ARCHIVUM MATHEMATICUM (BRNO)
Tomus 43 (2007), 67 – 74

PERIODIC SOLUTIONS OF SECOND ORDER NONLINEAR FUNCTIONAL DIFFERENCE EQUATIONS

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Abstract. Sufficient conditions for the existence of at least one $T-$periodic solution of second order nonlinear functional difference equations are established. We allow f to be at most linear, superlinear or sublinear in obtained results.

1. Introduction

The development of the study of periodic solution of functional difference equations is relatively rapid. There has been many approaches to study periodic solutions of difference equations, such as critical point theory, fixed point theorems in Banach spaces or in cones of Banach spaces, coincidence degree theory, Kaplan-Yorke method, and so on, one may see [3-7,11,13-15] and the references therein.

In papers [5,7,11,13,14], the authors studied the existence of periodic solutions of first order functional difference equations using different fixed point theorems in cones of Banach spaces. Zhu and Li in [15] used fixed point theorems in cones of Banach spaces to obtain positive periodic solutions of higher order functional difference equations. In [4], the authors studied the existence of periodic solutions of a second order nonlinear difference equation by using the critical point theory. Papers [1,2,8-10,12] concerned with the solvability (existence of positive solutions) of periodic boundary value problems for second order difference equations on a finite discrete segment.

In this paper, we, by using coincidence degree theory, study the second order nonlinear functional difference equation

$$\Delta^2 x(n-1) = f(n, x(n), x(n-\tau_1(n)), \dots, x(n-\tau_m(n))) , \quad n \in Z , \tag{1}$$

2000 *Mathematics Subject Classification*: 34B10, 34B15.
Key words and phrases: periodic solutions, second order functional difference equation, fixed-point theorem, growth condition.
The author was supported by the Science Foundation of Educational Committee of Hunan Province and the National Natural Sciences Foundation of P. R. China.
Received March 14, 2006.

where $\tau_i(n)$, $i = 1, \ldots, m$, are T-periodic sequences with $T \geq 1$, $f(n,u)$ is T-periodic about n for each $u = (x_0, \ldots, x_m, x_{m+1}) \in R^{m+2}$, and is continuous about u for each $n \in Z$.

The purpose is to establish sufficient conditions for the existence of at least one T-periodic solution of equation (1).

We suppose

(A_1) $f : Z \times R^{m+1} \to R$, $f(n, x_0, \ldots, x_{m+1})$ is continuous about $u = (x_0, \ldots, x_{m+1})$ and $T-$periodic about n;

(A_2) $\tau_i : Z \to Z$, $i = 1, \ldots, m$, are T-periodic;

This paper is organized as follows. In section 2, we give the main result and in Section 3, an example to illustrate the main result will be presented.

2. Main results

To get existence results for T-periodic solutions of equation (1), we need the following fixed point theorems.

Let X and Y be Banach spaces, $L :$ Dom $L \subset X \to Y$ be a Fredholm operator of index zero, $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Ker} Q = \operatorname{Im} L, \quad X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q\,.$$

It follows that

$$L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : \operatorname{Dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, Dom $L \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact.

Proposition 1 ([3]). *Let L be a Fredholm operator of index zero and let N be L-compact on Ω. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in \big[(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega\big] \times (0,1)$*;*

(ii) $Nx \notin \operatorname{Im} L$ *for every* $x \in \operatorname{Ker} L \cap \partial\Omega$*;*

(iii) $\deg\left(\wedge QN\big|_{\operatorname{Ker} L}, \ \Omega \cap \operatorname{Ker} L, 0\right) \neq 0$*, where* $\wedge : Y/\operatorname{Im} L \to \operatorname{Ker} L$ *is the isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$*.*

Let $X = \big\{x(n) : x(n+T) = x(n)$ for all $n \in Z\big\}$ be endowed with the norm $\|x\| = \max_{n \in [0,T-1]} \big|x(n)\big|$. It is easy to see that X is a Banach space.

For equation (1), set

$$L : \operatorname{Dom} L \cap X \to X\,, \quad L \bullet x(n) = \Delta^2 x(n-1)\,,$$

and

$$N : X \to X\,, \quad N \bullet x(n) = f\big(n, x(n), x(n-\tau_1(n)), \ldots, x(n-\tau_m(n))\big)\,,$$

for all $x \in X$ and $n \in N$. It is easy to check the results.

(i) $\operatorname{Ker} L = \big\{x(n) = c,\, n \in Z,\, c \in R\big\}$.

(ii) $\operatorname{Im} L = \big\{\, y \in X, \ \sum_{n=0}^{T-1} y(n) = 0\big\}$.

(iii) L is a Fredholm operator of index zero.

(iv) There are projectors $P:\ X \to X$ and $Q:\ Y \to Y$ such that $\operatorname{Ker} L = \operatorname{Im} P$, $\operatorname{Ker} Q = \operatorname{Im} L$. Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$, then N is L-compact on $\overline{\Omega}$.

The projectors $P : X \to X$ and $Q : X \to X$, the isomorphism $\wedge : \operatorname{Ker} L \to X/\operatorname{Im} L$ and the generalized inverse $K_p : \operatorname{Im} L \to D(L) \cap \operatorname{Im} P$ are as follows:

$$\begin{aligned} Px(n) &= x(0) \quad \text{for} \quad x \in X\,,\\ Q\big(y(n)\big) &= \frac{1}{T}\sum_{n=0}^{T-1} y(n)\,, \quad \text{for} \quad y \in X\,,\\ \wedge(c) &= c\,, \quad c \in R\,,\\ K_p\big(y(n)\big) &= \sum_{s=0}^{n-1}\sum_{j=0}^{s-1} y(j) - \frac{1}{T}\sum_{n=1}^{T}\sum_{s=0}^{n-1} y(s) \quad \text{for} \quad y \in X\,. \end{aligned}$$

(v) $x \in D(L)$ is a solution of equation (1) if and only if x is a solution of the operator equation $Lx = Nx$ in $D(L)$.

Suppose

(B) There a constant $M > 0$ so that

$$c\Big[\sum_{n=0}^{T-1} f\,(n, c, c, \ldots, c)\Big] > 0 \quad \text{for all} \quad |c| > M$$

or

$$c\Big[\sum_{n=0}^{T-1} f\,(n, c, c, \ldots, c)\Big] < 0 \quad \text{for all} \quad |c| > M\,.$$

Theorem 1. *Suppose that* (A_1), (A_2), (B) *hold and that there is numbers* $\beta > 0$, $\theta > 1$, *nonnegative sequences* $p_i(n)$ $(i = 0, \ldots, m)$, $r(n)$, *functions* $g(n, x_0, \ldots, x_m)$, $h(n, x_0, \ldots, x_m)$ *such that* $f(n, x_0, \ldots, x_m) = g(n, x_0, \ldots, x_m) + h(n, x_0, \ldots, x_m)$ *and*

$$g(n, x_0, x_1, \ldots, x_m)x_0 \geq \beta|x_0|^{\theta+1}\,,$$

and

$$|h(n, x_0, \ldots, x_m)| \leq \sum_{s=0}^{m} p_i(n)\,|x_i|^{\mu} + r(n)\,,$$

for all $n \in \{1, \ldots, T\}$, $(x_0, x_1, \ldots, x_m) \in R^{m+1}$. *Then equation* (1) *has at least one* $T-$*periodic solution if*

$$\|p_0\| + T\sum_{i=1}^{m}\Big(\sum_{n=1}^{T}[p_i(n)]^{\frac{\theta+1}{\theta-1}}\Big)^{\frac{\theta-1}{\theta+1}} < \beta\,.$$

Proof. To apply Proposition 1, we should define an open bounded subset Ω of X so that (i), (ii) and (iii) of Proposition 1 hold. To obtain Ω, we do three steps. The proof of this theorem is divided into four steps, which are as follows:
Step 1. Prove that the set $\big\{x : Lx = \lambda Nx,\ (x,\lambda) \in [(\operatorname{Dom} L \setminus \operatorname{Ker} L)] \times (0,1)\big\}$ is bounded.
Step 2. Prove that the set $\{x \in \operatorname{Ker} L :\ Nx \in \operatorname{Im} L\}$ is bounded.
Step 3. Prove the set $\big\{x \in \operatorname{Ker} L :\ \pm\lambda x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\big\}$ is bounded.
Step 4. Obtain open bounded set Ω such that (i), (ii) and (iii) of Proposition 1 hold. Using Proposition 1, we get the solution of equation (1).

Step 1. Let $\Omega_1 = \big\{x : Lx = \lambda Nx,\ (x,\lambda) \in [(\operatorname{Dom} L \setminus \operatorname{Ker} L)] \times (0,1)\big\}$. For $x \in \Omega_1$, we have $L \bullet x = \lambda N \bullet x$, $\lambda \in (0,1)$, so

$$\Delta^2 x(n-1) = \lambda f\big(n, x(n), x(n-\tau_1(n)), \dots, x(n-\tau_m(n))\big)\,. \tag{2}$$

So

$$\big[\Delta^2 x(n-1)\big]\, x(n) = \lambda f\big(n, x(n), x(n-\tau_1(n)), \dots, x(n-\tau_m(n))\big) x(n)\,.$$

Since

$$\begin{aligned}
2\sum_{n=1}^{T} \big[\Delta x(n)\big]\, x(n) &= 2\sum_{n=1}^{T} \big[x(n+1)x(n) - x(n)^2\big] \\
&= x(T+1)^2 - \sum_{i=1}^{T} \big[x(i+1) - x(i)\big]^2 - x(1)^2 \\
&\le 0\,,
\end{aligned}$$

and

$$\begin{aligned}
-2\sum_{n=1}^{T} \big[\Delta x(n-1)\big]\, x(n) &= -2\sum_{n=1}^{T} \big[x(n-1)x(n) - x(n)^2\big] \\
&= -x(T)^2 - \sum_{i=0}^{T-1} \big[x(i+1) - x(i)\big]^2 + x(0)^2 \\
&\le 0\,,
\end{aligned}$$

we get

$$\sum_{n=1}^{T} f\big(n, x(n), x(n-\tau_1(n)), \dots, x(n-\tau_m(n))\big) x(n) \le 0\,.$$

It follows that

$$\begin{aligned}
\beta \sum_{n=1}^{T} |x(n)|^{\theta+1} &\le \sum_{n=1}^{T} g\big(n, x(n), x(n-\tau_1(n)), \dots, x(n-\tau_m(n))\big) x(n) \\
&\le -\sum_{n=1}^{T} h\big(n, x(n), x(n-\tau_1(n)), \dots, x(n-\tau_m(n))\big) x(n)
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{n=1}^{T} \big|h\big(n, x(n), x(n-\tau_1(n)), \dots, x(n-\tau_m(n))\big)\big|\,|x(n)| \\
&\le \sum_{n=1}^{T} p_0(n)|x(n)|^{\theta+1} + \sum_{i=1}^{m}\sum_{n=1}^{T} p_i(n)|x(n-\tau_i(n))|^{\theta}|x(n)| + \sum_{n=1}^{T} r(n)|x(n)| \\
&\le \|p_0\| \sum_{n=1}^{T} |x(n)|^{\theta+1} + \sum_{i=1}^{m}\sum_{n=1}^{T} p_i(n)|x(n-\tau_i(n))|^{\theta}|x(n)| + \sum_{n=1}^{T} r(n)|x(n)|\,.
\end{aligned}
$$

For $x_i \ge 0, y_i \ge 0$, Holder inequality implies

$$
\sum_{i=1}^{s} x_i y_i \le \Big(\sum_{i=1}^{s} x_i^p\Big)^{1/p}\Big(\sum_{i=1}^{s} y_i^q\Big)^{1/q}, \quad 1/p + 1/q = 1, \;\; q > 0, \; p > 0\,.
$$

It follows that

$$
\begin{aligned}
\beta \sum_{n=1}^{T}|x(n)|^{\theta+1} &\le \|p_0\| \sum_{n=1}^{T} |x(n)|^{\theta+1} + \sum_{i=1}^{m}\Big[\sum_{n=1}^{T}(p_i(n)|x(n-\tau_i(n))|^{\theta})^{\frac{\theta+1}{\theta}}\Big]^{\frac{\theta}{\theta+1}} \\
&\quad \times \Big(\sum_{n=1}^{T} |x(n)|^{\theta+1}\Big)^{\frac{1}{\theta+1}} + \Big(\sum_{n=1}^{T}[r(n)]^{\frac{\theta+1}{\theta}}\Big)^{\frac{\theta}{\theta+1}}\Big(\sum_{n=1}^{T} |x(n)|^{\theta+1}\Big)^{\frac{1}{\theta+1}} \\
&\le \|p_0\| \sum_{n=1}^{T} |x(n)|^{\theta+1} + \sum_{i=1}^{m}\Big[\Big(\sum_{n=1}^{T}[p_i(n)]^{\frac{\theta+1}{\theta-1}}\Big)^{\frac{\theta-1}{\theta}}\Big(\sum_{n=1}^{T} |x(n-\tau_i(n))|^{\theta+1}\Big)\Big]^{\frac{\theta}{\theta+1}} \\
&\quad \times \Big(\sum_{n=1}^{T} |x(n)|^{\theta+1}\Big)^{\frac{1}{\theta+1}} + \Big(\sum_{n=1}^{T}[r(n)]^{\frac{\theta+1}{\theta}}\Big)^{\frac{\theta}{\theta+1}}\Big(\sum_{n=1}^{T} |x(n)|^{\theta+1}\Big)^{\frac{1}{\theta+1}} \\
&= \|p_0\| \sum_{n=1}^{T} |x(n)|^{\theta+1} + \sum_{i=1}^{m}\Big(\sum_{n=1}^{T}[p_i(n)]^{\frac{\theta+1}{\theta-1}}\Big)^{\frac{\theta-1}{\theta+1}}\Big(\sum_{n=1}^{T} |x(n-\tau_i(n))|^{\theta+1}\Big)^{\frac{\theta}{\theta+1}} \\
&\quad \times \Big(\sum_{n=1}^{T} |x(n)|^{\theta+1}\Big)^{\frac{1}{\theta+1}} + \Big(\sum_{n=1}^{T}[r(n)]^{\frac{\theta+1}{\theta}}\Big)^{\frac{\theta}{\theta+1}}\Big(\sum_{n=1}^{T} |x(n)|^{\theta+1}\Big)^{\frac{1}{\theta+1}} \\
&= \|p_0\| \sum_{n=1}^{T} |x(n)|^{\theta+1} + \sum_{i=1}^{m}\Big(\sum_{n=1}^{T}[p_i(n)]^{\frac{\theta+1}{\theta-1}}\Big)^{\frac{\theta-1}{\theta+1}} \\
&\quad \times \Big(\sum_{u\in\{n-\tau_i(n):\, n=1,\cdots,T\}} |x(u)|^{\theta+1}\Big)^{\frac{\theta}{\theta+1}}\Big(\sum_{n=1}^{T} |x(n)|^{\theta+1}\Big)^{\frac{1}{\theta+1}} \\
&\quad + \Big(\sum_{n=1}^{T}[r(n)]^{\frac{\theta+1}{\theta}}\Big)^{\frac{\theta}{\theta+1}}\Big(\sum_{n=1}^{T} |x(n)|^{\theta+1}\Big)^{\frac{1}{\theta+1}}
\end{aligned}
$$

$$
\begin{aligned}
&\le \|p_0\| \sum_{n=1}^{T} |x(n)|^{\theta+1} + T \sum_{i=1}^{m} \Big(\sum_{n=1}^{T} [p_i(n)]^{\frac{\theta+1}{\theta-1}} \Big)^{\frac{\theta-1}{\theta+1}} \Big(\sum_{u=1}^{T} |x(u)|^{\theta+1} \Big)^{\frac{\theta}{\theta+1}} \\
&\quad \times \Big(\sum_{n=1}^{T} |x(n)|^{\theta+1} \Big)^{\frac{1}{\theta+1}} + \Big(\sum_{n=1}^{T} [r(n)]^{\frac{\theta+1}{\theta}} \Big)^{\frac{\theta}{\theta+1}} \Big(\sum_{n=1}^{T} |x(n)|^{\theta+1} \Big)^{\frac{1}{\theta+1}} \\
&= \|p_0\| \sum_{n=1}^{T} |x(n)|^{\theta+1} + T \sum_{i=1}^{m} \Big(\sum_{n=1}^{T} [p_i(n)]^{\frac{\theta+1}{\theta-1}} \Big)^{\frac{\theta-1}{\theta+1}} \Big(\sum_{u=1}^{T} |x(u)|^{\theta+1} \Big) \\
&\quad + \Big(\sum_{n=1}^{T} [r(n)]^{\frac{\theta+1}{\theta}} \Big)^{\frac{\theta}{\theta+1}} \Big(\sum_{n=1}^{T} |x(n)|^{\theta+1} \Big)^{\frac{1}{\theta+1}} .
\end{aligned}
$$

We get

$$
\begin{aligned}
&\Big(\beta - \|p_0\| - T \sum_{i=1}^{m} \Big(\sum_{n=1}^{T} [p_i(n)]^{\frac{\theta+1}{\theta-1}} \Big)^{\frac{\theta-1}{\theta+1}} \Big) \sum_{u=1}^{T} |x(u)|^{\theta+1} \\
&\qquad \le \Big(\sum_{n=1}^{T} [r(n)]^{\frac{\theta+1}{\theta}} \Big)^{\frac{\theta}{\theta+1}} \Big(\sum_{n=1}^{T} |x(n)|^{\theta+1} \Big)^{\frac{1}{\theta+1}} .
\end{aligned}
$$

It follows that there is $M_1 > 0$ such that $\sum_{u=1}^{T} |x(u)|^{\theta+1} \le M_1$.

It follows from above discussion that $|x(n)| \le M_1^{1/(\theta+1)}$ for all $n \in \{1, \dots, T\}$. So Ω_1 is bounded. This completes the Step 1.

Step 2. Prove that the set $\Omega_2 = \{x \in \operatorname{Ker} L : \ Nx \in \operatorname{Im} L\}$ is bounded.

For $x \in \operatorname{Ker} L$, we have $x(n) = c$. Thus

$$
Nx(t) = f\,(n, c, c, \dots, c) \quad \text{for} \quad x \in X .
$$

$Nx \in \operatorname{Im} L$ implies that

$$
\sum_{n=0}^{T-1} f\,(n, c, c, \dots, c) = 0 .
$$

It follows from condition (B) that $|c| \le M$. Thus Ω_2 is bounded.

Step 3. Prove the set $\Omega_3 = \{x \in \operatorname{Ker} L : \ \pm\lambda x + (1-\lambda) QNx = 0, \ \lambda \in [0,1]\}$ is bounded.

If the first inequality of (B) holds, let

$$
\Omega_3 = \{x \in \operatorname{Ker} L : \ \lambda x + (1-\lambda) QNx = 0, \ \lambda \in [0,1]\} .
$$

We will prove that Ω_3 is bounded. To the contrary that Ω_3 is unbounded, there are sequences $x_n(k) = a_n$ and λ_n such that $\|a_n\|_\infty \to \infty$ as n tends to infinity. Thus we have $|a_n| > M$ for sufficiently large n. Since $x_n \in \Omega_3$, we get

$$
-(1-\lambda_n)\Big(\sum_{n=0}^{T-1} f\,(n, c, c, \dots, c) \Big) = \lambda_n c T .
$$

If $\lambda_n = 1$, then $a_n = 0$, a contradiction. Hence

$$-(1-\lambda_n)c\Big(\sum_{n=0}^{T-1} f\,(n, c, c, \dots, c)\Big) = \lambda_n c^2 T \le 0\,,$$

from (B), a contradiction.

If the second inequality of (B) holds, let

$$\Omega_3 = \{x \in \operatorname{Ker} L :\ -\lambda x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}\,.$$

Similarly, we can get a contradiction. So Ω_3 is bounded.

Step 4. Obtain open bounded set Ω such that (i), (ii) and (iii) of Proposition 1.

In the following, we shall show that all conditions of Proposition 1 are satisfied. Set Ω be a open bounded subset of X such that $\Omega \supset \cup_{i=1}^{3}\overline{\Omega_i}$. We know that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By the definition of Ω, we have $\Omega \supset \overline{\Omega_1}$ and $\Omega \supset \overline{\Omega_2}$, thus $Lx \ne \lambda Nx$ for $x \in (D(L)/\operatorname{Ker} L) \cap \partial\Omega$ and $\lambda \in (0,1)$; $Nx \notin \operatorname{Im} L$ for $x \in \operatorname{Ker} L \cap \partial\Omega$.

In fact, let $H(x,\lambda) = \pm\lambda x + (1-\lambda)QNx$. According the definition of Ω, we know $\Omega \supset \overline{\Omega_3}$, thus $H(x,\lambda) \ne 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L$, thus by homotopy property of degree,

$$\begin{aligned}\deg\big(QN \mid \operatorname{Ker} L, \Omega \cap \operatorname{Ker} L, 0\big) &= \deg\big(H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0\big)\\ &= \deg\big(H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0\big) = \deg\big(\pm\wedge, \Omega \cap \operatorname{Ker} L, 0\big) \ne 0\,.\end{aligned}$$

Thus by Proposition 1, $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$, which is a solution of equation (1). The proof is completed. □

3. An examples

In this section, we present an example to illustrate the main result in Section 2.

Example 1. Consider the following equation

$$\Delta^2 x(n-1) = \beta[x(n)]^{2k+1} + \sum_{i=1}^{m} p_i(n)[x(n-\tau_i(n))]^{2k+1} + r(n), \quad n \in Z\,, \tag{3}$$

where k is a positive integer, $\beta > 0$, $p_i(n)$, $r(n)$ are $2T$-periodic sequences. Corresponding to the assumptions of Theorem L, we set

$$g(n, x_0, \dots, x_m) = \beta[x_0]^{2k+1}\,,$$

and

$$h(, x_0, \dots, x_m) = \sum_{i=1}^{m} p_i(n)[x_i]^{2k+1} + r(n)$$

with $\theta = 2k+1$. It is easy to see that (A_1) and (A_2) hold, and

$$cf(n, c, \dots, c) = c^{2k+2}\Big(\beta + \sum_{i=1}^{m} p_i(n)\Big) + cr(n)$$

implies that there is $M > 0$ such that $cf(n, c, \ldots, c) > 0$ for all $n \in Z$ and $|c| > M$ if $\beta + \sum_{i=1}^{m} p_i(n) > 0$.

It follows from Theorem 1 that (3) has at least one $2T$-periodic solution if

$$\|p_0\| + T\sum_{i=1}^{m}\Big(\sum_{n=1}^{T}[p_i(n)]^{\frac{2k+2}{2k}}\Big)^{\frac{2k}{2k+2}} < \beta$$

and $\beta + \sum_{i=1}^{m} p_i(n) > 0$.

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