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ON LOCALLY LIPSCHITZ LOCALLY COMPACT TRANSFORMATION GROUPS OF MANIFOLDS

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Abstract. In this paper we show that a "locally Lipschitz" locally compact transformation group acting continuously and effectively on a connected paracompact locally Euclidean topological manifold is a Lie group. This is a contribution to the proof of the Hilbert-Smith conjecture. It generalizes the classical Bochner-Montgomery-Kuranishi Theorem[1, 9] and also the Repovš-Ščepin Theorem [17] which holds only for Riemannian manifolds.

A topological group G acting continuously and effectively (i.e. every non-trivial element of G acts non-trivially) on an n-dimensional topological manifold M is said to be locally Lipschitz if $\forall\, a \in M$, $\exists\,(U_a, \varphi_a)$ chart of M where $a \in U_a$ open $\subseteq M$, $\varphi_a \colon U_a \to \mathbf{R}^n$ an open embedding and $1 \in V$ open $\subseteq G$, $a \in U$ open $\subseteq U_a$ such that $V(U) \subseteq U_a$ and $\forall\, g \in V$, $d(gx, gy) \leq c_g d(x, y)$ $\forall\; x, y \in U$ for some $c_g \in \mathbf{R}$ where d is the transported Euclidean distance from $\mathbf{R}^n$ via the map φ_a. This definition generalizes the corresponding one given in [8].

In this paper we generalize the results of [8] and we show that a locally Lipschitz locally compact transformation group acting continuously and effectively on a connected paracompact locally Euclidean manifold is a Lie group, Theorem 2. This provides a contribution to the proof of the Hilbert-Smith conjecture. It generalizes the classical Bochner-Montgomery-Kuranishi Theorem [1, 9] where the group acts by C^1 diffeomorphisms on a smooth manifold and also the Repovš-Ščepin Theorem [17] which holds only for Riemannian manifolds.

The following theorem is all what we need to establish our result. It replaces the Hausdorff dimension argument in [17] by an elementary theorem in dimension theorem from Nagata book [16, p.83]. This will enable us to obtain the required generalization from Riemannian manifolds to topological manifolds.

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Theorem 1. *Let G be a zero-dimensional compact topological group acting continuously on a paracompact n-dimensional topological manifold M such that G is locally Lipschitz then* $\dim M/G = \dim M = n$ ($\dim$ = *covering dimension*).

Proof. Let $a \in M$ and let (U_a, φ_a) chart of M where $a \in U_a$ open $\subseteq M$, $\varphi_a \colon U_a \to \mathbf{R}^n$ an open embedding. Let $a \in U$ open $\subseteq U^-$ compact $\subseteq U_a$ and H open normal subgroup of G such that $H(U^-) \subseteq U_a$ [1, TGIII.36] such that $\forall\, g \in H$, $d(gx, gy) \leq c_g\, d(x, y)\ \forall\, x, y \in H(U^-)$ for some $c_g \in \mathbf{R}$, where d is the transported Euclidean distance from $\mathbf{R}^n$ via the map φ_a.

Since $H = \cup_{k \geq 1} \{g \in H : d(gx, gy) \leq k\, d(x, y)\ \forall x, y \in H(U^-)\}$, we may assume by a Baire category argument that $c_g = c\ \forall\, g \in H$.

Let β be the invariant Haar measure on H of total mass 1, then

$$d^-(x, y) = \int_H d(gx, gy)\, d\beta(g), \quad \forall\ x, y \in H(U^-)$$

defines an equivalent H-invariant metric on $H(U^-)$ and if $\pi_1 \colon M \to M/H$ is the canonical map, then $d*(\pi_1 x, \pi_1 y) = d^-(Hx, Hy)$ defines an admissible metric on $H(U^-)/H$.

Note that if $A \subseteq H(U^-)$, then $\operatorname{diam}(\pi_1, d*) \leq \operatorname{diam}(A, d^-) \leq c\ \operatorname{diam}(A, d)$ ($\operatorname{diam}(B, \delta)$ = diameter of set B under the metric δ) so that if

$$m_n(A, d) = \sup_{\varepsilon > 0} \inf \Big\{ \sum_{i \geq 1} \big(\operatorname{diam}(A_i, d) \big)^n : A = \bigcup_{i=1}^n A_i, \operatorname{diam}(A_i, d) < \varepsilon,\ \forall\ i \geq 1 \Big\}$$

[16, p.81], and if we identify $H(U^-)$ by its image in $\mathbf{R}^n$ under φ_a then $m_n(A, d) = \big(\frac{4}{\pi}\big)^{\frac{n}{2}} \Gamma\big(1 + \frac{n}{2}\big) \lambda_n^*(A)$ where λ_n^* (resp. λ_n) is the Lebesgue outer measure (resp. measure) on $\mathbf{R}^n$ [6, p.157].

We have $m_{n+1}\big(H(U^-), d\big) = 0$ since $\lambda_{n+1}\big(H(U^-)\big) = 0$ and

$$m_{n+1}\big(H(U^-)/H, d*\big) = 0$$

so that $\dim H(U^-)/H = \dim \pi_1(U^-) \leq n$ [16, p.83].

Now, if $\pi \colon M \to M/G$ and $\pi_2 \colon M/H \to (M/H)/(G/H)$ are the canonical maps, then

$$\begin{aligned}
\dim \pi(U^-) &= \dim \pi_2\big(G/H \pi_1(U^-)\big) && [2, \text{TG III.15}] \\
&= \dim G/H \pi_1(U^-) && [13, \text{Theorem 4.1}] \\
&= \dim \pi_1(U^-) && [15, \text{p.53}] \\
&\leq n
\end{aligned}$$

hence $n = \dim M \leq \dim M/G \leq n$ [15, p.62+p.130]. □

Now we can establish our main theorem which, in view of [10, Lemma 2.1], generalizes Repovš -Ščepin Theorem [17] which holds only for Riemannian manifolds.

Theorem 2 (Main Theorem). *Let G be a locally compact topological group acting continuously and effectively on a connected paracompact n-dimensional topological manifold M such that G is locally Lipschitz, then G is a Lie group.*

Proof. Note that G is metrizable [2, TG III 36 + TG III 76] and $\dim G < \infty$ [11], so that by the main approximation theorem [12, p.175] if G is not a Lie group, then G contains a zero-dimensional torsion free compact group by [2, TG III 60], [15, p.161], [14, Theorem 2.1] and [5, p.206] hence it contains the p-adic group $\mathbf{Z}_p = \lim\limits_{\leftarrow} \mathbf{Z}/p^n\mathbf{Z}$ for some prime number p [7, p.426] and

$$\begin{aligned} n = \dim M = \dim M/\mathbf{Z}_p &\quad \text{by Theorem 1} \\ = \dim_Z M/\mathbf{Z}_p &\quad [15, \text{p.206, p.210}] \\ = n + 2 &\quad [3, 18] \end{aligned}$$

($\dim_Z$ = cohomological dimension over $\mathbf{Z}$) which is absurd. □

Corollary 1 (Bochner-Montgomery-Kuranishi [1, 9])**.** *Let G be a locally compact topological group acting continuously and effectively on a connected paracompact n-dimensional C^1 manifold M such that G acts on M by C^1 diffeomorphisms, then G is a Lie group.*

Proof. Clearly by the mean value theorem [4, p.155] G is locally Lipschitz, hence our assertion follows from Theorem 2. □

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