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ON A GENERALIZED CLASS OF RECURRENT MANIFOLDS

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Abstract. The object of the present paper is to introduce a non-flat Riemannian manifold called *hyper-generalized recurrent manifolds* and study its various geometric properties along with the existence of a proper example.

1. Introduction

An n-dimensional Riemannian manifold M is said to be locally symmetric due to Cartan if its curvature tensor R satisfies $\nabla R = 0$, where ∇ denotes the Levi-Civita connection. During the last five decades the notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as recurrent manifolds by A. G. Walker [12], 2-recurrent manifolds by A. Lichnerowicz [6], Ricci recurrent manifolds by E. M. Patterson [8], concircular recurrent manifolds by T. Miyazawa [7], [13], weakly symmetric manifolds by L. Tamássy and T. Q. Binh [10], weakly Ricci symmetric manifolds by L. Tamássy and T. Q. Binh [11], conformally recurrent manifolds [1], projectively recurrent manifolds [2], generalized recurrent manifolds [3], generalized Ricci recurrent manifolds [4].

A non-flat n-dimensional Riemannian manifold (M^n, g) $(n \geq 2)$ is said to be a generalized recurrent manifold [3] if its curvature tensor R of type $(0,4)$ satisfies the following:

$$\nabla R = A \otimes R + B \otimes G \,, \tag{1.1}$$

where A and B are 1-forms of which B is non-zero, $\otimes$ is the tensor product, ∇ denotes the Levi-Civita connection, and G is a tensor of type $(0,4)$ given by

$$G(X,Y,Z,U) = g(X,U)g(Y,Z) - g(X,Z)g(Y,U)$$

for all $X, Y, Z, U \in \chi(M^n)$, $\chi(M^n)$ being the Lie algebra of smooth vector fields on M. Such a manifold is denoted by GK_n. Especially, if $B = 0$, the manifold reduces to a recurrent manifold, denoted by K_n ([12]).

The object of the present paper is to introduce a generalized class of recurrent manifolds called *hyper-generalized recurrent manifolds*.

A non-flat n-dimensional Riemannian manifold (M^n, g) $(n \geq 3)$ is said to be

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hyper-generalized recurrent manifold if its curvature tensor R of type $(0,4)$ satisfies the condition

$$\nabla R = A \otimes R + B \otimes (g \wedge S)\,, \tag{1.2}$$

where S is the Ricci tensor of type $(0,2)$, A, B are called associated 1-forms of which B is non-zero such that $A(X) = g(X,\sigma)$ and $B(X) = g(X,\rho)$, and the Kulkarni-Nomizu product $E \wedge F$ of two $(0,2)$ tensors E and F is defined by

$$\begin{aligned}(E \wedge F)(X_1, X_2, X_3, X_4) &= E(X_1,X_4)F(X_2,X_3) + E(X_2,X_3)F(X_1,X_4)\\ &\quad - E(X_1,X_3)F(X_2,X_4) - E(X_2,X_4)F(X_1,X_3)\,,\end{aligned}$$

$X_i \in \chi(M)$, $i = 1,2,3,4$. Such an n-dimensional manifold is denoted by HGK_n. Especially, if the manifold is Einstein with vanishing scalar curvature, then HGK_n reduces to a K_n. And if a HGK_n is Einstein with non-vanishing scalar curvature, then the manifold reduces to a GK_n [4]. Again, if a HGK_n is non-Einstein, then the manifold is neither K_n nor GK_n, and the existence of such manifold is given by a proper example in Section 3. Section 2 deals with some geometric properties of HGK_n.

An n-dimensional Riemannian manifold (M^n, g) $(n \geq 3)$ is said to be generalized Ricci-recurrent if its Ricci tensor is non-vanishing and satisfies the following:

$$\nabla S = A \otimes S + B \otimes g\,, \tag{1.3}$$

where A and B are 1-forms of which B is non-zero. Such a manifold is denoted by GRK_n.

In Section 2 it is shown that a HGK_n with non-vanishing scalar curvature is a GRK_n.

A non-flat Riemannian manifold (M^n, g) $(n > 3)$ is said to be generalized 2-recurrent [6] if its curvature tensor R satisfies

$$(\nabla\nabla R) = \alpha \otimes R + \beta \otimes G\,, \tag{1.4}$$

where α, β are tensors of type $(0,2)$. Again M is said to be generalized 2-Ricci recurrent if its Ricci tensor S is not identically zero and satisfies the following:

$$(\nabla\nabla S) = \alpha \otimes S + \beta \otimes g\,, \tag{1.5}$$

where α, β are tensors of type $(0,2)$.

In Section 2 it is shown that a HGK_n with non-zero constant scalar curvature is a generalized 2-Ricci recurent manifold.

As a special subgroup of the conformal transformation group, Y. Ishii [5] introduced the notion of the conharmonic transformation under which a harmonic function transforms into a harmonic function. The conharmonic curvature tensor $\overline{C}$ of type $(0,4)$ on a Riemannian manifold (M^n, g) $(n > 3)$ (this condition is assumed as for $n = 3$ the Weyl conformal tensor vanishes) is given by

$$\overline{C} = R - \frac{1}{n-2} g \wedge S\,. \tag{1.6}$$

If in (1.1) R is replaced by $\overline{C}$, then the manifold (M^n, g) $(n > 3)$ is called a generalized conharmonically recurrent and is denoted by $G\overline{C}K_n$. Every GK_n is a

$G\overline{C}K_n$ but not conversely. However, the converse is true if it is Ricci recurrent. It is shown that a $G\overline{C}K_n$ satisfying certain condition is a HGK_n. Also it is proved that a $G\overline{C}K_n$ is a K_n if it is GRK_n.

2. Some geometric properties of HGK_n

Let $\{e_i : i = 1, 2, \ldots, n\}$ be an orthonormal basis of the tangent space at any point of the manifold. We now prove the following:

Theorem 2.1. *In a Riemannian manifold (M^n, g) $(n \geq 3)$ the following results hold:*

(i) *A HGK_n with non-vanishing scalar curvature is a GRK_n.*

(ii) *In a HGK_n with non-zero and non-constant scalar curvature (r), the relation*

$$A(QX) + (n-2)B(QX) = \frac{r}{2}[A(X) + 2(n-2)B(X)]\,, \tag{2.1}$$

holds for all X, Q being the symmetric endomorphism corresponding to the Ricci tensor S of type $(0,2)$.

(iii) *In a HGK_n with non-zero constant scalar curvature*

(a) *the associated 1-forms A and B are related by $A + 2(n-1)B = 0$,*

(b) *$\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector σ as well as ρ.*

(iv) *In a non-Einstein HGK_n with vanishing scalar curvature the relations*

$$A(QX) = 0\,, \quad B(QX) = 0\,, \quad A(R(Z,X)\rho) = 0\,, \quad \textit{and}$$
$$A(X)B(R(Y,Z)V) + A(Y)B(R(Z,X)V) + A(Z)B(R(X,Y)V) = 0\,,$$

hold for all X, Y, Z, $V \in \chi(M^n)$.

(v) *A HGK_n $(n > 3)$ of non-vanishing scalar curvature is a $G\overline{C}K_n$.*

(vi) *A HGK_n of vanishing scalar curvature is a conharmonically recurrent manifold.*

(vii) *In a HGK_n with non-vanishing and constant scalar curvature, the associated 1-forms A and B are closed.*

(viii) *A HGK_n with non-zero constant scalar curvature is a generalized 2-Ricci recurent manifold.*

Proof of (i): After suitable contraction, (1.2) yields

$$\nabla S = A_1 \otimes S + B_1 \otimes g\,, \tag{2.2}$$

where A_1 and B_1 are 1-forms given by $A_1 = A + (n-2)B$ and $B_1 = rB$ of which $B_1 \neq 0$ as $r \neq 0$ and $B \neq 0$. This proves (i). □

Proof of (ii): From (2.2), it can be easily shown that the relation (2.1) holds. This proves (ii). □

Proof of (iii): From (2.2) it follows that

$$dr = r[A + 2(n-1)B]\,, \tag{2.3}$$

r being the scalar curvature of the manifold. If r is a non-zero constant, then (2.3) implies that

$$A + 2(n-1)B = 0\,, \tag{2.4}$$

which proves (a) of (iii).

By virtue of (2.4) and (2.1), we obtain

$$A(QX) = \frac{r}{n}A(X)\,, \quad \text{and} \quad B(QX) = \frac{r}{n}B(X)\,, \tag{2.5}$$

provided that r is a non-zero constant. This proves (b) of (iii). □

Proof of (iv): If $r = 0$, then (2.5) implies that $A(QX) = 0$ and $B(QX) = 0$ for all X. Again, by virtue of second Bianchi identity, (1.2) yields

$$\begin{aligned} A(X)R(Y,Z,U,V) + B(X)\{(g \wedge S)(Y,Z,U,V)\} + A(Y)R(Z,X,U,V) \\ + B(Y)\{(g \wedge S)(Z,X,U,V)\} + A(Z)R(X,Y,U,V) \\ + B(Z)\{(g \wedge S)(X,Y,U,V)\} = 0\,. \end{aligned} \tag{2.6}$$

Taking contraction over Y and V in (2.6), we obtain

$$\begin{aligned} A(R(Z,X)U) + [A(X) + (n-3)B(X)]S(Z,U) - [A(Z) + (n-3)B(Z)]S(X,U) \\ + r[B(X)g(Z,U) - B(Z)g(X,U)] + g(X,U)B(QZ) \\ - g(Z,U)B(QX) = 0\,. \end{aligned} \tag{2.7}$$

Again plugging $U = \rho$ in (2.7), we get

$$A(R(Z,X)\rho) = 0\,.$$

Setting $U = \rho$ in (2.6), we obtain

$$A(X)B(R(Y,Z)V) + A(Y)B(R(Z,X)V) + A(Z)B(R(X,Y)V) = 0\,.$$

Proof of (v): From (1.6) it follows that

$$\nabla\overline{C} = \nabla R - \frac{1}{n-2}\big(g \wedge (\nabla S)\big)\,, \tag{2.8}$$

which yields by virtue of (1.2) and (2.2) that

$$\nabla\overline{C} = A \otimes \overline{C} + D \otimes G\,, \tag{2.9}$$

where D is a non-zero 1-form given by

$$D(X) = -\frac{2r}{n-2}B(X)\,.$$

This proves the result. □

Proof of (vi): If $r = 0$, then $D = 0$ and hence (2.9) implies that

$$\nabla\overline{C} = A \otimes \overline{C}\,.$$

Hence the result. □

Proof of (vii): Differentiating (1.2) covariantly and then using (2.2) we obtain

$$
\begin{aligned}
(\nabla_Y \nabla_X R)(Z,W,U,V) = {} & [(\nabla_Y A)(X) + A(X)A(Y)]R(Z,W,U,V) \\
& + [(\nabla_Y B)(X) \\
& + A(X)B(Y) + B(X)A(Y) \\
& + (n-2)B(X)B(Y)](g \wedge S)(Z,W,U,V) \\
& + 2rB(X)B(Y)G(Z,W,U,V)\,.
\end{aligned} \tag{2.10}
$$

Interchanging X and Y and then subtracting the result we obtain

$$
\begin{aligned}
(\nabla_Y \nabla_X R)(Z,W,U,V) &= (\nabla_X \nabla_Y R)(Z,W,U,V) \\
&= [(\nabla_Y A)(X) - (\nabla_X A)(Y)]R(Z,W,U,V) \\
&\quad + [(\nabla_X B)(Y) - (\nabla_Y B)(X)](g \wedge S)(Z,W,U,V)\,.
\end{aligned} \tag{2.11}
$$

From Walker's lemma ([12], equation (26)) we have

$$
\begin{aligned}
(\nabla_X \nabla_Y R)(Z,W,U,V) - (\nabla_Y \nabla_X R)(Z,W,U,V) &+ (\nabla_Z \nabla_W R)(X,Y,U,V) \\
- (\nabla_W \nabla_Z R)(X,Y,U,V) &+ (\nabla_U \nabla_V R)(Z,W,X,Y) \\
- (\nabla_V \nabla_U R)(Z,W,X,Y) &= 0\,.
\end{aligned} \tag{2.12}
$$

By virtue of (2.11), (2.12) yields

$$
\begin{aligned}
P(X,Y)R(Z,W,U,V) + L(X,Y)(g \wedge S)(Z,W,U,V) &+ P(Z,W)R(X,Y,U,V) \\
+ L(Z,W)(g \wedge S)(X,Y,U,V) &+ P(U,V)R(Z,W,X,Y) \\
+ L(U,V)(g \wedge S)(Z,W,X,Y) &= 0\,,
\end{aligned} \tag{2.13}
$$

where $P(X,Y) = (\nabla_X A)(Y) - (\nabla_Y A)(X)$
and $L(X,Y) = (\nabla_X B)(Y) - (\nabla_Y B)(X)$.
If the scalar curvature is a non-zero constant, then we have the relation (2.4). Using (2.4) in (2.13) we obtain

$$
\begin{aligned}
P(X,Y)H(Z,W,U,V) + P(Z,W)H(X,Y,U,V) \\
+ P(U,V)H(Z,W,X,Y) = 0
\end{aligned} \tag{2.14}
$$

where $H = R - \frac{1}{2(n-1)}(g \wedge S)$, from which it follows that H is a symmetric $(0,4)$ tensor with respect to the first pair of two indices and the last pair of two indices. Consequently by virtue of Walker's lemma ([12], equation (27)) we obtain

$$P(X,Y) = L(X,Y) = 0$$

for all X, Y. And hence

$$(\nabla_X A)(Y) - (\nabla_Y A)(X) = 0\,,$$

$$(\nabla_X B)(Y) - (\nabla_Y B)(X) = 0\,.$$

Therefore $dA(X,Y) = 0$, $dB(X,Y) = 0$. This proves (vii). □

Proof of (viii): If the manifold is of non-zero constant scalar curvature, then from (2.2) it follows that

$$\begin{aligned}(\nabla_Y\nabla_X S)(Z,W) = {} & [(\nabla_Y A)(X) + (n-2)(\nabla_Y B)(X)]S(Z,W) \\ & + [A(X) + (n-2)B(X)][A(Y) + (n-2)B(Y)]S(Z,W) \\ & + rg(Z,W)[(\nabla_Y B)(X) + B(Y)\{A(X) + (n-2)B(X)\}] .\end{aligned} \tag{2.15}$$

Interchanging X, Y and subtracting the result, we obtain

$$\begin{aligned}(\nabla_X\nabla_Y S)(Z,W) - (\nabla_Y\nabla_X S)(Z,W) = {} & [P(X,Y) + (n-2)L(X,Y)] \\ & \times S(Z,W) + rg(Z,W)[L(X,Y) + A(Y)B(X) - A(X)B(Y)] .\end{aligned} \tag{2.16}$$

In view of (2.16) and (2.2) we obtain

$$(R(X,Y)\cdot S)(Z,W) = K(X,Y)g(Z,W) + N(X,Y)S(Z,W) , \tag{2.17}$$

where $K(X,Y) = r\,[A(Y)B(X) - A(X)B(Y) + XB(Y) - YB(X) - 2B([X,Y])]$ and

$$N(X,Y) = XA(Y) - YA(X) - 2A([X,Y]) + (n-2)\,[XB(Y) - YB(X) - 2B([X,Y])] \ .$$

The relation (2.17) implies that the manifold is a generalized 2-Ricci recurrent. This proves (viii). □

Theorem 2.2.

(i) *A $G\overline{C}K_n$ $(n > 3)$ is a HGK_n provided it satisfies*

$$\nabla S = -\frac{n-2}{2} B \otimes g . \tag{2.18}$$

(ii) *A $G\overline{C}K_n$ $(n > 3)$ is a GK_n if it is Ricci recurrent.*

(iii) *A $G\overline{C}K_n$ $(n > 3)$ is recurrent if it satisfies*

$$\nabla S = A \otimes S - \frac{n-2}{2} B \otimes g . \tag{2.19}$$

Proof of (i): If the manifold is $G\overline{C}K_n$ $(n > 3)$, then we have

$$\nabla \overline{C} = A \otimes \overline{C} + B \otimes G ,$$

which yields, by virtue of (1.6), that

$$\nabla R - \frac{1}{n-2}(g \wedge (\nabla S)) = A \otimes (R - \frac{1}{n-2} g \wedge S) + B \otimes G . \tag{2.20}$$

By virtue of (2.18), (2.20) takes the form

$$\nabla R = A \otimes R + C \otimes (g \wedge S) ,$$

where C is a 1-form given by $C = -\frac{1}{n-2}A$. This proves (i). □

Proof of (ii): If the manifold is Ricci recurrent ($\nabla S = A \otimes S$), then (2.20) takes the form (1.1) and hence the result. □

Proof of (iii): In view of (2.19), (2.20) reduces to

$$\nabla R = A \otimes R .$$

□

3. An example of $HGK_n (n > 3)$ which is not GK_n

In this section the existence of HGK_n is ensured by a proper example.

Example 3.1. We consider a Riemannian manifold $(\mathbb{R}^4, g)$ endowed with the metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (1+2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], \quad (i,j = 1,2,...,4) \tag{3.1}$$

where $q = \frac{e^{x^1}}{k^2}$ and k is a non-zero constant. This metric was first appeared in a paper of Shaikh and Jana [9]. The non-vanishing components of the Christoffel symbols of second kind, the curvature tensor and their covariant derivatives are

$$\Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = -\frac{q}{1+2q}, \quad \Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{q}{1+2q},$$

$$R_{1221} = R_{1331} = R_{1441} = \frac{q}{1+2q}, \quad R_{2332} = R_{2442} = R_{4334} = \frac{q^2}{1+2q},$$

$$R_{1221,1} = R_{1331,1} = R_{1441,1} = \frac{q(1-4q)}{(1+2q)^2},$$

$$R_{2332,1} = R_{2442,1} = R_{4334,1} = \frac{2q^2(1-q)}{(1+2q)^2}.$$

From the above components of the curvature tensor, the non-vanishing components of the Ricci tensor and scalar curvature are obtained as

$$S_{11} = \frac{3q}{(1+2q)^2}, \quad S_{22} = S_{33} = S_{44} = \frac{q}{(1+2q)}, \quad r = \frac{6q(1+q)}{(1+2q)^3} \neq 0.$$

We consider the 1-forms as follows:

$$A(\partial_i) = A_i = \begin{cases} \frac{2q^3-6q^2-6q+1}{(1+2q)(1-q^2)} & \text{for} \quad i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$B(\partial_i) = B_i = \begin{cases} \frac{q}{2(1-q^2)} & \text{for} \quad i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\partial_i = \frac{\partial}{\partial u^i}$, u^i being the local coordinates of $\mathbb{R}^4$.
In our $\mathbb{R}^4$, (1.2) reduces with these 1-forms to the following equations:

$$R_{1ii1,\,1} = A_1 R_{1ii1} + B_1[S_{ii}g_{11} + S_{11}g_{ii}] \quad \text{for} \quad i = 2,3,4, \tag{3.2}$$

$$R_{2ii2,\,1} = A_1 R_{2ii2} + B_1[S_{ii}g_{22} + S_{22}g_{ii}] \quad \text{for} \quad i = 3,4, \tag{3.3}$$

$$R_{4334,\,1} = A_1 R_{4334} + B_1[S_{44}g_{33} + S_{33}g_{44}]. \tag{3.4}$$

For $i = 2$,

$$\begin{aligned} \text{L.H.S. of (3.2)} = R_{1221,\,1} &= \frac{q(1-4q)}{(1+2q)^2} \\ &= A_1 R_{1221} + B_1[S_{22}g_{11} + S_{11}g_{22}] \\ &= \text{R.H.S. of (3.2)}\,. \end{aligned}$$

Similarly for $i = 3, 4$, it can be shown that the relation is true. By a similar argument it can be shown that (3.3) and (3.4) are also true. Hence the manifold under consideration is a HGK_4. Thus we can state the following:

Theorem 3.1. *Let* $(\mathbb{R}^4, g)$ *be a Riemannain manifold equipped with the metric given by* (3.1)*. Then* $(\mathbb{R}^4, g)$ *is a* HGK_4 *with non-vanishing and non-constant scalar curvature which is neither* GK_4 *nor* K_4*.*

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