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LIMIT AND INTEGRAL PROPERTIES OF PRINCIPAL SOLUTIONS FOR HALF-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. Some asymptotic properties of principal solutions of the half-linear differential equation

$$(a(t)\Phi(x'))' + b(t)\Phi(x) = 0\,, \tag{*}$$

$\Phi(u) = |u|^{p-2}u$, $p > 1$, is the p-Laplacian operator, are considered. It is shown that principal solutions of (*) are, roughly speaking, the smallest solutions in a neighborhood of infinity, like in the linear case. Some integral characterizations of principal solutions of (1), which completes previous results, are presented as well.

1. Introduction

Consider the half-linear equation

$$\big(a(t)\Phi(x')\big)' + b(t)\Phi(x) = 0\,, \tag{1}$$

where the functions a, b are continuous and positive for $t \geq 0$, and $\Phi(u) = |u|^{p-2}u$, $p > 1$.

When (1) is nonoscillatory, the asymptotic behavior of its solutions has been considered in many papers, see, e.g., [3, 4, 7, 9, 10, 11, 14, 15], the monographs [1, 8, 17] and references therein.

In particular, when (1) is nonoscillatory, the concept of a principal solution has been formulated for (1) in [11, 17], by extending the analogous one stated for the linear equation

$$\big(a(t)x'\big)' + b(t)x = 0\,, \tag{2}$$

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see, e.g., [13, Chapter 11]. More precisely, a nontrivial solution u of (1) is called *a principal solution* of (1) if for every nontrivial solution x of (1) such that $x \neq \lambda u$, $\lambda \in \mathbb{R}$, we have

$$\frac{u'(t)}{u(t)} < \frac{x'(t)}{x(t)} \quad \text{for large } t\,. \tag{3}$$

As in the linear case, the principal solution u exists and is unique up to a constant factor. Any nontrivial solution $x \neq \lambda u$ is called *nonprincipal solution.* Denote

$$J_a = \int_0^\infty \frac{dt}{\Phi^*\big(a(t)\big)}\,, \qquad J_b = \int_0^\infty b(t)\,dt\,,$$

where Φ^* is the inverse of the map Φ, i.e. $\Phi^*(u) = |u|^{p^*-2}u$, $p^* = p/(p-1)$.

The question concerning limit and integral characterizations of principal solutions, like in the linear case, has been posed in [7] and partially solved in [3] under any of the following assumptions

$$\begin{gathered} \text{i) } J_a = \infty\,, \quad p \geq 2\,, \qquad \text{ii) } J_b = \infty\,, \quad 1 < p \leq 2\,, \\ \text{iii) } J_a + J_b < \infty\,. \end{gathered} \tag{4}$$

In this paper we continue such a study, by assuming

$$J_a + J_b = \infty\,. \tag{5}$$

We will characterize principal solutions of (1) by means of some limit or integral properties, which extend our quoted results in [3].

The paper is organized as follows. In Section 2 some preliminary results, concerning the classification of solutions of (1), are given. In Section 3 principal solutions of (1) are characterized by showing that they are, roughly speaking, the smallest solutions in a neighborhood of infinity, like in the linear case. Some integral characterizations of principal solutions of (1) are presented in Section 4, completing in such a way our previous results in [3]. Some open problems complete the paper.

2. Preliminaries

We start this section by recalling some basic results, which will be useful in the sequel.

It is easy to verify that the quasi-derivative $y = x^{[1]}$ of any solution x of (1), where $x^{[1]}(t) = a(t)\Phi\big(x'(t)\big)$, is a solution of the so-called reciprocal equation

$$\Big(\Phi^*\Big(\frac{1}{b(t)}\Big)\Phi^*(y')\Big)' + \Phi^*\Big(\frac{1}{a(t)}\Big)\Phi^*(y) = 0\,, \tag{6}$$

which is obtained from (1) by interchanging the function a with $\Phi^*(1/b)$ and b with $\Phi^*(1/a)$. Conversely, the quasiderivative $y^{[1]}(t) = \Phi^*\left(1/b(t)\right)\Phi^*(y'(t))$ of any solution y of (6) is a solution of (1). Observe that J_a $[J_b]$ for (1) plays the same role as J_b $[J_a]$ for (6) and vice versa.

In view of positiveness of a and b, (1) and (6) have the same character with respect to the oscillation, i.e. (1) is nonoscillatory if and only if (6) is nonoscillatory. When $J_a = J_b = \infty$, then (1) is oscillatory (see, e.g., [8, Th.1.2.9.]). If either $J_a = \infty, J_b < \infty$ or $J_a < \infty, J_b = \infty$, then both oscillation and nonoscillation can occur (see, e.g., [8, §3.1]).

Principal solutions of (1) and (6) are related, as the following result, which can be proved by using the same argument as in [3, Theorem 1], shows.

Proposition 1. *Let* (1) *be nonoscillatory and assume* (5)*. A solution u of* (1) *is a principal solution if and only if $v = u^{[1]}$ is a principal solution of* (6)*.*

When (1) is nonoscillatory, taking into account that (6) is nonoscillatory too, we have that any nontrivial solution x of (1) belongs to one of the following two classes:

$$\mathbb{M}^+ = \{x \text{ solution of } (1) : \exists t_x \geq 0 : \; x(t)x'(t) > 0 \text{ for } t > t_x\}$$
$$\mathbb{M}^- = \{x \text{ solution of } (1) : \exists t_x \geq 0 : \; x(t)x'(t) < 0 \text{ for } t > t_x\}.$$

The following holds.

Proposition 2. *Let* (1) *be nonoscillatory and assume* (5)*. Let $\mathbb{S}$ be the set of nontrivial solutions of* (1)*. Then*

$$J_a = \infty \iff \mathbb{S} \equiv \mathbb{M}^+ \, ; \qquad J_b = \infty \iff \mathbb{S} \equiv \mathbb{M}^- \, .$$

Moreover, (1) *does not have solutions x such that*

$$\lim_{t\to\infty} x(t) = c_x \, , \quad \lim_{t\to\infty} x^{[1]}(t) = d_x \, , \qquad 0 < |c_x| < \infty \, , \quad 0 < |d_x| < \infty \, . \tag{7}$$

Proof. The first statement follows by using a similar argument as in [3, Lemma 1] (see also [8, Lemmas 4.1.3, 4.1.4]). Now let us prove (7). Assume $J_a = \infty$ and let x be a solution of (1) satisfying (7). Then $x \in \mathbb{M}^+$ and, without loss of generality, suppose $x(t) > 0, x'(t) > 0$ for large t. From $x^{[1]}(t) = a(t)\Phi(x'(t))$ we obtain for large t

$$x'(t) \sim \frac{1}{\Phi^*(a(t))} \, , \tag{8}$$

where the symbol $g_1(t) \sim g_2(t)$ means that $g_1(t)/g_2(t)$ has a finite nonzero limit, as $t \to \infty$. From (8) we obtain that x is unbounded (as $t \to \infty$), which is a contradiction. The case $J_b = \infty$ can be treated by using a similar argument. $\square$

Notice that if the assumption (5) is not verified, then both statements in Proposition 2 fail, as it follows, for instance, from [12, Theorem 3] and applying this result to the reciprocal equation (6).

In virtue of the positiveness of the functions a, b, and Proposition 2, both classes $\mathbb{M}^+, \mathbb{M}^-$ can be divided, *a-priori,* into the following subclasses:

$$\mathbb{M}^+_{\ell,0} = \left\{x \in \mathbb{M}^+ : \lim_{t\to\infty} x(t) = c_x,\, \lim_{t\to\infty} x^{[1]}(t) = 0, 0 < |c_x| < \infty\right\},$$

$$\mathbb{M}^+_{\infty,0} = \left\{x \in \mathbb{M}^+ : \lim_{t\to\infty} |x(t)| = \infty,\, \lim_{t\to\infty} x^{[1]}(t) = 0\right\},$$

$$\mathbb{M}^+_{\infty,\ell} = \left\{x \in \mathbb{M}^+ : \lim_{t\to\infty} |x(t)| = \infty,\, \lim_{t\to\infty} x^{[1]}(t) = d_x,\ 0 < |d_x| < \infty\right\},$$

$$\mathbb{M}^-_{0,\ell} = \left\{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = 0,\, \lim_{t\to\infty} x^{[1]}(t) = d_x,\ 0 < |d_x| < \infty\right\},$$

$$\mathbb{M}^-_{0,\infty} = \left\{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = 0,\, \lim_{t\to\infty} |x^{[1]}(t)| = \infty\right\},$$

$$\mathbb{M}^-_{\ell,\infty} = \left\{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = c_x,\, \lim_{t\to\infty} |x^{[1]}(t)| = \infty,\ 0 < |c_x| < \infty\right\}.$$

The existence of solutions in these subclasses depends on the convergence or divergence of the following integrals:

$$J_1 = \lim_{T\to\infty} \int_0^T \Phi^*\Big(\frac{1}{a(t)}\Big)\Phi^*\Big(\int_0^t b(s)\,ds\Big)\,dt\,,$$

$$J_2 = \lim_{T\to\infty} \int_0^T \Phi^*\Big(\frac{1}{a(t)}\Big)\Phi^*\Big(\int_t^T b(s)\,ds\Big)\,dt\,,$$

and

$$Y_1 = \lim_{T\to\infty} \int_0^T b(t)\,\Phi\Big(\int_t^T \Phi^*\big(\frac{1}{a(s)}\big)\,ds\Big)dt\,,$$

$$Y_2 = \lim_{T\to\infty} \int_0^T b(t)\,\Phi\Big(\int_0^t \Phi^*\Big(\frac{1}{a(s)}\Big)\,ds\Big)dt\,.$$

Clearly, for the linear equation (2) we have $J_1 = Y_1, J_2 = Y_2$. Observe that the integral J_1 for (1) plays the same role as Y_2 for (6) and vice versa; analogously J_2 for (1) plays the same role as Y_1 for (6) and vice versa.

The following holds.

Lemma A. *Concerning the mutual behavior of* J_1, Y_1*, the only possible cases are the following:*

$$\begin{array}{ll} J_1 = Y_1 = \infty & \text{for}\quad 1 < p \\ J_1 = \infty,\ Y_1 < \infty & \text{for}\quad 2 < p \\ J_1 < \infty,\ Y_1 = \infty & \text{for}\quad 1 < p < 2 \\ J_1 < \infty,\ Y_1 < \infty & \text{for}\quad 1 < p. \end{array}$$

Analogously for J_2, Y_2*, the only possible cases are*

$$\begin{array}{ll} J_2 = Y_2 = \infty & \text{for}\quad 1 < p \\ J_2 = \infty,\ Y_2 < \infty & \text{for}\quad 2 < p \\ J_2 < \infty,\ Y_2 = \infty & \text{for}\quad 1 < p < 2 \\ J_2 < \infty,\ Y_2 < \infty & \text{for}\quad 1 < p. \end{array}$$

Moreover, if $J_2 + Y_2 = \infty$*, then* $J_a = \infty$*, and, if* $J_1 + Y_1 = \infty$*, then* $J_b = \infty$*.*

Proof. The possible cases for J_i, Y_i $(i = 1, 2)$ follow from [6, Corollary 1 and Examples 1, 2]. The relations between J_i, Y_i and J_a, J_b follow from [2, Lemma 2]. $\square$

The following holds.

Theorem A. i$_1$) *Assume $J_a = \infty$. Then*

$$\mathbb{M}^+_{\ell,0} \neq \emptyset \Longleftrightarrow J_2 < \infty\,, \quad \mathbb{M}^+_{\infty,\ell} \neq \emptyset \Longleftrightarrow Y_2 < \infty\,.$$

i$_2$) *Assume $J_b = \infty$. Then*

$$\mathbb{M}^-_{0,\ell} \neq \emptyset \Longleftrightarrow Y_1 < \infty\,, \quad \mathbb{M}^-_{\ell,\infty} \neq \emptyset \Longleftrightarrow J_1 < \infty\,.$$

Proof. Claim i$_1$) follows, for instance, from [14, Th.s 4.1 and 4.2] (see also [12, Section 4], [16, Th. 4.3], in which a more general equation is considered). Claim i_2) follows by applying i_1) to the reciprocal equation (6). $\square$

3. Limit characterization

When (1) is nonoscillatory, in [7] the question, whether principal solutions are smallest solutions in a neighborhood of infinity also in the half-linear case, has been posed. This problem has been solved in [3, Theorem 2] under any of assumptions in (4).

To extend such a result, the following uniqueness result plays an important role.

Theorem B. *Let $\eta \neq 0$ be a given constant.*

i$_1$) *Assume $J_a = \infty$, $J_2 < \infty$. Then there exists a unique solution x of (1) such that $x \in \mathbb{M}^+$ and $\lim_{t\to\infty} x(t) = \eta$.*

i$_2$) *Assume $J_b = \infty$, $Y_1 < \infty$. Then there exists a unique solution x of (1) such that $x \in \mathbb{M}^-$ and $\lim_{t\to\infty} x^{[1]}(t) = \eta$.*

Proof. Claim i$_1$) follows from [14, Theorem 4.3] (see also [8, Theorem 4.1.7]). Claim i_2) follows by applying i$_1$) to the reciprocal equation (6). $\square$

The following holds.

Theorem 1. *Let u be a solution of* (1) *and assume either* i$_1$) $J_a = \infty, J_2 < \infty$ *or* i$_2$) $J_b = \infty$, $Y_1 < \infty$. *Then u is a principal solution if and only if for any nontrivial solution x of* (1) *such that $x \neq \lambda u$, $\lambda \in \mathbb{R}$, we have*

$$\lim_{t\to\infty} \frac{u(t)}{x(t)} = 0\,. \tag{9}$$

Proof. If (9) holds for any nontrivial solution x of (1) such that $x \neq \lambda u$, $\lambda \in \mathbb{R}$, then, by using the same argument as in [3, Theorem 2], u is a principal solution of (1).

Conversely, suppose that u is a principal solution and let us show that (9) holds for any nontrivial solution x of (1) such that $x \neq \lambda u$, $\lambda \in \mathbb{R}$ if either i_1) or i_2) holds.

Assume case i$_1$). By Theorem A, we have $\mathbb{M}^+_{\ell,0} \neq \emptyset$ and so (1) is nonoscillatory. Without loss of generality, suppose u eventually positive. We claim that u is bounded (as $t \to \infty$). Assume that u is unbounded and consider $x \in \mathbb{M}^+_{\ell,0}$ such that x is eventually positive. From (3), the ratio u/x is eventually positive decreasing, which yields a contradiction because $\lim_{l\to\infty}[u(t)/x(t)] = \infty$. Then u is bounded and so $u \in \mathbb{M}^+_{\ell,0}$. For any nontrivial solution x of (1), such that $x \neq \lambda u$, in view of Theorem B, we obtain that x is unbounded and so (9) holds.

Now assume case i$_2$). Again by Theorem A, we have $\mathbb{M}^-_{0,\ell} \neq \emptyset$ and so (1) is nonoscillatory. Without loss of generality, suppose u and x eventually positive. In view of Proposition 2, we have $u^{[1]}(t) < 0, x^{[1]}(t) < 0$ for large t. From (3), we obtain for large t

$$\frac{u^{[1]}(t)}{x^{[1]}(t)} > \Phi\Big(\frac{u(t)}{x(t)}\Big) > 0\,. \tag{10}$$

Applying Proposition 1, $u^{[1]}$ is a principal solution of (6). Since for (6) the case i_1) holds, we obtain

$$\lim_{t\to\infty} \frac{u^{[1]}(t)}{x^{[1]}(t)} = 0$$

and so, from (10), the assertion follows. □

From Theorem B, Theorem 1 and Theorem 2 in [3], we obtain the following.

Corollary 1. *The set of principal solutions of* (1) *is either* $\mathbb{M}^+_{\ell,0}$ *or* $\mathbb{M}^-_{0,\ell}$ *according to either* $J_a = \infty, J_2 < \infty$*, or* $J_b = \infty, Y_1 < \infty$*, respectively.*

Remark 1. Summarizing Theorem 1 and [3, Theorem 2] (which holds under any of assumptions in (4)), and taking into account Lemma A, we obtain that, if (1) is nonoscillatory, then the limit characterization of principal solutions (9) holds in any case except the following two cases

$$J_2 = Y_2 = \infty\,,\ 1 < p < 2\,; \quad J_1 = Y_1 = \infty\,,\ p > 2\,. \tag{11}$$

When any of these cases occurs (and (1) is nonoscillatory), we conjecture that the limit characterization (9) continues to hold, as the following example suggests.

Example 1. Consider the Euler type equation ($t \geq 1$)

$$\big(\Phi(x')\big)' + \Big(\frac{\gamma}{t}\Big)^p \Phi(x) = 0\,, \tag{12}$$

where $\gamma = (p-1)/p$, $1 < p < 2$. Obviously, $J_a = J_2 = \infty$ and $u(t) = t^\gamma$ is a solution of (12). Moreover, any nontrivial solution $x \neq \lambda u$, $\lambda \in \mathbb{R}$, satisfies

$$x(t) \sim t^\gamma (\log t)^{2/p}\,,$$

and $u(t) = t^\gamma$ is a principal solution of (12) (see, e.g., [8, Example 4.2.1. iii)]). Obviously, (9) is satisfied.

4. Integral characterizations

It is well-known, see e.g. [13, Ch. XI, Theorem 6.4], that, if the linear equation (2) is nonoscillatory, then principal solutions u of (2) can be equivalently characterized by one of the following conditions (in which x denotes an arbitrary nontrivial solution of (2), linearly independent of u):

$$(\pi_1) \qquad \lim_{t\to\infty} \frac{u(t)}{x(t)} = 0;$$

$$(\pi_2) \qquad \frac{u'(t)}{u(t)} < \frac{x'(t)}{x(t)} \qquad \text{for large } t;$$

$$(\pi_3) \qquad \int^{\infty} \frac{dt}{a(t)u^2(t)} = \infty.$$

The characterizations (π_1), (π_2) depend on all the solutions of (2). Even if this is not a serious disadvantage in the linear case, because of the reduction of order formula, the characterization (π_3) seems prefereable, since it is, roughly speaking, self-contained.

In this section we study the possible extensions of the integral characterization (π_3) to the half-linear case.

In [7] principal solutions u of (1) have been characterized by means of the following integral

$$Q_u := \int^{\infty} \frac{u'(t)}{u^2(t)u^{[1]}(t)}\, dt. \tag{13}$$

In particular, when b may change its sign, the following holds.

Theorem C [7, Theorem 3.1]**.** *Suppose that (1) is nonoscillatory and let* $1 < p \le 2$. *If* x *is a nonprincipal solution of* (1)*, then* $Q_x < \infty$.

When $b(t) > 0$, such a result has been partially extended in [3] by the following way.

Theorem D [3, Theorems 3, 4]**.** *Let* (1) *be nonoscillatory and assume any of conditions*

$$\text{i) } J_a = \infty\,,\ p \ge 2\,, \qquad \text{ii) } J_b = \infty\,,\ 1 < p \le 2\,.$$

A solution u *of (1) is a principal solution if and only if* $Q_u = \infty$.

In addition in [3, Corrigendum] an example is given, illustrating that the characterization (13) cannot be extended to the case $J_a = \infty$, $1 < p < 2$, without any additional assumptions.

Here we extend Theorems C, D by introducing a new integral characterization of principal solutions. Consider the integral

$$R_u := \int^{\infty} \frac{b(t)\Phi(u(t))}{u(t)(u^{[1]}(t))^2}\, dt\,, \tag{14}$$

which arises considering Q_y, where $y = u^{[1]}$ is a solution of the reciprocal equation (6). Concerning the characterization of nonprincipal solutions, the following result extends Theorem C.

Theorem 2. *Let* (1) *be nonoscillatory and assume* (5). *If x is a nonprincipal solution of* (1), *then $Q_x < \infty$ and $R_x < \infty$.*

To prove this result, the following lemma is useful.

Lemma 1. *Assume that* (1) *is nonoscillatory and* (5) *holds. If x is a nonprincipal solution of* (1), *then*

$$\limsup_{t\to\infty} x(t)x^{[1]}(t) = \infty \quad \text{or} \quad \liminf_{t\to\infty} x(t)x^{[1]}(t) = -\infty \, ,$$

according to $J_a = \infty$ or $J_b = \infty$, respectively.

Proof. Let $J_a = \infty$. Assume that there exists a constant $h > 0$ such that for large t

$$x(t)x^{[1]}(t) < h \, .$$

Because x is a nonprincipal solution, in view of Theorem A and Corollary 1, x is unbounded. Then

$$Q_x = \int^{\infty} \frac{x'(t)}{x^2(t)x^{[1]}(t)}\, dt \ge \frac{1}{h}\int^{\infty} \frac{x'(t)}{x(t)}\, dt = \infty \, ,$$

which contradicts Theorem C or Theorem D, according to $1 < p \le 2$ or $p \ge 2$, respectively.

Now let $J_b = \infty$. Consider the reciprocal equation (6): applying the first part of the proof and using Proposition 1, we obtain $\limsup_{t\to\infty} y(t)y^{[1]}(t) = \infty$ for any nonprincipal solution y of (6). Because $y(t)y^{[1]}(t) = -x(t)x^{[1]}(t)$, the assertion follows. □

Proof of Theorem 2. Taking into account Lemma 1 and using the identity

$$\int_T^t \frac{x'(s)}{x^2(s)x^{[1]}(s)}\, ds = \frac{1}{x(T)x^{[1]}(T)} - \frac{1}{x(t)x^{[1]}(t)} + \int_T^t \frac{b(s)\Phi(x(s))}{x(s)(x^{[1]}(s))^2}\, ds \, ,$$

we obtain

$$Q_x = \frac{1}{x(T)x^{[1]}(T)} + R_x$$

and so both integrals Q_x, R_x have the same behavior. Thus, if $1 < p \le 2$, the assertion follows from Theorem C and if $p > 2$, the assertion follows applying again Theorem C to the reciprocal equation (6). □

Concerning principal solutions, the following holds.

Theorem 3. *Let* (1) *be nonoscillatory and let u be a principal solution of* (1).

i$_1$) *Assume $J_a = \infty$. In addition, when $J_2 = \infty$, suppose $p \ge 2$. Then $R_u = \infty$.*

i$_2$) *Assume $J_b = \infty$. In addition, when $Y_1 = \infty$, suppose $1 < p \le 2$. Then $Q_u = \infty$.*

Proof. Claim i$_1$). Since (1) is nonoscillatory, we have $J_b < \infty$ (see, e.g., [8, Theorem 1.2.9]). By Proposition 2 we have $\mathbb{S} \equiv \mathbb{M}^+$. Without loss of generality, assume $u(t) > 0, u^{[1]}(t) > 0$ for $t \geq T \geq 0$. We have

$$\int_T^t \frac{u'(s)}{u^2(s)u^{[1]}(s)}\, ds = \frac{1}{u(T)u^{[1]}(T)} - \frac{1}{u(t)u^{[1]}(t)} + \int_T^t \frac{b(s)\Phi(u(s))}{u(s)(u^{[1]}(s))^2}\, ds$$
$$< \frac{1}{u(T)u^{[1]}(T)} + \int_T^t \frac{b(s)\Phi(u(s))}{u(s)(u^{[1]}(s))^2}\, ds\,.$$

Then

$$Q_u \leq \frac{1}{u(T)u^{[1]}(T)} + R_u\,. \tag{15}$$

When $p \geq 2$, from Theorem D we have $Q_u = \infty$ and so (15) yields $R_u = \infty$. Now let $1 < p < 2$. By assumptions and Lemma A we have $J_2 < \infty$ and so, in view of Corollary 1, $u \in \mathbb{M}^+_{\ell,0}$. By using the l'Hospital rule, we have

$$u^{[1]}(t) \sim \int^\infty b(s) ds\,. \tag{16}$$

Thus, taking into account that $J_b < \infty$ we obtain

$$R_u \sim \int^\infty \frac{b(t)}{(u^{[1]}(t))^2}\, dt = \int^\infty \frac{b(t)}{\left(\int_t^\infty b(s)\, ds\right)^2}\, dt = \infty.$$

Claim i$_2$). The assertion follows by applying claim i$_1$) to the reciprocal equation (6) and using Proposition 1. □

From Theorems 2, 3 we obtain the following.

Corollary 2. *Let* (1) *be nonoscillatory and assume* (5)*. In addition, when* $J_2 = \infty$*, suppose* $p \geq 2$ *and when* $Y_1 = \infty$*, suppose* $1 < p \leq 2$*. A solution* u *of* (1) *is a principal solution if and only if* $Q_u + R_u = \infty$*.*

Notice that, when $J_a + J_b < \infty$, the integral characterization (13) fails, as, for instance, Example 2 in [3] shows. The same example illustrates that also the integral characterization (14) fails.

We close this section by studying the behavior of integrals Q_u, R_u, where u is a principal solution of (1). The following holds.

Theorem 4. *Let* u *be a principal solution of* (1)*.*

i$_1$) *Assume* $J_a = \infty, J_2 < \infty$*. Then* $Q_u = \infty$ *if and only if*

$$\int_0^\infty \Big(\frac{1}{a(t)}\Big)^{1/(p-1)} \Big(\int_t^\infty b(s)\, ds\Big)^{(2-p)/(p-1)} dt = \infty\,. \tag{17}$$

i$_2$) *Assume* $J_b = \infty$*,* $Y_1 < \infty$*. Then* $R_u = \infty$ *if and only if*

$$\int^\infty b(t)\Big(\int_t^\infty \Big(\frac{1}{a(s)}\Big)^{1/(p-1)} ds\Big)^{p-2} dt = \infty\,. \tag{18}$$

Proof. Without loss of generality, assume $u(t) > 0$ for large t.

Claim i_1). Integrating (1) on (t, ∞) and taking into account that, in view of Corollary 1, $u \in \mathbb{M}^{+}_{\ell,0}$, (16) holds and so

$$u'(t)^{p-2} \sim \Big(\frac{1}{a(t)}\Big)^{(p-2)/(p-1)} \Big(\int_t^\infty b(s)\,ds\Big)^{(p-2)/(p-1)}.$$

Thus

$$\frac{u'(t)}{u^{[1]}(t)} = \frac{1}{a(t)}\,\frac{1}{u'(t)^{p-2}} \sim \Big(\frac{1}{a(t)}\Big)^{1/(p-1)} \Big(\int_t^\infty b(s)\,ds\Big)^{(2-p)/(p-1)}, \tag{19}$$

from which the assertion follows.

Claim i_2). Integrating the equality

$$u'(t) = \Phi^*\Big(\frac{u^{[1]}(t)}{a(t)}\Big)$$

on (t, ∞) and taking into account that, in view of Corollary 1, $u \in \mathbb{M}^{-}_{0,\ell}$, we have

$$u(t) \sim \int_t^\infty \Phi^*\Big(\frac{1}{a(s)}\Big)\,ds\,,$$

and therefore

$$\frac{b(t)\Phi(u(t))}{u(t)} \sim b(t)\Big(\int_t^\infty \Big(\frac{1}{a(s)}\Big)^{1/(p-1)} ds\Big)^{p-2},$$

from which the assertion follows. ☐

Remark 2. Using the previous results and integral relations stated in [5, Lemma 1], it is easy to show when the integrals Q_u, R_u have the same behavior for any principal solution u of (1).

We start by considering the case $J_a = \infty$. If $p \geq 2$, from Theorems D and 3 we have $Q_u = R_u = \infty$. Now consider the case $J_2 < \infty$, $1 < p < 2$ (and $J_a = \infty$). By applying [5, Lemma 1] with $\mu = (p-1)/(2-p)$ and $\lambda = p-1$ and taking into account $\mu > \lambda$, we obtain

$$Y_2 = \infty \Longrightarrow \int^\infty \Big(\frac{1}{a(t)}\Big)^{1/(p-1)} \Big(\int_t^\infty b(s)\,ds\Big)^{(2-p)/(p-1)} dt = \infty\,.$$

Thus, if $Y_2 = \infty$, in virtue of Theorems 3, 4, we have $Q_u = R_u = \infty$. Observe that when $J_a = \infty, J_2 < \infty$, $Y_2 < \infty$, $1 < p < 2$, the condition (17) can fail, as the example in [3, Corrigendum] shows. In such a circumstance, again from Theorems 3, 4, we have $Q_u < \infty$, $R_u = \infty$ and so the integrals Q_u, R_u have a different behavior.

In the case $J_b = \infty$ the situation is similar. By applying the above argument to the reciprocal equation (6) we obtain that $Q_u = R_u = \infty$ when $1 < p \leq 2$. The same conclusion holds if $J_1 < \infty$, $Y_1 = \infty$ and $1 < p < 2$. Finally, when $J_b = \infty$, $J_1 < \infty$, $Y_1 < \infty$, $p > 2$, the condition (18) can fail, and it is easy to produce an example in which $Q_u = \infty$, $R_u < \infty$.

Remark 3. Analogously to the limit characterization, it remains an open problem to find an integral characterization of principal solutions in both cases (11).

When b may change its sign, the limit and integral characterization of the principal solutions have been partially solved in [4] provided $J_a < \infty$. These problems remain open in the opposite case $J_a = \infty$ as well.

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