ARCHIVUM MATHEMATICUM (BRNO)
Tomus 44 (2008), 115–118

CANONICAL 1-FORMS ON HIGHER ORDER ADAPTED FRAME BUNDLES

Jan Kurek and Włodzimierz M. Mikulski

Abstract. Let $(M,\mathcal{F})$ be a foliated $m+n$-dimensional manifold M with n-dimensional foliation $\mathcal{F}$. Let V be a finite dimensional vector space over $\mathbf{R}$. We describe all canonical ($\mathcal{F}ol_{m,n}$-invariant) V-valued 1-forms $\Theta\colon TP^r(M,\mathcal{F}) \to V$ on the r-th order adapted frame bundle $P^r(M,\mathcal{F})$ of $(M,\mathcal{F})$.

All manifolds and maps are assumed to be of class $\mathcal{C}^\infty$.

A definition of foliations can be found in [2]. Let $\mathcal{F}ol_{m,n}$ be the category of foliated $m+n$-dimensional manifolds with n-dimensional foliations and their foliation respecting local diffeomorphisms. Let $(M,\mathcal{F})$ be a $\mathcal{F}ol_{m,n}$-object. Then we have an adapted r-th order frame bundle

$$P^r(M,\mathcal{F}) = \left\{ j^r_0\varphi \mid \varphi\colon (\mathbf{R}^{m+n},\mathcal{F}^{m,n}) \to (M,\mathcal{F}) \text{ is a } \mathcal{F}ol_{m,n}\text{-map}\right\}$$

over M of $(M,\mathcal{F})$ with the target projection, where $\mathcal{F}^{m,n} = \left\{\{a\}\times\mathbf{R}^n\right\}_{a\in\mathbf{R}^m}$ is the n-dimensional canonical foliation on $\mathbf{R}^{m+n}$. We see that $P^r(M,\mathcal{F})$ is a principal bundle with the standard Lie group $G^r_{m,n} = P^r(\mathbf{R}^{m+n},\mathcal{F}^{m,n})_0$ (with the multiplication given by the composition of jets) acting on the right on $P^r(M,\mathcal{F})$ by the composition of jets. Every $\mathcal{F}ol_{m,n}$-map $\psi\colon (M_1,\mathcal{F}_1) \to (M_2,\mathcal{F}_2)$ induces a local fibred diffeomorphism (even a principal bundle local isomorphism) $P^r\psi\colon P^r(M_1,\mathcal{F}_1) \to P^r(M_2,\mathcal{F}_2)$ given by $P^r\psi(j^r_0\varphi) = j^r_0(\psi\circ\varphi)$.

Definition 1. Let V be a finite dimensional vector space over $\mathbf{R}$. We recall that a $\mathcal{F}ol_{m,n}$-canonical V-valued 1-form Θ on P^r is a family of $\mathcal{F}ol_{m,n}$-invariant V-valued 1-forms $\Theta_{(M,\mathcal{F})} : TP^r(M,\mathcal{F}) \to V$ on $P^r(M,\mathcal{F})$ for any $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$. The invariance means that the V-valued 1-forms $\Theta_{(M_1,\mathcal{F}_1)}$ and $\Theta_{(M_2,\mathcal{F}_2)}$ are $P^r\Phi$-related ($P^r\Phi^*\Theta_{(M_2,\mathcal{F}_2)} = \Theta_{(M_1,\mathcal{F}_1)}$) for any $\mathcal{F}ol_{m,n}$-map $\Phi\colon (M_1,\mathcal{F}_1) \to (M_2,\mathcal{F}_2)$.

It is rather-known the following $\mathcal{F}ol_{m,n}$-canonical $\mathbf{R}^{m+n}$-valued 1-form on $P^1(M,\mathcal{F})$.

Example 1. For every $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$ we define an $\mathbf{R}^{m+n}$-valued 1-form $\theta_{(M,\mathcal{F})}$ on $P^1(M,\mathcal{F})$ as follows. Consider the target projection $\beta\colon P^1(M,\mathcal{F}) \to M$

2000 *Mathematics Subject Classification:* Primary: 58A20; Secondary: 58A32.
Key words and phrases: foliated manifold, infinitesimal automorphism, higher order adapted frame bundle, canonical 1-form.
Received July 24, 2007, revised October 2007. Editor I. Kolář.

given by $\beta(j^r_0\varphi) = \varphi(0)$, an element $u = j^1_0\psi \in P^1(M,\mathcal{F})$ and a tangent vector $X = j^1_0 c \in T_u\big(P^1(M,\mathcal{F})\big)$. We define the form $\theta = \theta_{(M,\mathcal{F})}$ by

$$\theta(X) = u^{-1} \circ T\beta(X) = j^1_0(\psi^{-1} \circ \beta \circ c) \in T_0\mathbf{R}^{m+n} = \mathbf{R}^{m+n}\,.$$

Let us notice that if $n = 0$ then $(M,\mathcal{F}) = M$ and $P^1(M,\mathcal{F}) = P^1(M)$ and $\theta_{(M,\mathcal{F})} = \theta_M$ is the well-known canonical $\mathbf{R}^m$-valued 1-form on the frame bundle P^1M.

To present a general example of $\mathcal{F}ol_{m,n}$-canonical V-valued 1-forms on P^r we need the following lemma.

Lemma 1. *Let $(M,\mathcal{F})$ be a $\mathcal{F}ol_{m,n}$-object. Then any vector $v \in T_wP^r(M,\mathcal{F})$, $w \in \big(P^r(M,\mathcal{F})\big)_x$, $x \in M$ is of the form $\mathcal{P}^rX_w$ for some infinitesimal automorphism $X \in \mathcal{X}(M,\mathcal{F})$, where $\mathcal{P}^rX \in \mathcal{X}\big(P^r(M,\mathcal{F})\big)$ is the flow lifting of X to $P^r(M,\mathcal{F})$. Moreover j^r_xX is uniquely determined.*

Remark 1. We inform that a vector field X on M is an infinitesimal automorphism of $(M,\mathcal{F})$ iff the flow $\{\mathrm{Expt}X\}$ of X is formed by local $\mathcal{F}ol_{m,n}$-maps $(M,\mathcal{F}) \to (M,\mathcal{F})$ or (equivalently) $[X,Y]$ is tangent to $\mathcal{F}$ for any Y tangent to $\mathcal{F}$. The space $\mathcal{X}(M,\mathcal{F})$ of all infinitesimal automorphisms of $(M,\mathcal{F})$ is a Lie subalgebra in $\mathcal{X}(M)$. Given an infinitesimal automorphism $X \in \mathcal{X}(M,\mathcal{F})$, the flow lifting $\mathcal{P}^rX$ is a vector field on $P^r(M,\mathcal{F})$ such that if $\{\Phi_t\}$ is the flow of X then $\{P^r(\Phi_t)\}$ is the flow of $\mathcal{P}^rX$. (Since Φ_t are $\mathcal{F}ol_{m,n}$-maps we can apply functor P^r.)

Proof of Lemma 1. We can of course assume that $(M,\mathcal{F}) = (\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ and $x = 0$. Since $P^r(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ is in usual way a principal subbundle of $P^r(\mathbf{R}^{m+n})$, then by well-known manifold version of the lemma, we find $X \in \mathcal{X}(\mathbf{R}^{m+n})$ such that $v = \mathcal{P}^rX_w$ and j^r_0X is determined uniquely. An infinitesimal automorphism $Y \in \mathcal{X}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ gives $\mathcal{P}^rY_w$ which is tangent to $P^r(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$. On the other hand the dimension of $P^r(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ and the dimension of the space of r-jets j^r_0Y of $Y \in \mathcal{X}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ are equal. Then the lemma follows from the dimension argument because flow operators are linear. □

Example 2. Let $\lambda\colon J^{r-1}_0\big(T_{\mathrm{Inf-Aut}}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})\big) \to V$ be an $\mathbf{R}$-linear map, where $J^{r-1}_0\big(T_{\mathrm{Inf-Aut}}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})\big)$ is the vector space of all $(r-1)$-jets j^{r-1}_0X at $0 \in \mathbf{R}^{m+n}$ of infinitesimal automorphisms $X \in \mathcal{X}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$. Given a $\mathcal{F}ol_{m,n}$-object $(M,\mathcal{F})$, we define a V-valued 1-form $\Theta^\lambda_{(M,\mathcal{F})}\colon TP^r(M,\mathcal{F}) \to V$ on $P^r(M,\mathcal{F})$ as follows. Let $v \in T_wP^r(M,\mathcal{F})$, $w = j^r_0\varphi \in (P^r(M,\mathcal{F}))_x$, $x \in M$. By Lemma 1, $v = \mathcal{P}^rX_w$ for some infinitesimal automorphism $X \in \mathcal{X}(M,\mathcal{F})$, and j^r_xX is uniquely determined. Then it is determined the $(r-1)$-jet $j^{r-1}_0\big((\varphi^{-1})_*X\big)$ at 0 of the image $(\varphi^{-1})_*X$ of X by φ^{-1}. We put

$$\Theta^\lambda_{(M,\mathcal{F})}(v) := \lambda\big(j^{r-1}_0((\varphi^{-1})_*X)\big)\,.$$

Clearly, $\Theta^\lambda = \{\Theta^\lambda_{(M,\mathcal{F})}\}$ is a $\mathcal{F}ol_{m,n}$-canonical V-valued 1-form on P^r.

The main result of the present short note is the following classification theorem.

Theorem 1. *Any $\mathcal{F}ol_{m,n}$-canonical V-valued 1-form on P^r is Θ^λ for some unique $\mathbf{R}$-linear map $\lambda\colon J^{r-1}_0\big(T_{\mathrm{Inf-Aut}}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})\big) \to V$.*

In the proof of Theorem 1 we use the following fact.

Lemma 2. *Let $X, Y \in \mathcal{X}(M,\mathcal{F})$ be infinitesimal automorphisms of $(M,\mathcal{F})$ and $x \in M$ be a point. Suppose that $j^{r-1}_x X = j^{r-1}_x Y$ and X_x is not-tangent to $\mathcal{F}$. Then there exists a (locally defined) $\mathcal{F}ol_{m,n}$-map $\Phi\colon (M,\mathcal{F}) \to (M,\mathcal{F})$ such that $j^r_x(\Phi) = j^r_x(\mathrm{id}_M)$ and $\Phi_* X = Y$ near x.*

Proof. A direct modification of the proof of Lemma 42.4 in [1]. □

Proof of Theorem 1. Let Θ be a $\mathcal{F}ol_{m,n}$-canonical V-valued 1-form on P^r. We must define $\lambda\colon J^{r-1}_0\big(T_{\mathrm{Inf-Aut}}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})\big) \to V$ by

$$\lambda(\xi) := \Theta_{(\mathbf{R}^{m+n},\mathcal{F}^{m,n})}\big(\mathcal{P}^r \tilde{X}_{j^r_0(\mathrm{id}_{\mathbf{R}^{m+n}})}\big)$$

for all $\xi \in J^{r-1}_0\big(T_{\mathrm{Inf-Aut}}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})\big)$, where given ξ in question, $\tilde{X}$ is a unique (a unique germ at 0 of) infinitesimal automorphism of $(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ such that $j^{r-1}_0\tilde{X} = \xi$ and the coefficients of $\tilde{X}$ with respect to the basis of the canonical vector fields $\frac{\partial}{\partial x^i} \in \mathcal{X}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ $(i = 1,\dots,m+n)$ are polynomials of degree $\leq r-1$.

We are going to show that $\Theta = \Theta^\lambda$. Because of the $\mathcal{F}ol_{m,n}$-invariance it remains to show that

$$(*) \qquad \Theta_{(\mathbf{R}^{m+n},\mathcal{F}^{m,n})}(v) = \Theta^\lambda_{(\mathbf{R}^{m+n},\mathcal{F}^{m,n})}(v)$$

for any $v \in T_{j^r_0(\mathrm{id}_{\mathbf{R}^{m+n}})}P^r(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$.

By the definition of λ and Θ^λ we have $(*)$ for any v of the form $\mathcal{P}^r\tilde{X}_{j^r_0(\mathrm{id}_{\mathbf{R}^{m+n}})}$, where $\tilde{X}$ is an infinitesimal automorphism of $(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ such that the coefficients of $\tilde{X}$ with respect to the basis of canonical vector fields on $\mathbf{R}^{m+n}$ are polynomials of degree $\leq r-1$.

Now, let v be arbitrary in question. Then by Lemma 1, v is of the form $v = \mathcal{P}^r X_{j^r_0(\mathrm{id}_{\mathbf{R}^{m+n}})}$ for some infinitesimal automorphism X of $(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$. Clearly (because of a density argument), we can additionally assume that X_0 is not tangent to $\mathcal{F}^{m,n}$. Let $\tilde{X}$ be an infinitesimal automorphism of $(\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ such that $j^{r-1}_0\tilde{X} = j^{r-1}_0 X$ and the coefficients of $\tilde{X}$ with respect to the basis of constant vector fields on $\mathbf{R}^{m+n}$ are polynomials of degree $\leq r-1$. Let $\tilde{v} = \mathcal{P}^r\tilde{X}_{j^r_0(\mathrm{id}_{\mathbf{R}^{m+n}})}$. Then (we have observed above) it holds $\Theta_{(\mathbf{R}^{m+n},\mathcal{F}^{m,n})}(\tilde{v}) = \Theta^\lambda_{(\mathbf{R}^{m+n},\mathcal{F}^{m,n})}(\tilde{v})$. On the other hand by Lemma 2, there is a $\mathcal{F}ol_{m,n}$-map $\Phi\colon (\mathbf{R}^{m+n},\mathcal{F}^{m,n}) \to (\mathbf{R}^{m+n},\mathcal{F}^{m,n})$ such that $j^r_0\Phi = j^r_0(\mathrm{id}_{\mathbf{R}^{m+n}})$ and $\Phi_*\tilde{X} = X$ near 0. Since $j^r_0\Phi = \mathrm{id}$, Φ preserves $j^r_0(\mathrm{id}_{\mathbf{R}^{m+n}})$. Then since $\Phi_*\tilde{X} = X$, Φ sends $\tilde{v}$ into v. Then because of the invariance of Θ and Θ^λ with respect to Φ, we obtain $\Theta_{(\mathbf{R}^{m+n},\mathcal{F}^{m,n})}(v) = \Theta_{(\mathbf{R}^{m+n},\mathcal{F}^{m,n})}(\tilde{v}) = \Theta^\lambda_{(\mathbf{R}^{m+n},\mathcal{F}^{m+n})}(\tilde{v}) = \Theta^\lambda_{(\mathbf{R}^{m+n},\mathcal{F}^{m,n})}(v)$. □

In the case $r = 1$, we have $J^0_0(T_{\mathrm{Inf-Aut}}(\mathbf{R}^{m+n},\mathcal{F}^{m,n})) \tilde{=} \mathbf{R}^{m+n}$. Then by Theorem 1, the vector space of $\mathcal{F}ol_{m,n}$-canonical V-valued 1-forms on P^1 is $(m+n)\dim(V)$-dimensional. Then (because of a dimension argument) we have.

Corollary 1. *Any $\mathcal{F}ol_{m,n}$-canonical V-valued 1-form $\Theta = \{\Theta_{(M,\mathcal{F})}\}$ on P^1 is of the form*

$$\Theta_{(M,\mathcal{F})} = \lambda \circ \theta_{(M,\mathcal{F})}\colon TP^1(M,\mathcal{F}) \to V$$

for some unique linear map $\lambda\colon \mathbf{R}^{m+n} \to V$*, where* $\theta = \{\theta_{(M,\mathcal{F})}\}$ *is the canonical* $\mathbf{R}^{m+n}$*-valued* 1*-form on* P^1 *from Example 1.*

Example 3. It is easy to see that

$$J_0^{r-1}\big(T_{\mathrm{Inf-Aut}}(\mathbf{R}^{m+n}, \mathcal{F}^{m,n})\big) \tilde{=} \mathbf{R}^{m+n} \oplus \mathcal{L}ie\,(G^{r-1}_{m,n})\,.$$

Thus by Example 2 for $\lambda = \mathrm{id}_{\mathbf{R}^{m+n}\oplus\mathcal{L}ie\,(G^{r-1}_{m,n})}$ we have a $\mathcal{F}ol_{m,n}$-canonical $\mathbf{R}^{m,n} \oplus \mathcal{L}ie\,(G^{r-1}_{m,n})$-valued 1-form

$$\theta^r_{(M,\mathcal{F})} := \Theta^{\mathrm{id}_{\mathbf{R}^{m+m}\oplus\mathcal{L}ie\,(G^{r-1}_{m,n})}}\colon TP^r(M,\mathcal{F}) \to \mathbf{R}^{m+n} \oplus \mathcal{L}ie\,(G^{r-1}_{m,n})$$

on P^r. For $r = 1$, we have $\theta^1 = \theta$ as in Example 1. In particular, for $n = 0$ we obtain the well-known canonical $\mathbf{R}^m \oplus \mathcal{L}ie\,(G^{r-1}_m)$-valued 1-form

$$\theta^r_M\colon P^rM \to \mathbf{R}^m \oplus \mathcal{L}ie\,(G^{r-1}_m)$$

on the r-order frame bundle P^rM.

By similar arguments as for Corollary 1 we have.

Corollary 2. *Any* $\mathcal{F}ol_{m,n}$*-canonical* V*-valued* 1*-form* $\Theta = \{\Theta_{(M,\mathcal{F})}\}$ *on* P^r *is of the form*

$$\Theta_{(M,\mathcal{F})} = \lambda \circ \theta^r_{(M,\mathcal{F})}\colon TP^r(M,\mathcal{F}) \to V$$

for some unique linear map $\lambda\colon \mathbf{R}^{m+n} \oplus \mathcal{L}ie\,(G^{r-1}_{m,n}) \to V$*, where* θ^r *is as in Example 3.*

In particular (*for* $n = 0$)*, any canonical* V*-valued* 1*-form* $\Theta = \{\Theta_M\}$ *on* P^rM *is of the form*

$$\Theta_M = \lambda \circ \theta^r_M\colon TP^rM \to V$$

for some unique linear map $\lambda\colon \mathbf{R}^m \oplus \mathcal{L}ie\,(G^{r-1}_m) \to V$.

Remark. Recently, we obtained (by a modification of the above paper) a similar result on gauge invariant vector valued 1-forms on higher order principal prolongations of principal bundles. The paper will appear in Lobachevskii Math. J. 2008.

References

[1] Kolář, I., Michor, P. W., Slovák, J., *Natural Operations in Differential Geometry*, Springer Verlag, 1993.

[2] Wolak, R. A., *Geometric structures on foliated manifolds*, Publications del Departamento de Geometria y Topologia, Universidad de Santiago de Compostella **76** (1989).

Institute of Mathematics, Maria Curie-Sklodowska Univesity
Pl. M. Curie-Sklodowskiej 1, Lublin, Poland
E-mail: kurek@hektor.umcs.lublin.pl

Institute of Mathematics, Jagiellonian University
ul. Reymonta 4, Kraków, Poland
E-mail: mikulski@im.uj.edu.pl