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LEPAGE FORMS THEORY APPLIED

Michal Lenc, Jana Musilová, and Lenka Czudková

Abstract. In the presented paper we apply the theory of Lepage forms on jet prolongations of fibred manifold with one-dimensional base to the relativistic mechanics. Using this geometrical theory, we obtain and discuss some well-known conservation laws in their general form and apply them to a concrete physical example.

1. Introduction

In variational physical theories conservation laws are closely related to invariance transformations connected with the corresponding Lagrange structure. This fact is expressed by Noether theorems. Although it is usual in physics to treat these problems directly in coordinates, the variational theories give a correct and effective coordinate free way for to solve them. Our approach is presented in more general form for mechanics as well as field theory in [5]. It is based on the theory of Lepage forms and Lepage equivalents of Lagrangians on fibred manifolds developed by Krupka (see e.g. [1]). Here we show the effectiveness of this approach by means of some examples from mechanics. In mechanics itself this could look too simply. Nevertheless, the use of such a method is very useful in situations where the base of an underlying fibred manifold is multidimensional and thus the direct coordinate calculations are not too lucid (moreover, they are sometimes incorrect from the mathematical point of view). Immediate applications occur readily in classical field theories, including classical bosonic strings.

2. Underlying structures

2.1. Notations.

Let (Y, π, X) be a fibred manifold with one-dimensional base X, total space Y of dimension $m+1$ and the surjective submersion $\pi\colon Y \to X$. Denote by $(J^r Y, \pi_r, X)$, $r \geq 0$, the r-jet prolongation of (Y, π, X) where we put $J^0 Y = Y$, $\pi_0 = \pi$, and $\pi_{s,r}\colon J^s Y \to J^r Y$, $0 \leq r < s$, canonical projections. Let (V, ψ) be a local fibred chart on Y where $V \subset Y$ is an open set and $\psi = (t, q^\sigma)$, $1 \leq \sigma \leq m$. The pair (U, φ), $U = \pi(V)$, $\varphi = (t)$, is the associated fibred chart on X. A smooth mapping $\gamma\colon U \to Y$ such that $\pi \circ \gamma = \mathrm{Id}_U$ is called a local section of π on U. Thus, the pair

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(V_r, ψ_r) where $V_r = \pi_{r,0}^{-1}(V)$, $\psi_r = (t, q^\sigma, q_1^\sigma, \dots, q_r^\sigma)$ and $q_k^\sigma = \frac{\mathrm{d}^k q^\sigma \gamma\varphi^{-1}}{\mathrm{d}t^k}$, $1 \le k \le r$, is the associated fibred chart on (J^rY, π_r, X). For the simplicity we will use also the notation $q_1^\sigma = \dot{q}^\sigma$ and $q_2^\sigma = \ddot{q}^\sigma$.

A vector field ξ on Y is called *π-projectable* if there is a vector field ξ_0 on X such that $T\pi\,\xi = \xi_0 \circ \pi$ and it is called *π-vertical* if $T\pi\,\xi = 0$. For π-projectable vector field it holds $\xi = \xi^0(t)\frac{\partial}{\partial t} + \xi^\sigma(t, q^\nu)\frac{\partial}{\partial q^\sigma}$ and for its s-jet prolongation $J^s\xi = \xi^0(t)\frac{\partial}{\partial t} + \xi^\sigma(t, q^\nu)\frac{\partial}{\partial q^\sigma} + \sum_{k=1}^{s} \xi_k^\sigma(t, q^\sigma, q_1^\sigma, \dots, q_k^\sigma)\frac{\partial}{\partial q_k^\sigma}$ where $\xi_k^\sigma = \frac{\mathrm{d}\xi_{k-1}^\sigma}{\mathrm{d}t} - q_k^\sigma \frac{\mathrm{d}\xi^0}{\mathrm{d}t}$. Analogously, a *$\pi_s$-projectable vector field*, *$\pi_{s,r}$-projectable vector field*, *π_s-vertical vector field* and *$\pi_{s,r}$-vertical vector field* are defined.

A differential form η on J^sY is called *π_s-horizontal* if $i_\xi\eta = 0$ for every π_s-vertical vector field on J^sY and it is called *contact* if $J^s\gamma^*\eta = 0$ for every section γ of π. Analogously, a *$\pi_{s,r}$-horizontal form* is defined. Differential forms $\omega^\sigma = \mathrm{d}q^\sigma - q_1^\sigma\mathrm{d}t$, $\omega_2^\sigma = \mathrm{d}q_1^\sigma - q_2^\sigma\mathrm{d}t, \dots, \omega_{s-1}^\sigma = \mathrm{d}q_{s-1}^\sigma - q_s^\sigma\mathrm{d}t$ and $\mathrm{d}q_s^\sigma$ form the basis of 1-forms on J^sY adapted to the contact structure. Recall that every k-form η on J^sY can be uniquely decomposed as follows: $\pi_{s+1,s}^*\eta = \mathrm{p}_{k-1}\eta + \mathrm{p}_k\eta$ where $\mathrm{p}_{k-1}\eta$ and $\mathrm{p}_k\eta$ are called *$(k-1)$-contact component* and *k-contact component* of η, respectively. A k-form η is called *k-contact* if $\mathrm{p}_{k-1}\eta = 0$, and is called $(k-1)$-contact if $\mathrm{p}_k\eta = 0$. For $k = 1$ we have $\mathrm{p}_{k-1}\eta = \mathrm{p}_0\eta = \mathrm{h}\eta$ the *horizontal component* of η.

2.2. **Lagrange structures.**

Let $W \subset Y$ be an open set. A horizontal 1-form Λ on $W_r = \pi_{r,0}^{-1}(W)$ is called *Lagrangian of order* r, i.e. $\Lambda = L(t, q^\sigma, q_1^\sigma, \dots, q_r^\sigma)\,\mathrm{d}t + \frac{\mathrm{d}\chi\left(t, q^\sigma, \dots, q_{r-1}^\sigma\right)}{\mathrm{d}t}\,\mathrm{d}t$. The pair (π, Λ) is called *Lagrange structure* and $\mathrm{h}\,\mathrm{d}\chi = \frac{\mathrm{d}\chi}{\mathrm{d}t}\,\mathrm{d}t$ is trivial Lagrangian. Let Ω be a compact submanifold of X with boundary $\partial\Omega$. Denote by $\Gamma_{\Omega,W}$ the set of sections γ of π defined on a neighbourhood of Ω such that $\gamma(\Omega) \subset W$. The mapping

$$\Lambda_\Omega\colon \Gamma_{\Omega,W} \ni \gamma \to \Lambda_\Omega(\gamma) = \int_\Omega J^r\gamma^*\Lambda \tag{1}$$

is called the *variational function* or the *action function of the Lagrangian* Λ *over* Ω. For the π-projectable vector field ξ on Y we have so-called *first variation of the action function* Λ_Ω *induced by* ξ

$$(\partial_{J^r\xi}\Lambda_\Omega)\colon \Gamma_{\Omega,W} \ni \gamma \to (\partial_{J^r\xi}\Lambda_\Omega)(\gamma) = \int_\Omega J^r\gamma^*\partial_{J^r\xi}\Lambda\,. \tag{2}$$

The section γ of π is called *extremal* of the Lagrange structure (π, Λ) on Ω if $\int_\Omega J^r\gamma^*\partial_{J^r\xi}\Lambda = 0$ for every π-projectable vector field ξ defined in a neighbourhood of $\gamma(\Omega)$ such that $\operatorname{supp}\xi \subset \pi^{-1}(\Omega)$.

3. Lepage equivalents and conservation laws

3.1. **Lepage equivalents.**

Let us recall the concept of Lepage equivalent in mechanics (cf. [3], for the quite general definition concerning a field theory see [5], [1]). Let $W \subset Y$ be an open set. A 1-form ϱ on $W_s = \pi_{s,0}^{-1}(W)$ is called a *Lepage 1-form*, if $\mathrm{p}_1\mathrm{d}\varrho$ is a $\pi_{s+1,0}$-horizontal

$(n+1)$-form. Recall that a 1-form ϱ on W_s is a Lepage 1-form, if and only if it holds $\mathrm{h}i_\xi\,\mathrm{d}\varrho = 0$ for every $\pi_{s,0}$-vertical vector field ξ on W_s (see [1]).

Let $\Lambda = \bar{L}\,\mathrm{d}t = L\mathrm{d}t + \frac{\mathrm{d}\chi}{\mathrm{d}t}\,\mathrm{d}t$ be a r-th order Lagrangian (including a possible r-th order trivial Lagrangian, i.e. the total derivative of a function χ on W_{r-1}). The *Lepage equivalent* of Λ is such a Lepage form θ_Λ for which $\mathrm{h}\theta_\Lambda = \Lambda$ (up to a possible projection). For mechanics, the Lepage equivalent is unique and it is of order $2r-1$. It holds

$$\theta_\Lambda = \bar{L}\mathrm{d}t + \sum_{i=0}^{r-1}\left(\sum_{k=0}^{r-i-1}(-1)^k\frac{\mathrm{d}^k}{\mathrm{d}t^k}\,\frac{\partial\bar{L}}{\partial q^\sigma_{i+1+k}}\right)\omega^\sigma_i\,. \tag{3}$$

For example, for a first order Lagrangian we have

$$\theta_\Lambda = L\mathrm{d}t + \frac{\partial L}{\partial\dot{q}^\sigma}\omega^\sigma + \mathrm{d}\chi\,. \tag{4}$$

Now, using the formula $\partial_\Xi\eta = i_\Xi\mathrm{d}\eta + \mathrm{d}i_\Xi\eta$ and taking into account the properties of the Lepage equivalent of Λ we obtain the *infinitesimal first variational formula*

$$J^r\gamma^*\partial_{J^r\xi}\Lambda = J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\,\mathrm{d}\theta_\Lambda + \mathrm{d}J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\theta_\Lambda\,. \tag{5}$$

Integrating (5) and applying the Stokes theorem we obtain

$$\int_\Omega J^r\gamma^*\partial_{J^r\xi}\Lambda = \int_\Omega J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\,\mathrm{d}\theta_\Lambda + \int_{\partial\Omega} J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\theta_\Lambda\,. \tag{6}$$

The $\pi_{2r,0}$-horizontal form $E_\Lambda = \mathrm{p}_1\,\mathrm{d}\theta_\Lambda$ is the well-known *Euler-Lagrange form* of Λ. The following theorem is an immediate consequence of preceding considerations.

Theorem. *Let Λ be a Lagrangian of order r on $W_r = \pi^{-1}_{r,0}(W)$ and let θ_Λ be its Lepage equivalent. Then, the section $\gamma \in \Gamma_{\Omega,W}$ is the extremal of Lagrange structure (π,Λ) on Ω if and only if for every π-vertical vector field ξ on W such that* $\operatorname{supp}(\xi\circ\gamma)\subset\Omega$ *it holds*

$$J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\,\mathrm{d}\theta_\Lambda = 0\,, \tag{7}$$

or equivalently, E_Λ vanishes along $J^{2r}\gamma$.

3.2. **Invariance transformations and conservation laws.**

A local automorphism α on Y is called an *invariance transformation of Lagrangian* Λ if it holds

$$J^r\alpha^*\Lambda - \Lambda = 0\,. \tag{8}$$

A π-projectable vector field ξ on Y is called a *generator of invariance transformations of Lagrangian* Λ if its local one-parametrical group consists of invariance transformations of Λ. Then

$$\partial_{J^r\xi}\Lambda = 0\,. \tag{9}$$

Thus, for the extremal γ on Ω and for a generator of invariance transformations ξ we have, using (6), (7) and (9),

$$\int_{\partial\Omega} J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\theta_\Lambda = 0\,. \tag{10}$$

The last integral represents the flow of the quantity $J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\theta_\Lambda$ through the boundary $\partial\Omega$ of Ω. It holds

$$J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\theta_\Lambda = J^{2r}\gamma^*\pi^*_{2r,2r-1} i_{J^{2r-1}\xi}\theta_\Lambda = \mathrm{h} i_{J^{2r-1}\xi}\theta_\Lambda\,.$$

We call

$$\Psi\left(\xi\right) = \mathrm{h} i_{J^{2r-1}\xi}\theta_\Lambda \tag{11}$$

the *elementary flow*, as a quantity obeying a conservation law along extremals. The definition relation (11) includes possible trivial Lagrangians as well, i.e. it contains the "free" term $\mathrm{h} i_{J^{2r-1}\xi}\,\mathrm{d}\chi$ (see above).

A local automorphism α on Y is called an *invariance transformation of Euler-Lagrange form* $E_\Lambda = E_\sigma\omega^\sigma \wedge \mathrm{d}t$ if it holds

$$J^{2r}\alpha^* E_\Lambda - E_\Lambda = 0\,. \tag{12}$$

A π-projectable vector field ξ on Y is called a *generator of invariance transformations of Euler-Lagrange form* E_Λ, if its local one-parametrical group of transformations consists of invariance transformations of E_Λ, i.e.

$$\partial_{J^{2r}\xi}E_\Lambda = 0\,. \tag{13}$$

Because of the identity

$$J^{2r}\alpha^* E_\Lambda = E_{J^r\alpha^*\Lambda}$$

(see e.g. [1]), it is evident that every invariance transformation of Λ is an invariance transformation of E_Λ, and for every invariance transformation of E_Λ the Lagrangian $\widetilde{\Lambda} = J^r\alpha^*\Lambda - \Lambda$, or alternatively $\widetilde{\Lambda} = \partial_{J^r\xi}\Lambda$, is trivial. Thus, it holds $\partial_{J^r\xi}\Lambda = -\,\mathrm{h}\,\mathrm{d}\eta$, where η is a function on W_{r-1} and sign "-" is formal. The corresponding flows (the quantities remaining constant along extremals) can be obtained as follows. It holds

$$J^r\gamma^*\partial_{J^r\xi}\Lambda = J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\,\mathrm{d}\theta_\Lambda + \mathrm{d}J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\theta_\Lambda\,.$$

The left-hand side can be written as $-J^r\gamma^*\mathrm{h}\,\mathrm{d}\eta = -J^r\gamma^*\,\mathrm{d}\eta = -\mathrm{d}J^r\gamma^*\eta$. The first term on the right-hand side vanishes along extremals. So we have

$$\mathrm{d}J^{2r-1}\gamma^*\left(i_{J^{2r-1}\xi}\theta_\Lambda + \eta\right) = 0\,,$$

$$\Psi(\xi) = \mathrm{h} i_{J^{2r-1}\xi}\theta_\Lambda + \eta\,, \tag{14}$$

where η is a function on W_{r-1}. Recall that ξ is a generator of invariance transformations of E_Λ.

In the following, we will focus on invariance transformations of Lagrangians only.

4. Lagrangians for relativistic particles

4.1. **First order Lagrange structures.**

As typical Lagrangians of relativistic particles are of the first order, let us discuss this case in general. The corresponding Lagrange structure is

$$(\pi,\Lambda),\quad (Y,\pi,X),\quad \dim X = 1,\quad \dim Y = 5,\quad \Lambda = L(\tau, x^\mu, \dot{x}^\mu)\,\mathrm{d}\tau + \frac{\mathrm{d}\chi}{\mathrm{d}\tau}\,\mathrm{d}\tau$$

where $\chi = \chi(\tau, x^\mu)$. The base X is a space of "non-physical" parameter, every fibre over $\tau \in X$ is the time-space with metrics $g = (g_{\alpha\beta})$, $1 \le \alpha, \beta \le 4$ where $g_{\alpha\beta} = g_{\alpha\beta}(x^\mu)$. The expression of Λ includes the *minimal Lagrangian* $L\,\mathrm{d}\tau$ and an arbitrary trivial Lagrangian $\mathrm{h}\,\mathrm{d}\chi = \frac{\mathrm{d}\chi}{\mathrm{d}\tau}\,\mathrm{d}\tau$. In examples we also put $x^4 = ct$, where t is the time coordinate (see Sec. 4.3). Following the previous section we have

$$\Psi(\xi) = \mathrm{h}i_{J^1\xi}\theta_\Lambda\,, \qquad \theta_\Lambda = L\,\mathrm{d}\tau + \frac{\partial L}{\partial \dot{x}^\mu}\,\omega^\mu + \mathrm{d}\chi\,, \qquad \xi = \xi^0\frac{\partial}{\partial\tau} + \xi^\mu\frac{\partial}{\partial x^\mu}\,,$$

$$\Psi(\xi) = \left(L - \dot{x}^\mu\frac{\partial L}{\partial \dot{x}^\mu}\right)\xi^0 + \frac{\partial L}{\partial \dot{x}^\mu}\xi^\mu + \Psi_0, \qquad \Psi_0 = \frac{\partial\chi}{\partial\tau}\xi^0 + \frac{\partial\chi}{\partial x^\mu}\xi^\mu \tag{15}$$

where ξ is a generator of invariance transformations of the Lagrangian. It is given by

$$0 = \partial_{J^1\xi}\Lambda = i_{J^1\xi}\mathrm{d}\Lambda + \mathrm{d}i_{J^1\xi}\Lambda\,.$$

After some simple calculations we obtain the condition for components of generators of invariance transformations — *Noether equation*

$$\frac{\partial L}{\partial x^\mu}\xi^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\frac{\mathrm{d}\xi^\mu}{\mathrm{d}\tau} - \dot{x}^\mu\frac{\partial L}{\partial \dot{x}^\mu}\frac{\mathrm{d}\xi^0}{\mathrm{d}\tau} + \frac{\partial L}{\partial\tau}\xi^0 + L\frac{\mathrm{d}\xi^0}{\mathrm{d}\tau} + \frac{\mathrm{d}\Psi_0}{\mathrm{d}\tau} = 0\,. \tag{16}$$

4.2. **Generalized quadratic Lagrangian.**

Let us generalize the standard quadratic Lagrangian by considering metrics $f_X = f_{\tau\tau}(\tau)\,\mathrm{d}\tau\otimes\mathrm{d}\tau$ on the base X of the underlying fibred manifold, i.e.

$$f = f_{\tau\tau}(\tau)\,\mathrm{d}\tau\otimes\,\mathrm{d}\tau + g_{\alpha\beta}(x^\mu)\,\mathrm{d}x^\alpha\otimes\,\mathrm{d}x^\beta\,.$$

Denoting $F = \det(f_{\tau\tau})$ we can write

$$\Lambda = L\,\mathrm{d}\tau + \frac{\mathrm{d}\chi}{\mathrm{d}\tau}\,\mathrm{d}\tau = -\frac{1}{2}\sqrt{F}\left(f^{\tau\tau}g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta + m^2c^2\right)\,\mathrm{d}\tau + \frac{\mathrm{d}\chi}{\mathrm{d}\tau}\,\mathrm{d}\tau\,. \tag{17}$$

The Lepage equivalent is then

$$\theta_\Lambda = \frac{1}{2}\sqrt{F}\left\{\left(f^{\tau\tau}\,g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta - m^2c^2\right)\,\mathrm{d}\tau - 2f^{\tau\tau}\,g_{\alpha\beta}\,\dot{x}^\beta\,\mathrm{d}x^\alpha\right\} + \mathrm{d}\chi\,.$$

Putting L into (15) we obtain the flows

$$\Psi(\xi) = \frac{1}{2}\sqrt{F}\left\{\left(f^{\tau\tau}\,g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta - m^2c^2\right)\xi^0 - 2f^{\tau\tau}\,g_{\alpha\beta}\,\dot{x}^\alpha\xi^\beta\right\} + \Psi_0 \tag{18}$$

where $\xi = \xi^0\frac{\partial}{\partial\tau} + \xi^\mu\frac{\partial}{\partial x^\mu}$ are again invariance transformations given by the Noether equation (16). Using the condition

$$f_{\tau\tau}\cdot f^{\tau\tau} = 1 \quad\Longrightarrow\quad \dot{f}_{\tau\tau}\,f^{\tau\tau} + \dot{f}^{\tau\tau}\,f_{\tau\tau} = 0$$

we obtain after some calculations the following condition for invariance transformations

$$\partial_{J^1\xi}\Lambda = -\frac{1}{2}\sqrt{F}\left\{\dot{x}^\alpha\dot{x}^\beta\left[\left(f^{\tau\tau}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\,\xi^\mu + f^{\tau\tau}\,g_{\mu\alpha}\frac{\partial\xi^\mu}{\partial x^\beta} + f^{\tau\tau}\,g_{\mu\beta}\,\frac{\partial\xi^\mu}{\partial x^\alpha}\right)\right.\right.$$
$$\left. -f^{\tau\tau}\,g_{\alpha\beta}\left(\frac{\mathrm{d}\xi^0}{\mathrm{d}\tau} - \frac{1}{2}\,\xi^0\,\frac{\dot{f}^{\tau\tau}}{f^{\tau\tau}}\right)\right] + \dot{x}^\alpha\left(2f^{\tau\tau}\,g_{\mu\alpha}\frac{\partial\xi^\mu}{\partial\tau} - \frac{2}{\sqrt{F}}\frac{\partial\Psi_0}{\partial x^\alpha}\right)$$
$$\left. + m^2c^2\left(\frac{\mathrm{d}\xi^0}{\mathrm{d}\tau} - \frac{1}{2}\,\xi^0\,\frac{\dot{f}^{\tau\tau}}{f^{\tau\tau}} - \frac{2}{m^2c^2\sqrt{F}}\frac{\partial\Psi_0}{\partial\tau}\right)\right\} = 0\,.$$

All coefficients of this polynomial in velocities must vanish, i.e.

$$\frac{\mathrm{d}\xi^0}{\mathrm{d}\tau} - \frac{1}{2}\xi^0\,\frac{\dot{f}^{\tau\tau}}{f^{\tau\tau}} - \frac{2}{m^2c^2\sqrt{F}}\frac{\partial\Psi_0}{\partial\tau} = 0\,, \qquad 2f^{\tau\tau}\,g_{\mu\alpha}\frac{\partial\xi^\mu}{\partial\tau} - \frac{2}{\sqrt{F}}\frac{\partial\Psi_0}{\partial x^\alpha} = 0$$
$$\left(\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\,\xi^\mu + g_{\mu\alpha}\frac{\partial\xi^\mu}{\partial x^\beta} + g_{\mu\beta}\,\frac{\partial\xi^\mu}{\partial x^\alpha}\right) - g_{\alpha\beta}\left(\frac{\mathrm{d}\xi^0}{\mathrm{d}\tau} - \frac{1}{2}\xi^0\,\frac{\dot{f}^{\tau\tau}}{f^{\tau\tau}}\right) = 0\,.$$

Recall that Ψ_0 is given by (15), i.e. it contains unknown components of invariance transformations.

For minimal Lagrangian, i.e. $\chi = 0$, we obtain

$$\frac{\mathrm{d}\xi^0}{\mathrm{d}\tau} - \frac{1}{2}\xi^0\,\frac{\dot{f}^{\tau\tau}}{f^{\tau\tau}} = 0,\; g_{\alpha\beta}\,\frac{\partial\xi^\alpha}{\partial\tau} = 0,\; \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\xi^\mu + g_{\mu\beta}\frac{\partial\xi^\mu}{\partial x^\alpha} + g_{\alpha\mu}\frac{\partial\xi^\mu}{\partial x^\beta} = 0\,. \tag{19}$$

We can see that the third condition is the equation for Killing vector field, the solution of the first two equations is

$$\xi^0(\tau) = K\,\sqrt{f^{\tau\tau}} = \frac{K}{\sqrt{F}} \quad \text{where } K \text{ is a constant,} \qquad \frac{\partial\xi^\alpha}{\partial\tau} = 0\,.$$

Thus, we can choose metrics $f_{\tau\tau}$ arbitrarily. The choice of a general Lagrangian changes the conditions (19) slightly, the main difference is the equation for ξ^μ, now the equation for the component of a homothetic Killing field.

4.3. **A relativistic particle as a non-holonomic mechanical system.**

In the preceding sections we have described a non-zero mass particle in the special relativity theory as a system on the five-dimensional fibred manifold $\pi\colon \mathbf{R}\times\mathbf{R}^4 \to \mathbf{R}$, where $\mathbf{R}^4$ is the Minkowski space-time. The evolution space of such a particle is the prolongation $\pi_1\colon \mathbf{R}\times\mathbf{R}^4\times\mathbf{R}^4 \to \mathbf{R}$. This corresponds to a "non-physical" parameter τ measured along the base of the underlying fibred manifold, trajectories of the particle being four-dimensional curves. Such a situation corresponds to a four-dimensional observer. On the other hand, the existence of the standard condition on four-velocity can be considered as a constraint condition in evolution space. Thus, a relativistic particle can be considered as a first order mechanical system subjected to the non-holonomic constraint. This enables us to adapt the description of the particle motion to a three-dimensional observer. Such an approach was applied in [4] on the base of the geometrical theory of non-holonomic constraints formulated in [2].

The standard condition for four-velocity is

$$(\dot x^4)^2 - \sum_{p=1}^{3} (\dot x^p)^2 = 1 \implies \dot x^4 = \pm \sqrt{1 + \sum_{p=1}^{3} (\dot x^p)^2}\,, \tag{20}$$

where (τ, x^μ), $1 \le \mu \le 4$, are coordinates on $\mathbf{R} \times \mathbf{R}^4$, x^ℓ for $1 \le \ell \le 3$ are cartesian coordinates and $x^4 = ct$, where c is the light speed and t is time. Moreover, $\dot x^\mu = \mathrm{d}x^\mu / \mathrm{d}\tau$. The constraint condition (20) defines a *constraint submanifold* $\mathcal{Q}$ in the evolution space $\pi_1 \colon \mathbf{R} \times \mathbf{R}^4 \times \mathbf{R}^4 \to \mathbf{R}$. Excluding points with $\dot x^4 = 0$ the condition (20) gives a constraint submanifold $\mathcal{Q}$ as a union of two connected components $\mathcal{Q}_+$ and $\mathcal{Q}_-$, given by signs "+" and "-", respectively. Without loss of generality we choose the component $\mathcal{Q}_+$.

Choosing appropriate coordinates, one can express the constraint condition (as well as equations of motion of the particle) in a form adapted to a three-dimensional observer. For $\dot x^4 \ne 0$ consider new coordinates $(\tau, x^\ell, t, v^\ell, \dot x^4)$, $1 \le \ell \le 3$, defined by the transformation equations

$$\dot x^\ell = \frac{1}{c} v^\ell \dot x^4\,. \tag{21}$$

These coordinates are global, but they are not fibred coordinates for original projection π of the underlying fibred manifold. Note that (t, x^ℓ, v^ℓ) are coordinates on $\mathbf{R} \times \mathbf{R}^3 \times \mathbf{R}^3$, adapted to the fibration $\mathbf{R} \times \mathbf{R}^3 \to \mathbf{R}$ of $\mathbf{R}^4$. Here $\vec r = (x^1, x^2, x^3)$ is a usual position vector and $\vec v = (v^1, v^2, v^3)$ is the usual velocity. In new coordinates we have for the constraint (20)

$$\left(1 - \frac{v^2}{c^2}\right) (\dot x^4)^2 = 1 \Longrightarrow \dot x^4 = c\,\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\,. \tag{22}$$

5. Quadratic Lagrangian

Now, let us put in (17) $f^{\tau\tau} = m^2$ and $\chi = 0$, i.e.

$$\Lambda = L\,\mathrm{d}\tau, \qquad L = -\frac{m}{2}\, g_{\alpha\beta}\, \dot x^\alpha\, \dot x^\beta - \frac{1}{2} mc^2. \tag{23}$$

As the two special cases of invariance transformations we can consider

1) $\xi^0 = 1$, $\xi^\mu = 0$, i.e. the vector field is $\xi = \xi_0 = \frac{\partial}{\partial \tau}$,
2) $\xi^0 = 0$, i.e. the vector field $\xi_V = \xi^\mu \frac{\partial}{\partial x^\mu}$ is vertical, ξ^μ being components of the Killing vector field.

We obtain corresponding flows

$$\Psi(\xi_0) = -\frac{m}{2} \left(g_{\alpha\beta} \dot x^\alpha \dot x^\beta - c^2 \right), \quad \Psi(\xi_V) = -m g_{\alpha\beta} \dot x^\beta\, \xi^\alpha\,.$$

Denote

$$\Psi_\tau = \frac{m}{2} \left(g_{\alpha\beta} \dot x^\alpha \dot x^\beta - c^2 \right) \qquad \text{and} \qquad \Psi_\alpha = m g_{\alpha\beta} \dot x^\beta\,. \tag{24}$$

The quantity (Ψ_α) represents the four-momentum of the particle, as it will be shown below. Recall that the problem is regular, because of regularity of metrics g.

Let us to consider the Hamilton formulation of our problem, i.e. calculate the momenta and Hamiltonian.

$$p_\mu = -\frac{\partial L}{\partial \dot{x}^\mu} = m g_{\alpha\mu}\dot{x}^\alpha \quad \Longrightarrow \quad \dot{x}^\alpha = \frac{1}{m} p_\mu\, g^{\mu\alpha}\,,$$
$$H = L + p_\mu \dot{x}^\mu = \frac{1}{2m}\, g^{\mu\nu} p_\mu\, p_\nu - \frac{1}{2} mc^2\,.$$

Hamiltonian equations thus read (after Legendre transformation $(\tau, x^\mu, \dot{x}^\mu) \to (\tau, , x^\mu, p_\mu))$

$$\frac{\mathrm{d}p_\alpha}{\mathrm{d}\tau} = -\frac{\partial H}{\partial x^\alpha} = -\frac{1}{2m}\frac{\partial g^{\mu\nu}}{\partial x^\alpha}\, p_\mu p_\nu, \qquad \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = \frac{\partial H}{\partial p_\alpha} = \frac{1}{m} g^{\alpha\beta}\, p_\beta\,. \tag{25}$$

Note that in Minkowski metrics $p_\mu = \left(Ec^{-1}, -\vec{p}\right)$ where $\vec{p}$ is the usual three dimensional momentum.

Example. Consider Friedman-Robertson-Walker metrics (cf. [6])

$$g_{44} = 1\,, \quad g_{ik} = -a^2\left(x^4\right)\left[\delta_{ik} + \delta_{im}\,\delta_{kn}\,\frac{Kx^m x^n}{1-Kr^2}\right] \quad \text{where} \quad K = 0, \pm 1\,.$$

After some calculations we obtain for the Killing vectors
1) $\xi^{[J]i} = \sqrt{1-Kr^2}\,\delta^{Ji}$, $\xi^{[J]4} = 0$, $J = 1, 2, 3$,
2) $\xi^{[J]i} = \delta_{lm}\varepsilon^{iJm}x^l$, $\xi^{[J]4} = 0$, $J = 1, 2, 3$

and for the flows

$$\Psi\left(\xi\right) = \frac{m}{2}\left[\left(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta - c^2\right)\xi^0 - 2g_{\alpha\beta}\dot{x}^\beta\xi^\alpha\right]\,.$$

Specially, consider the Killings vectors $\xi^{[J]i} = \sqrt{1-Kr^2}\,\delta^{Ji}$. Using the coordinates for a three-dimensional observer introduced in sec. 4.3 and relations (21) and (22), we have three conservation laws

$$m\frac{\sqrt{1-Kr^2}}{\sqrt{1-\frac{v^2}{c^2}}}v_i = C_i \tag{26}$$

with $v_i = -g_{ij}v^j$. Defining $v_i v^i = v^2$ we obtain, after some calculations,

$$\frac{m^2v^2}{1-\frac{v^2}{c^2}} = \frac{1}{a^2\left(t\right)\left(1-Kr^2\right)}\left(\delta^{ij} - Kx^ix^j\right)C_iC_j\,. \tag{27}$$

As for momentum it holds $p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$, one gets important and well-known relation

$$p\left(t\right)a\left(t\right) = p\left(t_0\right)a\left(t_0\right)\,. \tag{28}$$

The momentum is inversely proportional to the Robertson-Walker scale factor.

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Institute of Theoretical Physics and Astrophysics
Faculty of Science, Masaryk University
Kotlářská 2, 611 37 Brno, Czech Republic
E-mail: lenc@physics.muni.cz, janam@physics.muni.cz, czudkova@physics.muni.cz