associated with $E$ and $E^*$, respectively, and operators $S\colon \mathcal{A} \to E$ and $T\colon \mathcal{A}_1 \to E^*$ such that $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is a fusion bi-Banach frame* *for $E$ if every sequence $\{x_n\} \subset E$ such that $x_n \in G_n$ and $x_n \neq 0$, $n \in \mathbb{N}$ satisfies* $\bigcap\limits_{n=1}^{\infty} [x_{n+1}, x_{n+2}, \ldots] = \{0\}$.

**Proof.** Since $\big[\bigcup\limits_{n=1}^{\infty} G_n\big] = E$, there exist an associated Banach space $\mathcal{A}_1$ and a bounded linear operator $T\colon \mathcal{A}_1 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for $E^*$ with respect to $\mathcal{A}_1$. Let, if possible, there exist no Banach space $\mathcal{A}$ associated with $E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for $E$ with respect to $\mathcal{A}$ where $S\colon \mathcal{A} \to E$ is a bounded linear operator. Now, since $\big[\bigcup\limits_{n=1}^{\infty} G_n\big] = E$ and $v_i v_j = 0$ for all $i \neq j$, $u_n = \sum\limits_{i=1}^{n} v_i$ is a bounded linear projection of $E$ onto $\big[\bigcup\limits_{i=1}^{n} G_i\big]$ along $\Big[\bigcup\limits_{i=n+1}^{\infty} G_i\Big]$, $n \in \mathbb{N}$. Write $E = \big[\bigcup\limits_{i=1}^{n} G_i\big] \oplus \big[\bigcup\limits_{i=n+1}^{\infty} G_i\big]$, $n \in \mathbb{N}$. Then

$$\{x \in E : v_i(x) = 0, i = 1, 2, \ldots, n\} = \Big[\bigcup_{i=n+1}^{\infty} G_i\Big], \quad n \in \mathbb{N}.$$

Since $(\{G_n, v_n\}, S)$ is not a fusion Banach frame for $E$ with respect to any associated Banach space, there exists $0 \neq x \in \bigcap\limits_{n=1}^{\infty} \big[\bigcup\limits_{i=n+1}^{\infty} G_i\big]$. So, there exists $y_1 = \sum\limits_{i=1}^{m_1} z_i$ where $z_i \in G_i$ $(1 \leq i \leq m_1)$ such that $\operatorname{dist}(x, y_1) < 1$, that is, $\operatorname{dist}\big(x, \big[\bigcup\limits_{i=1}^{m_1} G_i\big]\big) < 1$. Also, $x \in \big[\bigcup\limits_{i=m_1+1}^{\infty} G_i\big]$. So, we can choose $m_2 > m_1$ and $y_2 = \sum\limits_{i=m_1+1}^{m_2} z_i$, where $z_i \in G_i$ $(m_1 + 1 \leq i \leq m_2)$ such that $\operatorname{dist}\big(x, \big[\bigcup\limits_{i=m_1+1}^{m_2} G_i\big]\big) < \frac{1}{2}$. Proceeding like this, for each $n \in \mathbb{N}$, we get a sequence $\{z_n\} \subset E$ and an increasing sequence $\{m_n\}$ of positive integers such that $z_n \in G_n$, $n \in \mathbb{N}$ and $\operatorname{dist}\big(x, \big[\bigcup\limits_{i=m_{n-1}+1}^{m_n} G_i\big]\big) < \frac{1}{n}$.
Thus $x \in [z_{n+1}, z_{n+2}, \ldots]$, $n \in \mathbb{N}$. Consider a sequence $\{x_n\} \subset E$ with $0 \neq x_n \in G_n$, $n \in \mathbb{N}$ such that $x_n = z_n$ whenever $z_n \neq 0$. Then $x \in [x_{n+1}, x_{n+2}, \ldots]$, $n \in \mathbb{N}$. Hence $\bigcap\limits_{n=1}^{\infty} [x_{n+1}, x_{n+2}, \ldots] \neq \{0\}$. □

Finally, we give a characterization of fusion bi-Banach frames in terms of a sequence in $bv_0$, where $bv_0$ is the linear space of all sequences $\{\alpha_n\}$ of scalars with $\lim\limits_{n\to\infty} \alpha_n = 0$ and for which the norm $\|\{\alpha_n\}\| = \sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|$ is finite.

**Theorem 2.6.** *Let $E$ be a Banach space and $(\{G_n, v_n\}, S)$ be a fusion Banach frame for $E$, where the projections $\{v_n\}$ on $E$ are such that $v_i v_j = 0$ for all $i \neq j$. Then $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is a fusion bi-Banach frame for $E$ if and only* *if for every $x \in E$, there exist $\{\alpha_j\} \in bv_0$ and $z \in E$ such that $v_n(x) = \alpha_n v_n(z)$, $n \in \mathbb{N}$ and $\sup\limits_{1\le n<\infty} \big\| \sum\limits_{i=1}^{n} v_i(z)\big\| < \infty$.*

**Proof.** Let $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ be a fusion bi-Banach frame for $E$. For each $k \in \mathbb{N}$, write $u_k = \sum\limits_{i=1}^{k} v_i$. Then $\lim\limits_{k\to\infty} u_k(x) = x$, $x \in E$. Therefore, there exists a sequence $\{m_n\}$ of positive integers such that

$$\|x - u_k(x)\| < \frac{1}{4^{n+1}}\,, \qquad k \ge m_n\,, \quad n \in \mathbb{N}\,.$$

Take $y_n = \sum\limits_{i=m_{n-1}+1}^{m_n} v_i(x)$, $n \in \mathbb{N}$. Then $\|y_n\| \le \frac{2}{4^n}$, $n \in \mathbb{N}$.
So, $\sum\limits_{n=1}^{\infty} 2^{n-1}\|y_n\| \le \sum\limits_{n=1}^{\infty} 2^{-n}$. Thus, the series $\sum\limits_{n=1}^{\infty} 2^{n-1}y_n$ converges.
Put $z = \sum\limits_{n=1}^{\infty} 2^{n-1}y_n$ and $\alpha_j = 2^{1-n}, m_{n-1}+1 \le j \le m_n$, $n \in \mathbb{N}$.
Therefore, $\{\alpha_j\} \in bv_0$. Also, we have

$$v_j(z) = 2^{n-1}v_j(x)\,, \qquad m_{n-1}+1 \le j \le m_n\,,\ n \in \mathbb{N}\,.$$

Hence, $v_j(x) = \alpha_j v_j(z)$, $j \in \mathbb{N}$.

Conversely, for integers $p < q$, we have

$$\begin{aligned}\Big\|\sum_{i=p}^{q} v_i(x)\Big\| &= \Big\|\sum_{i=p}^{q}\alpha_i\Big(\sum_{j=1}^{i} v_j(z) - \sum_{j=1}^{i-1} v_j(z)\Big)\Big\| \\ &\le \Big(|\alpha_p| + \sum_{i=p}^{q-1}|\alpha_i - \alpha_{i+1}| + |\alpha_q|\Big)\sup_{1\le n<\infty}\Big\|\sum_{j=1}^{n} v_j(z)\Big\|\end{aligned}$$

Since, $\{\alpha_j\} \in bv_0$, $\big\{\sum\limits_{i=1}^{n} v_i(x)\big\}$ is a Cauchy sequence and hence converges.
Also, since $\{v_n\}$ is total on $E$ and

$$v_j\Big(x - \lim_{n\to\infty}\sum_{i=1}^{n} v_i(x)\Big) = 0\,, \quad \text{for all} \quad j \in \mathbb{N}\,,$$

it follows that $x = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} v_i(x)$. Therefore, $\Big[\bigcup\limits_{n=1}^{\infty} G_i\Big] = E$. Thus, $\{v_n^*\}$ is total over $E^*$ and so by Lemma 1.2, there exist a Banach space $\mathcal{A}_1$ associated with $E^*$ and an operator $T\colon \mathcal{A}_1 \to E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for $E^*$ with respect to $\mathcal{A}_1$. Hence, $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is a fusion bi-Banach frame for $E$. □

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