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ON THE NUMBER OF ZEROS OF BOUNDED NONOSCILLATORY SOLUTIONS TO HIGHER-ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The higher-order nonlinear ordinary differential equation

$$x^{(n)} + \lambda p(t) f(x) = 0\,, \quad t \ge a\,,$$

is considered and the problem of counting the number of zeros of bounded nonoscillatory solutions $x(t;\lambda)$ satisfying $\lim_{t\to\infty} x(t;\lambda) = 1$ is studied. The results can be applied to a singular eigenvalue problem.

1. Introduction

In this paper the higher-order ordinary differential equation

$$x^{(n)} + \lambda p(t) f(x) = 0\,, \quad t \ge a\,, \tag{1.1}$$

is considered under the hypotheses that

$$n \ge 2 \quad \text{is even;} \tag{1.2}$$

$$\lambda > 0 \quad \text{is a parameter;} \tag{1.3}$$

$$p(t) \quad \text{is continuous on } [a,\infty),\ a > 0 \text{ and } p(t) > 0 \text{ for } t \ge a\,; \tag{1.4}$$

$$\begin{cases} f(x) & \text{is locally Lipschitz continuous on } \mathbf{R} \text{ and} \\ xf(x) > 0 & \text{for } x \ne 0. \end{cases} \tag{1.5}$$

Moreover we assume that $f(x)$ satisfies the additional conditions as follows: there exists γ such that $0 < \gamma < 1$ and

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$$
\begin{cases}
0 < \liminf\limits_{x\to\infty} \dfrac{f(x)}{|x|^\gamma \operatorname{sgn} x} \le \limsup\limits_{x\to\infty} \dfrac{f(x)}{|x|^\gamma \operatorname{sgn} x} < \infty\,, \\
0 < \liminf\limits_{x\to-\infty} \dfrac{f(x)}{|x|^\gamma \operatorname{sgn} x} \le \limsup\limits_{x\to-\infty} \dfrac{f(x)}{|x|^\gamma \operatorname{sgn} x} < \infty\,,
\end{cases} \tag{1.6}
$$

and

$$
\liminf_{x\to 0} \frac{f(x)}{x} > 0\,. \tag{1.7}
$$

Since $f(x)$ is locally Lipschitz continuous on $\mathbf{R}$, it is continuous on $\mathbf{R}$. This fact, together with the condition $xf(x) > 0$ ($x \ne 0$), implies $f(0) = 0$. Further, the Lipschitz continuity of $f(x)$ near $x = 0$ implies that there are $x_0 > 0$ and $L > 0$ such that $|f(x)| \le L|x|$ for $|x| \le x_0$. Thus we have

$$
\limsup_{x\to 0} \frac{f(x)}{x} < \infty\,. \tag{1.8}
$$

Let

$$
f^*(x) = \begin{cases} x\,, & |x| \le 1\,, \\ |x|^\gamma \operatorname{sgn} x\,, & |x| \ge 1\,. \end{cases} \tag{1.9}
$$

Then, by the conditions (1.6)–(1.8) there exist $L_1 > 0$ and $L_2 > 0$ such that

$$
L_1 x f^*(x) \le x f(x) \le L_2 x f^*(x) \quad \text{for all} \quad x \in \mathbf{R}\,. \tag{1.10}
$$

In this sense, the nonlinear term $f(x)$ in (1.1) behaves like $c_1 x$ near $x = 0$ and like $c_2|x|^\gamma \operatorname{sgn} x$ ($0 < \gamma < 1$) near $x = \pm\infty$, where c_1 and c_1 are positive constants. It should be noted that the nondecreasing condition of $f(x)$ itself is not assumed.

It is known (see, for example, Kiguradze and Chanturia [6, Corollary 8.2]) that if $p(t)$ satisfies the integral condition

$$
\int_a^\infty t^{n-1} p(t)\, dt < \infty\,, \tag{1.11}
$$

then, for each $\lambda > 0$, equation (1.1) has a bounded nonoscillatory solution $x(t;\lambda)$ such that

$$
\lim_{t\to\infty} x(t;\lambda) = 1\,. \tag{1.12}
$$

The main interest of this paper is the problem of counting the number of zeros of solutions $x(t;\lambda)$ satisfying (1.12).

For the second-order nonlinear equations, the problem of counting the number of zeros of solutions is studied in the framework of regular eigenvalue problems on compact interval (see, for example, [8, 9] and the papers cited therein). Regular eigenvalue problems for higher-order linear differential equations are also considered in several papers (see [1] and the papers cited in [1]). They are summarized and discussed in the book of Elias [2]. For a class of higher-order nonlinear differential equations, a regular eigenvalue problem is discussed by Elias and Pinkus [4].

In the recent paper [7], the higher-order linear equation

$$x^{(n)} + \lambda p(t)x = 0\,, \quad t \geq a\,, \tag{1.13}$$

was considered in an infinite interval $[a, \infty)$ and the following result was obtained: Suppose that (1.11) holds, and let $x(t;\lambda)$ be solutions of (1.13) satisfying (1.12). Then there exists a sequence $\{\lambda(k)\}_{k=0}^{\infty}$ of positive parameters with the properties that (i) $0 = \lambda(0) < \lambda(1) < \cdots < \lambda(k-1) < \lambda(k) < \ldots$, $\lim_{k\to\infty} \lambda(k) = \infty$; (ii) if $\lambda \in \big(\lambda(k-1), \lambda(k)\big)$, then $x(t;\lambda)$ has at most $k-1$ zeros in (a,∞); (iii) if $\lambda = \lambda(k)$, then $x(t;\lambda)$ has exactly $k-1$ zeros in (a,∞) and $x(a;\lambda) = 0$. Very recently, a more general problem has been considered by Elias [3] and the above result in [7] has been extended vastly.

In this paper it will be proved that the above result in [7] remains valid for the nonlinear equation (1.1). As an underlying fact we first prove the following theorem.

Theorem 1.1. *Consider the nonlinear equation* (1.1) *under the conditions* (1.2)*−*(1.7)*. Suppose that* (1.11) *holds. Then, for each* $\lambda > 0$*, equation* (1.1) *has a solution* $x(t;\lambda)$ *satisfying* (1.12)*, and* $x(t;\lambda)$ *is globally defined and is uniquely determined on* $[a,\infty)$*. Furthermore,* $x(t;\lambda)$ *and its derivatives* $x^{(i)}(t;\lambda)$ $(i = 1,2,\ldots,n-1)$ *with respect to* t *are continuous functions of* $(t,\lambda) \in [a,\infty)\times(0,\infty)$.

The main result in this paper is the following theorem.

Theorem 1.2. *Consider the nonlinear equation* (1.1) *under the conditions* (1.2)*−*(1.7)*. Suppose that* (1.11) *holds, and, for each* $\lambda > 0$*, let* $x(t;\lambda)$ *be a solution of* (1.1) *satisfying* (1.12)*. Then there exists a sequence* $\{\lambda(k)\}_{k=0}^{\infty}$ *of positive parameters with the properties that*

- (i) $0 = \lambda(0) < \lambda(1) < \cdots < \lambda(k-1) < \lambda(k) < \cdots$, $\lim\limits_{k\to\infty} \lambda(k) = \infty$*;*
- (ii) *if* $\lambda \in \big(\lambda(k-1), \lambda(k)\big)$*,* $k = 1,2,\ldots$*, then* $x(t;\lambda)$ *has at most* $k-1$ *zeros in the open interval* (a,∞)*;*
- (iii) *if* $\lambda = \lambda(k), k = 1,2,\ldots$*, then* $x(t;\lambda)$ *has exactly* $k-1$ *zeros in the open interval* (a,∞) *and* $x(a;\lambda) = 0$*.*

The proofs of Theorems 1.1 and 1.2 are given in the next section.

Later, it will be seen that the solution $x(t;\lambda)$ satisfies

$$\lim_{t\to\infty} x^{(i)}(t;\lambda) = 0 \quad (i = 1,2,\ldots,n-1)\,. \tag{1.14}$$

Therefore Theorem 1.2 can be applied to the singular eigenvalue problem

$$\begin{cases} x^{(n)} + \lambda p(t)f(x) = 0\,, \quad t \geq a\,, \\ x(a) = 0, \quad \lim\limits_{t\to\infty} x^{(i)}(t) = 0 \quad (i = 1,2,\ldots,n-1)\,, \end{cases} \tag{1.15}$$

and the following corollary is easily verified.

Corollary 1.1. *Consider the problem* (1.15) *under the conditions* (1.2) − (1.7)*. If* (1.11) *holds, then there exists a sequence* $\{\lambda(k)\}_{k=1}^{\infty}$ *such that* $0 < \lambda(1) < \cdots < \lambda(k-1) < \lambda(k) < \cdots$, $\lim_{k\to\infty} \lambda(k) = \infty$*, and for* $\lambda = \lambda(k)$ $(k = 1,2,\ldots)$ *the* *problem* (1.15) *has a nontrivial solution* $x = x\big(t; \lambda(k)\big)$ *having exactly* $k-1$ *zeros in the open interval* (a, ∞).

To the author's knowledge, for higher-order nonlinear eigenvalue problems in infinite interval there is no literature showing the existence of a sequence of eigenfunctions with the prescribed numbers of zeros.

The author has a great interest in the Emden-Fowler nonlinear differential equation

$$x^{(n)} + \lambda p(t)|x|^{\gamma}\operatorname{sgn} x = 0\,, \quad t \ge a\,, \quad \text{where} \quad \gamma > 0\,, \gamma \ne 1\,. \tag{1.16}$$

It is expected that Corollary 1.1 remains true for (1.16). However the problem for (1.16) is still open.

2. Proofs of Theorems

Let us set about the proof of Theorem 1.1.

Proof of Theorem 1.1. As stated in the above, the existence of $x(t;\lambda)$ for each fixed $\lambda > 0$ is well known. Here, we show the existence and the uniqueness of $x(t;\lambda)$ and the continuity of $x^{(i)}(t;\lambda)$ $(i = 0, 1, \ldots, n-1)$ simultaneously. Note first that a solution $x(t;\lambda)$ of (1.1) satisfying (1.12) is written in the form

$$x(t;\lambda) = 1 - \lambda \int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!}\, p(s)\, f\big(x(s;\lambda)\big)\, ds$$

for all large t. Let $\Lambda > 0$ be an arbitrary number. From the Lipschitz continuity of $f(x)$ on the interval $[1/2, 1]$, there exists $L > 0$ such that

$$|f(x) - f(y)| \le L|x - y| \quad \text{for all} \quad x, y \quad \text{with} \quad 1/2 \le x, y \le 1\,. \tag{2.1}$$

Let

$$M = \max\big\{f(x) \,:\, 1/2 \le x \le 1\big\} \ \ (>0)\,. \tag{2.2}$$

Then, take $T = T(\Lambda) \ge a$ so that

$$\Lambda L \int_T^{\infty} s^{n-1} p(s)\, ds \le \frac{(n-1)!}{2} \tag{2.3}$$

and

$$\Lambda M \int_T^{\infty} s^{n-1} p(s)\, ds \le \frac{(n-1)!}{2}\,. \tag{2.4}$$

By (1.11), it is possible to take such a T. Denote by $C_b\big([T,\infty) \times (0,\Lambda]\big)$ the Banach space which consists of all bounded continuous functions $x(t;\lambda)$ on $[T,\infty) \times (0,\Lambda]$ with the norm

$$\|x\| = \sup\big\{|x(t;\lambda)| \,:\, (t,\lambda) \in [T,\infty) \times (0,\Lambda]\big\}\,.$$

We define the subset X of $C_b\big([T,\infty) \times (0,\Lambda]\big)$ and the mapping $\Phi \,:\, X \to C_b\big([T,\infty) \times (0,\Lambda]\big)$ by

$$X = \Big\{x \in C_b\big([T,\infty) \times (0,\Lambda]\big) \,:\, \frac{1}{2} \le x(t;\lambda) \le 1 \quad \text{for} \quad (t,\lambda) \in [T,\infty) \times (0,\Lambda]\Big\}$$

and

$$(\Phi x)(t;\lambda) = 1 - \lambda \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\; p(s)\, f\big(x(s;\lambda)\big)\, ds\,, \qquad (t,\lambda) \in [T,\infty) \times (0,\Lambda]\,,$$

respectively. Using (2.1) – (2.4), we find that $\Phi(X) \subseteq X$ and

$$\|\Phi x - \Phi y\| \le \frac{1}{2}\,\|x-y\| \quad \text{for all} \quad x,y \in X\,.$$

Then the contraction mapping principle ensures the existence and uniqueness of a fixed point $x = x(t;\lambda) \in X$ of Φ. It is easy to see that, for each $\lambda \in (0,\Lambda]$, this fixed point $x(t;\lambda)$ is a solution of (1.1) on the interval $[T,\infty)$ and satisfies (1.12). It is immediate to see that this $x(t;\lambda)$ is a unique solution of (1.1) on $[T,\infty)$ satisfying (1.12). Further, $x(t;\lambda)$ is a continuous function of $(t,\lambda) \in [T,\infty)\times(0,\Lambda]$, and so the derivatives of $x(t;\lambda)$ with respect to t,

$$x^{(i)}(t;\lambda) = (-1)^{i-1}\lambda \int_t^\infty \frac{(s-t)^{n-i-1}}{(n-i-1)!}\; p(s)\, f\big(x(s;\lambda)\big)\, ds\,, \qquad i = 1,2,\ldots,n-1\,,$$

are also continuous functions of $(t,\lambda) \in [T,\infty)\times(0,\Lambda]$.

Since (1.6) with $0<\gamma<1$ is assumed to hold, it follows from Wintner's theorem ([5, p.29]) that the maximal interval of existence of solutions of (1.1) is the entire interval $[a,\infty)$. Since $f(x)$ is locally Lipschitz continuous on $\mathbf{R}$, we find that, for each $\lambda \in (0,\Lambda]$, the solution $x(t;\lambda)$ of (1.1) on $[T,\infty)$ constructed as the above is uniquely continued to the left of T as a solution of (1.1) on $[a,\infty)$. This solution of (1.1) on $[a,\infty)$ is denoted by $x(t;\lambda)$ again, where $\lambda \in (0,\Lambda]$.

Recall that the solution $x(t;\lambda)$ and its derivatives $x^{(i)}(t;\lambda)$, $i = 1,2,\ldots,n-1$, are continuous functions of $(t,\lambda) \in [T,\infty)\times(0,\Lambda]$. Then, by a fundamental theorem concerning the continuity of solutions of initial value problems with parameters (see, for example, [5, p.94]), we see that the solution $x(t;\lambda)$ and its derivatives $x^{(i)}(t;\lambda)$, $i = 1,2,\ldots,n-1$, are continuous functions of $(t,\lambda) \in [a,\infty)\times(0,\Lambda]$. Since $\Lambda > 0$ is arbitrary, this proves the continuity of $x^{(i)}(t;\lambda)$ on $[a,\infty)\times(0,\infty)$, $i = 0,1,2,\ldots,n-1$. The proof of Theorem 1.1 is complete. $\square$

In what follows, $x(t;\lambda)$ denotes the solution of (1.1) satisfying (1.12). It is worth noting that, for each $\lambda > 0$,

$$x(t;\lambda) = 1 - \lambda \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\; p(s)\, f\big(x(s;\lambda)\big)\, ds\,, \quad t \ge a\,, \tag{2.5}$$

and

$$x^{(i)}(t;\lambda) = (-1)^{i-1}\lambda \int_t^\infty \frac{(s-t)^{n-i-1}}{(n-i-1)!}\; p(s)\, f\big(x(s;\lambda)\big)\, ds\,, \quad t \ge a\,, \qquad i = 1,2,\ldots,n-1\,. \tag{2.6}$$

In particular, we have (1.14).

The following lemma is clear from the proof of Theorem 1.1.

Lemma 2.1. *Let $\Lambda > 0$ be fixed. Then, there exists $T = T(\Lambda) \geq a$ such that, for $\lambda \in (0, \Lambda]$, $x(t;\lambda)$ has no zeros in the interval $[T, \infty)$.*

It is possible to take $\lambda_* > 0$ such that

$$\lambda_* L \int_a^\infty s^{n-1} p(s)\, ds \leq \frac{(n-1)!}{2} \quad \text{and} \quad \lambda_* M \int_a^\infty s^{n-1} p(s)\, ds \leq \frac{(n-1)!}{2}\,,$$

where L and M are given by (2.1) and (2.2), respectively. Then, by the proof of Theorem 1.1, we see that $1/2 \leq x(t;\lambda) \leq 1$ for $t \in [a, \infty)$ and $\lambda \in (0, \lambda_*]$. Thus we have the following.

Lemma 2.2. *There exists $\lambda_* > 0$ such that, for $\lambda \in (0, \lambda_*]$, $x(t;\lambda)$ has no zeros in the interval $[a, \infty)$.*

Since $f(x)$ is locally Lipschitz continuous on $\mathbf{R}$, we see that if $x(t)$ is a solution of (1.1) such that

$$x(t_0) = x'(t_0) = \cdots = x^{(n-1)}(t_0) = 0\,, \quad \text{where} \quad t_0 \geq a\,,$$

then $x(t) \equiv 0$ for all $t \geq a$. Therefore, for each $\lambda > 0$, $x(t;\lambda)$ has a finite number of zeros in any finite subinterval of $[a, \infty)$. This fact, combined with Lemma 2.1, implies that, for each $\lambda > 0$, $x(t;\lambda)$ has a finite number of zeros in $[a, \infty)$.

The following lemma can be proved exactly as in the linear case ([7, Proposition 2.4]).

Lemma 2.3. *For each $\lambda > 0$, the zeros of $x(t;\lambda)$ are simple.*

The following lemma gives an estimate for $x(t;\lambda)$.

Lemma 2.4. *We have*

$$\big|x(t;\lambda)\big| \leq \Big[1 + (1-\gamma)\lambda L_2 \int_a^\infty \frac{s^{n-1}}{(n-1)!}\, p(s)\, ds\Big]^{1/(1-\gamma)} \tag{2.7}$$

for all $(t,\lambda) \in [a,\infty) \times (0,\infty)$. Here, γ and L_2 are the constants appearing in (1.6) *and* (1.10)*, respectively. Note that $0 < \gamma < 1$.*

Proof. Let $\lambda > 0$ be fixed. From (2.5) it follows that

$$\big|x(t;\lambda)\big| \leq 1 + \lambda \int_t^\infty \frac{s^{n-1}}{(n-1)!}\, p(s)\, \big|f(x(s;\lambda))\big|\, ds \tag{2.8}$$

for $t \geq a$. Denote by $y(t)$ the right-hand side of (2.8). We have $y(t) \geq 1$ $(t \geq a)$, $\lim_{t\to\infty} y(t) = 1$ and

$$\big|x(t;\lambda)\big| \leq y(t)\,, \quad t \geq a\,. \tag{2.9}$$

Then,

$$\begin{aligned} y'(t) &= -\lambda \frac{t^{n-1}}{(n-1)!}\, p(t)\, \big|f(x(t;\lambda))\big| \geq -\lambda L_2 \frac{t^{n-1}}{(n-1)!}\, p(t)\, f^*\big(|x(t;\lambda)|\big) \\ &\geq -\lambda L_2 \frac{t^{n-1}}{(n-1)!}\, p(t)\, f^*\big(y(t)\big) = -\lambda L_2 \frac{t^{n-1}}{(n-1)!}\, p(t)\, y(t)^\gamma \end{aligned}$$

for $t \ge a$, where $f^*(x)$ is given by (1.9), and the inequality (1.10) has been used. Thus, for $\tau \ge t \ge a$,

$$\frac{1}{1-\gamma}\, y(\tau)^{1-\gamma} - \frac{1}{1-\gamma}\, y(t)^{1-\gamma} \ge -\lambda L_2 \int_t^\tau \frac{s^{n-1}}{(n-1)!}\, p(s)\, ds\,.$$

Letting $\tau \to \infty$ in the above, we get

$$\frac{1}{1-\gamma} - \frac{1}{1-\gamma}\, y(t)^{1-\gamma} \ge -\lambda L_2 \int_t^\infty \frac{s^{n-1}}{(n-1)!}\, p(s)\, ds\,,$$

and so

$$y(t) \le \left[1 + (1-\gamma)\lambda L_2 \int_a^\infty \frac{s^{n-1}}{(n-1)!}\, p(s)\, ds\right]^{1/(1-\gamma)}, \quad t \ge a\,. \tag{2.10}$$

Then, (2.9) and (2.10) yield (2.7). The proof of Lemma 2.4 is complete. □

Lemma 2.5. *Let $[b,c] \subseteq [a,\infty)$ and let $\{\lambda_j\}$ be a sequence such that $0 < \lambda_j \to \infty$ as $j \to \infty$. Suppose that, for each j, $x = \varphi(t;\lambda_j)$ is a solution of* (1.1) *with $\lambda = \lambda_j$ on $[b,c]$ such that $\varphi(t;\lambda_j)$ does not change signs and $\varphi(t;\lambda_j) \not\equiv 0$ on $[b,c]$. Then, there exist a constant $m > 0$ and a subinterval $[b',c'] \subseteq [b,c]$ and a subsequence $\{\lambda_{j'}\} \subseteq \{\lambda_j\}$ such that*

$$\bigl|\varphi(t;\lambda_{j'})\bigr| \ge m\lambda_{j'}^{1/(1-\gamma)} \quad on \quad [b',c'] \tag{2.11}$$

for all j'.

Proof. A part of the proof of this lemma is done along the proof of the result of Elias [1, Lemma 4]. Without loss of generality, we may suppose that either

$$\varphi(t;\lambda_j) \ge 0\,, \not\equiv 0 \quad \text{on} \quad [b,c] \quad \text{for all} \quad j\,, \tag{2.12}$$

or

$$\varphi(t;\lambda_j) \le 0\,, \not\equiv 0 \quad \text{on} \quad [b,c] \quad \text{for all} \quad j\,. \tag{2.13}$$

In the case where (2.13) holds, let $\tilde{f}(x) = -f(-x)$. Then, $x = -\varphi(t;\lambda_j)$ is a nonnegative and nontrivial solution of the equation

$$x^{(n)} + \lambda p(t)\tilde{f}(x) = 0 \tag{2.14}$$

with $\lambda = \lambda_j$ on $[b,c]$. Observe that, for the equation (2.14), all the hypotheses corresponding to (1.2) – (1.7) are satisfied. Thus it is enough to discuss the case where (2.12) holds.

Take b_1, c_1 and d arbitrarily so that $b < b_1 < d < c_1 < c$, and fix these three numbers. We first consider the sequence $\{\varphi(t;\lambda_j)\}$ on the interval $[b,d]$. By Taylor's formula with remainder, we have

$$\begin{aligned}\varphi(t;\lambda_j) &= \sum_{l=0}^{n-1} \frac{\varphi^{(l)}(d;\lambda_j)}{l!}\,(t-d)^l + \int_d^t \frac{(t-s)^{n-1}}{(n-1)!}\,\varphi^{(n)}(s;\lambda_j)\, ds\\ &= \sum_{l=0}^{n-1} (-1)^l\, \frac{\varphi^{(l)}(d;\lambda_j)}{l!}\,(d-t)^l - \lambda_j \int_t^d \frac{(s-t)^{n-1}}{(n-1)!}\, p(s) f\bigl(\varphi(s;\lambda_j)\bigr)\, ds\,,\end{aligned}$$

where the hypothesis (1.2) has been used. Then it follows from (2.12) and Cauchy's inequality that

$$\begin{aligned}\lambda_j \int_t^d \frac{(s-t)^{n-1}}{(n-1)!}\, p(s) f\big(\varphi(s;\lambda_j)\big)\, ds &\le \sum_{l=0}^{n-1} (-1)^l\, \frac{\varphi^{(l)}(d;\lambda_j)}{l!}\, (d-t)^l \\ &\le \Big\{ \sum_{l=0}^{n-1} \big[\varphi^{(l)}(d;\lambda_j)\big]^2 \Big\}^{1/2} \Big\{ \sum_{l=0}^{n-1} \Big[\frac{(d-t)^l}{l!}\Big]^2 \Big\}^{1/2}\end{aligned}$$

for $t \in [b,d]$. Set

$$D(\lambda_j) = \Big\{ \sum_{l=0}^{n-1} \big[\varphi^{(l)}(d;\lambda_j)\big]^2 \Big\}^{1/2} \quad \text{and} \quad K = \Big\{ \sum_{l=0}^{n-1} \Big[\frac{(d-b)^l}{l!}\Big]^2 \Big\}^{1/2}.$$

The preceding inequality gives

$$\lambda_j \int_b^d \frac{(s-b)^{n-1}}{(n-1)!}\, p(s)\, f\big(\varphi(s;\lambda_j)\big)\, ds \le K D(\lambda_j)\,,$$

and so

$$\lambda_j \int_{b_1}^d \frac{(s-b)^{n-1}}{(n-1)!}\, p(s)\, f\big(\varphi(s;\lambda_j)\big)\, ds \le K D(\lambda_j)\,.$$

This yields

$$\lambda_j\, \frac{(b_1-b)^{n-1}}{(n-1)!} \int_{b_1}^d p(s)\, f\big(\varphi(s;\lambda_j)\big)\, ds \le K D(\lambda_j)\,,$$

or

$$\lambda_j \int_{b_1}^d p(s)\, f\big(\varphi(s;\lambda_j)\big)\, ds \le (b_1-b)^{-n+1} (n-1)!\, K D(\lambda_j)\,.$$

Let $L = (b_1-b)^{-n+1}(n-1)! K\,(>0)$. Then,

$$\lambda_j \int_t^d p(s)\, f\big(\varphi(s;\lambda_j)\big)\, ds \le L D(\lambda_j)\,, \quad b_1 \le t \le d\,. \tag{2.15}$$

Integrating this inequality over $[t,d]$ ($b_1 \le t \le d$), we see that

$$\lambda_j \int_t^d (s-t) p(s)\, f\big(\varphi(s;\lambda_j)\big)\, ds \le L D(\lambda_j)(d-t)\,, \quad b_1 \le t \le d\,. \tag{2.16}$$

Repeated integrations of (2.16) give

$$\lambda_j \int_t^d \frac{(s-t)^i}{i!}\, p(s)\, f\big(\varphi(s;\lambda_j)\big)\, ds \le L D(\lambda_j)\, \frac{(d-t)^i}{i!}\,, \quad b_1 \le t \le d\,, \tag{2.17}$$

for $i = 0, 1, 2, \ldots, n-1$.

By Taylor's formula with remainder again, we get

$$\begin{aligned}\varphi^{(i)}(t;\lambda_j) &= \sum_{l=0}^{n-i-1} \frac{\varphi^{(i+l)}(d;\lambda_j)}{l!}\,(t-d)^l + \int_d^t \frac{(t-s)^{n-i-1}}{(n-i-1)!}\,\varphi^{(n)}(s;\lambda_j)\,ds \\ &= \sum_{l=0}^{n-i-1} (-1)^l\,\frac{\varphi^{(i+l)}(d;\lambda_j)}{l!}\,(d-t)^l \\ &\quad + (-1)^{n-i-1}\lambda_j \int_t^d \frac{(s-t)^{n-i-1}}{(n-i-1)!}\,p(s)\,f\big(\varphi(s;\lambda_j)\big)\,ds\end{aligned}$$

for $i = 0, 1, 2, \dots, n-1$. Then, using (2.17), we see that

$$\begin{aligned}\big|\varphi^{(i)}(t;\lambda_j)\big| &\le \Big\{\sum_{l=0}^{n-i-1} \big[\varphi^{(i+l)}(d;\lambda_j)\big]^2\Big\}^{1/2} \Big\{\sum_{l=0}^{n-i-1} \Big[\frac{(d-t)^l}{l!}\Big]^2\Big\}^{1/2} \\ &\quad + \lambda_j \int_t^d \frac{(s-t)^{n-i-1}}{(n-i-1)!}\,p(s)\,f\big(\varphi(s;\lambda_j)\big)\,ds \\ &\le D(\lambda_j)\Big\{\sum_{l=0}^{n-i-1} \Big[\frac{(d-b_1)^l}{l!}\Big]^2\Big\}^{1/2} + LD(\lambda_j)\,\frac{(d-b_1)^{n-i-1}}{(n-i-1)!}\end{aligned}$$

for all $t \in [b_1, d]$ and $i = 0, 1, 2, \dots, n-1$. This implies that the sequences

$$\Big\{\frac{\varphi^{(i)}(t;\lambda_j)}{D(\lambda_j)}\Big\}_j, \quad i = 0, 1, 2, \dots, n-1\,,$$

are uniformly bounded on the interval $[b_1, d]$. Therefore, the sequences

$$\big\{\varphi^{(i)}(t;\lambda_j)/D(\lambda_j)\big\}_j, \quad i = 0, 1, 2, \dots, n-2\,,$$

are uniformly bounded and equicontinuous on $[b_1, d]$. From this we can deduce that there exist a C^{n-2}-class function $\varphi_0(t)$ on $[b_1, d]$ and a subsequence $\{\lambda_{j'}\} \subseteq \{\lambda_j\}$ — let us denote the subsequence $\{\lambda_{j'}\}$ by $\{\lambda_j\}$ again — such that

$$\frac{\varphi^{(i)}(t;\lambda_j)}{D(\lambda_j)} \to \varphi_0^{(i)}(t) \quad \text{uniformly on} \quad [b_1, d] \quad \text{as} \quad j \to \infty\,, \tag{2.18}$$

where $i = 0, 1, 2, \dots, n-2$. By (2.12), we have $\varphi_0(t) \ge 0$ for $t \in [b_1, d]$.

Consider the case where $\varphi_0(t) \equiv 0$ on $[b_1, d]$. In this case it follows from (2.18) that, for $i = 0, 1, 2, \dots, n-2$,

$$\frac{\varphi^{(i)}(t;\lambda_j)}{D(\lambda_j)} \to 0 \quad \text{uniformly on} \quad [b_1, d] \quad \text{as} \quad j \to \infty\,. \tag{2.19}$$

In particular,

$$\frac{\varphi^{(i)}(d;\lambda_j)}{D(\lambda_j)} \to 0 \quad \text{as} \quad j \to \infty \quad (i = 0, 1, 2, \dots, n-2)\,.$$

Since

$$\sum_{i=0}^{n-1} \Big\{\frac{\varphi^{(i)}(d;\lambda_j)}{D(\lambda_j)}\Big\}^2 = 1\,,$$

we have

$$\left\{\frac{\varphi^{(n-1)}(d;\lambda_j)}{D(\lambda_j)}\right\}^2 \to 1 \quad \text{as} \quad j \to \infty\,.$$

Then there is a subsequence $\{\lambda_{j'}\} \subseteq \{\lambda_j\}$ — we denote the subsequence $\{\lambda_{j'}\}$ by $\{\lambda_j\}$ again — such that either

$$\frac{\varphi^{(n-1)}(d;\lambda_j)}{D(\lambda_j)} = \frac{\varphi^{(n-1)}(d;\lambda_j)}{\left\{\sum_{i=0}^{n-1}\left[\varphi^{(i)}(d;\lambda_j)\right]^2\right\}^{1/2}} \to +1 \quad \text{as} \quad j \to \infty \tag{2.20}$$

or

$$\frac{\varphi^{(n-1)}(d;\lambda_j)}{D(\lambda_j)} = \frac{\varphi^{(n-1)}(d;\lambda_j)}{\left\{\sum_{i=0}^{n-1}\left[\varphi^{(i)}(d;\lambda_j)\right]^2\right\}^{1/2}} \to -1 \quad \text{as} \quad j \to \infty \tag{2.21}$$

holds. Assume that (2.20) occurs. Since $\varphi^{(n-1)}(t;\lambda_j)/D(\lambda_j)$ is nonincreasing on $[b_1, d]$, we see that, for all sufficiently large j,

$$\frac{\varphi^{(n-1)}(t;\lambda_j)}{D(\lambda_j)} \geq \frac{\varphi^{(n-1)}(d;\lambda_j)}{D(\lambda_j)} \geq \frac{1}{2} \quad \text{on} \quad [b_1, d]$$

and so

$$\frac{\varphi^{(n-2)}(d;\lambda_j)}{D(\lambda_j)} - \frac{\varphi^{(n-2)}(b_1;\lambda_j)}{D(\lambda_j)} \geq \frac{1}{2}\,(d - b_1)\,.$$

Then, letting $j \to \infty$ and noting (2.19), we get $0 \geq (d - b_1)/2$, which is a contradiction. Therefore, (2.20) does not occur. Consequently, for the case where $\varphi_0(t) \equiv 0$ on $[b_1, d]$, we must have (2.21).

Next consider the case where $\varphi_0(t) \geq 0, \not\equiv 0$ on $[b_1, d]$. In this case there are a positive number $\delta > 0$ and an interval $[t_1, t_2] \subseteq [b_1, d]$ such that $3\delta \leq \varphi_0(t) \leq 4\delta$ on $[t_1, t_2]$. Therefore, for all sufficiently large j,

$$2\delta D(\lambda_j) \leq \varphi(t;\lambda_j) \leq 5\delta D(\lambda_j) \quad \text{on} \quad [t_1, t_2]\,. \tag{2.22}$$

Assume that $\liminf D(\lambda_j) = 0$ as $j \to \infty$. Then, for some subsequence $\{\lambda_{j'}\} \subseteq \{\lambda_j\}$, we have $\lim_{j'\to\infty} D(\lambda_{j'}) = 0$. By (2.22), $\lim_{j'\to\infty} \varphi(t;\lambda_{j'}) = 0$ uniformly on $[t_1, t_2]$. Then, from (1.10) and (2.22), we may suppose that

$$f(\varphi(t;\lambda_{j'})) \geq L_1\varphi(t;\lambda_{j'}) \geq 2\delta L_1 D(\lambda_{j'}) \quad \text{on} \quad [t_1, t_2]$$

for all large j', where L_1 is a positive constant appearing (1.10). By (2.15), we get

$$2\delta L_1 \lambda_{j'} \int_{t_1}^{t_2} p(s)\,ds \leq L$$

for all large j'. But this gives a contradiction as $j' \to \infty$.

Next assume that $0 < \liminf D(\lambda_j) < \infty$ as $j \to \infty$. There are $D_0 \in (0, \infty)$ and a subsequence $\{\lambda_{j'}\} \subseteq \{\lambda_j\}$ such that $\lim_{j'\to\infty} D(\lambda_{j'}) = D_0 \in (0, \infty)$. By (2.22), we have $\delta D_0 \leq \varphi(t;\lambda_{j'}) \leq 6\delta D_0$ on $[t_1, t_2]$ for all large j'. By (2.15) again, we get

$$M_0\lambda_{j'} \int_{t_1}^{t_2} p(s)\,ds \leq 2LD_0$$

for all large j', where $M_0 = \min\{f(x) : \delta D_0 \le x \le 6\delta D_0\}$ (>0). This also gives a contradiction as $j' \to \infty$.

Thus we must have $\liminf D(\lambda_j) = \infty$ as $j \to \infty$, and hence $\lim D(\lambda_j) = \infty$ as $j \to \infty$. By (2.22), we may suppose that $\varphi(t;\lambda_j) \ge 1$ on $[t_1, t_2]$ for all j. Then, by (1.10),

$$f\big(\varphi(t;\lambda_j)\big) \ge L_1 \varphi(t;\lambda_j)^\gamma \quad \text{on} \quad [t_1, t_2]\,, \tag{2.23}$$

where $L_1 > 0$ and $0 < \gamma < 1$. Using (2.15), (2.22) and (2.23), we can compute as follows:

$$\begin{aligned}\lambda_j \int_{t_1}^{t_2} p(s)\,(2\delta)^\gamma ds &\le \lambda_j \int_{t_1}^{t_2} p(s)\Big(\frac{\varphi(s;\lambda_j)}{D(\lambda_j)}\Big)^\gamma ds \\ &\le \lambda_j D(\lambda_j)^{-\gamma}\,\frac{1}{L_1}\int_{t_1}^{t_2} p(s)\, f\big(\varphi(s;\lambda_j)\big)\,ds \le \frac{L}{L_1} D(\lambda_j)^{1-\gamma}\,,\end{aligned}$$

which implies that

$$D(\lambda_j) \ge m' \lambda_j^{1/(1-\gamma)} \tag{2.24}$$

with

$$m' = \Big[\frac{L_1}{L}\,(2\delta)^\gamma \int_{t_1}^{t_2} p(s)\,ds\Big]^{1/(1-\gamma)} > 0\,.$$

By (2.22) and (2.24), we obtain

$$\varphi(t;\lambda_j) \ge m \lambda_j^{1/(1-\gamma)} \quad \text{on} \quad [t_1, t_2]\ \big(\subseteq [b,d]\big)\,, \tag{2.25}$$

where $m = 2\delta m'$.

By the above arguments we find that if the sequence $\{\varphi(t;\lambda_j)\}$ is considered on $[b,d]$, then either (2.21) or (2.25) holds. For the case where (2.25) holds, the conclusion of the lemma is proved. For the case where (2.21) holds, we consider the sequence $\{\varphi(t;\lambda_j)\}$ on the interval $[d,c]$. Put

$$\psi(t;\lambda_j) = \varphi(2d - t;\lambda_j)\,, \quad t \in [2d-c, d]\,. \tag{2.26}$$

We have

$$\psi^{(i)}(t;\lambda_j) = (-1)^i \varphi^{(i)}(2d-t;\lambda_j)\,, \quad i = 1, 2, \ldots, n\,, \tag{2.27}$$

and, in particular, $x = \psi(t;\lambda_j)$ is a solution of the equation

$$x^{(n)} + \lambda p(2d-t) f(x) = 0 \quad \text{with} \quad \lambda = \lambda_j \tag{2.28}$$

on $[2d-c, d]$, where the hypothesis (1.2) has been used. Observe that (2.28) is the same form as the original equation (1.1). Since $\psi(t;\lambda_j) \ge 0$, $\not\equiv 0$ on $[2d-c, d]$, exactly as in the previous arguments, we find that either

$$\frac{\psi^{(n-1)}(d;\lambda_j)}{\Big\{\sum_{i=0}^{n-1}\big[\psi^{(i)}(d;\lambda_j)\big]^2\Big\}^{1/2}} \to -1 \quad \text{as} \quad j \to \infty$$

or there exist a number $m > 0$ and an interval $[t_1, t_2]$ $\big(\subseteq [2d-c, d]\big)$ such that

$$\psi(t;\lambda_j) \ge m\lambda_j^{1/(1-\gamma)} \quad \text{on} \quad [t_1, t_2]\,.$$

By the relations (2.26) and (2.27) these are equivalent to

$$\frac{\varphi^{(n-1)}(d;\lambda_j)}{\Big\{\sum_{i=0}^{n-1}\big[\varphi^{(i)}(d;\lambda_j)\big]^2\Big\}^{1/2}} \to +1 \quad \text{as} \quad j \to \infty \tag{2.29}$$

and

$$\varphi(t;\lambda_j) \ge m\lambda_j^{1/(1-\gamma)} \quad \text{on} \quad [2d-t_2, 2d-t_1] \ \big(\subseteq [d, c]\big)\,, \tag{2.30}$$

respectively. Of course, (2.29) contradicts (2.21). Thus, for the case where (2.21) holds, we must have (2.30). This finishes the proof of Lemma 2.5. □

Lemma 2.6. *Let $\{\lambda_j\}$ be a sequence such that $0 < \lambda_j \to \infty$ as $j \to \infty$. Suppose that, for each j, $x(t;\lambda_j)$ has a zero $z(\lambda_j)$ in $[a,\infty)$, and that*

$$\lim_{j\to\infty} z(\lambda_j) = \infty\,. \tag{2.31}$$

Then, for all sufficiently large j, $x(t;\lambda_j)$ has another zero in $\big[a, z(\lambda_j)\big)$.

Proof. Assume, contrary to our claim, that there is a subsequence $\{\lambda_{j'}\} \subseteq \{\lambda_j\}$ such that, for any j', $x(t;\lambda_{j'})$ has no zeros in $\big[a, z(\lambda_{j'})\big)$. Since $z(\lambda_{j'}) \to \infty$ as $j' \to \infty$, we can take an interval $[b,c]$ such that $[b,c] \subseteq \big[a, z(\lambda_{j'})\big)$ for all j'. By Lemma 2.5, there exist a constant $m > 0$ and a subinterval $[b', c'] \subseteq [b, c]$ and a subsequence $\{\lambda_{j''}\} \subseteq \{\lambda_{j'}\}$ — we denote $\{\lambda_{j''}\}$ by $\{\lambda_j\}$ again — such that

$$\big|x(t;\lambda_j)\big| \ge m\lambda_j^{1/(1-\gamma)} \quad \text{on} \quad [b', c'] \tag{2.32}$$

for all j. It follows from (2.5) that

$$\begin{aligned} x(t;\lambda_j) = 1 &- \lambda_j \int_t^{z(\lambda_j)} \frac{(s-t)^{n-1}}{(n-1)!}\, p(s)\, f\big(x(s;\lambda_j)\big)\, ds \\ &- \lambda_j \int_{z(\lambda_j)}^{\infty} \frac{(s-t)^{n-1}}{(n-1)!}\, p(s)\, f\big(x(s;\lambda_j)\big)\, ds\,, \quad t \ge a\,. \end{aligned}$$

If $t \in [b', c']$, then

$$\operatorname{sgn} x(t;\lambda_j) = \operatorname{sgn} x(s;\lambda_j) \quad \text{for all} \quad s \in \big[t, z(\lambda_j)\big)\,.$$

Hence we have

$$\begin{aligned} \big|x(t;\lambda_j)\big| = \operatorname{sgn} x(t;\lambda_j) &- \lambda_j \int_t^{z(\lambda_j)} \frac{(s-t)^{n-1}}{(n-1)!}\, p(s)\, \big|f(x(s;\lambda_j))\big|\, ds \\ &- \lambda_j \operatorname{sgn} x(t;\lambda_j) \int_{z(\lambda_j)}^{\infty} \frac{(s-t)^{n-1}}{(n-1)!}\, p(s)\, f\big(x(s;\lambda_j)\big)\, ds \end{aligned}$$

for $t \in [b', c']$, and so

$$\big|x(t;\lambda_j)\big| \le 1 + \lambda_j \int_{z(\lambda_j)}^{\infty} \frac{(s-t)^{n-1}}{(n-1)!}\, p(s)\, \big|f(x(s;\lambda_j))\big|\, ds$$

for $t \in [b', c']$. Then, using (2.32), (2.7) and (1.10), we find that

$$m\lambda_j^{1/(1-\gamma)} \le 1 + L_2 \lambda_j B(\lambda_j) \int_{z(\lambda_j)}^{\infty} \frac{s^{n-1}}{(n-1)!}\, p(s)\, ds\,,$$

where

$$B(\lambda_j) = \Big[1 + (1-\gamma)\lambda_j L_2 \int_a^{\infty} \frac{s^{n-1}}{(n-1)!}\, p(s)\, ds\Big]^{\gamma/(1-\gamma)}.$$

This gives

$$0 < m \le \frac{1}{\lambda_j^{1/(1-\gamma)}} + L_2 \Big[\frac{1}{\lambda_j} + (1-\gamma) L_2 \int_a^{\infty} \frac{s^{n-1}}{(n-1)!}\, p(s)\, ds\Big]^{\gamma/(1-\gamma)} \times \int_{z(\lambda_j)}^{\infty} \frac{s^{n-1}}{(n-1)!}\, p(s)\, ds\,.$$

However, by the condition (2.31), the right-hand side of the above tends to 0 as $j \to \infty$. This is a contradiction. The proof of Lemma 2.6 is complete. □

Lemma 2.7. *Let $\{\lambda_j\}$ be a sequence with $0 < \lambda_j \to \infty$ as $j \to \infty$. Suppose that $z_2(\lambda_j)$ and $z_1(\lambda_j)$ are successive zeros of $x(t; \lambda_j)$ such that $a < z_2(\lambda_j) < z_1(\lambda_j)$ and*

$$\lim_{j\to\infty} z_1(\lambda_j) = \infty\,.$$

Then we have

$$\lim_{j\to\infty} z_2(\lambda_j) = \infty\,.$$

Proof. Assume to the contrary that $\liminf_{j\to\infty} z_2(\lambda_j) < \infty$. There are a subsequence $\{\lambda_{j'}\} \subseteq \{\lambda_j\}$ and an interval $[b, c]$ such that $[b, c] \subseteq \big(z_2(\lambda_{j'}), z_1(\lambda_{j'})\big)$ for all j'. Then, analogously to the proof of Lemma 2.6, we are lead to a contradiction. The proof of Lemma 2.7 is complete. □

Lemma 2.8. *Let $\{\lambda_j\}$ be a sequence with $0 < \lambda_j \to \infty$ as $j \to \infty$, and let k be an arbitrary positive integer. Then, for all sufficiently large j, $x(t; \lambda_j)$ has at least k zeros in the interval (a, ∞).*

Proof. Assume that there is a subsequence $\{\lambda_{j'}\} \subseteq \{\lambda_j\}$ such that, for all j', $x(t; \lambda_{j'})$ has no zeros in (a, ∞). Then,

$$0 < x(t; \lambda_{j'}) < 1 \quad \text{on} \quad (a, \infty) \quad \text{for all} \quad j'\,. \tag{2.33}$$

From Lemma 2.5 there exist a constant $m > 0$ and an interval $[b', c'] \subseteq (a, \infty)$ and a subsequence $\{\lambda_{j''}\}$ of $\{\lambda_{j'}\}$ such that

$$x(t; \lambda_{j''}) \ge m\lambda_{j''}^{1/(1-\gamma)} \quad \text{on} \quad [b', c'] \quad \text{for all} \quad j''\,,$$

and consequently $\lim_{j''\to\infty} x(t; \lambda_{j''}) = \infty$ for each $t \in [b', c']$. This is a contradiction to (2.33). Therefore, for all sufficiently large j, $x(t; \lambda_j)$ has at least one zero in (a, ∞).

For all sufficiently large j, let $z_1(\lambda_j)$ be the largest zero of $x(t;\lambda_j)$. Assume that $\liminf_{j\to\infty} z_1(\lambda_j) < \infty$. There are a subsequence $\{\lambda_{j'}\} \subseteq \{\lambda_j\}$ and a number b such that $b > z_1(\lambda^1_{j'})$ for all j'. We have

$$0 < x(t;\lambda_{j'}) < 1 \quad \text{on} \quad (b,\infty) \quad \text{for all} \quad j'.$$

Then, exactly as in the above, we obtain a contradiction. Thus we must have $\lim_{j\to\infty} z_1(\lambda_j) = \infty$. It is concluded by Lemma 2.6 that, for all sufficiently large j, $x(t;\lambda_j)$ has another zero $z_2(\lambda_j)\ \big(<z_1(\lambda_j)\big)$, and, by Lemma 2.7, we have

$$\lim_{j\to\infty} z_2(\lambda_j) = \infty.$$

We repeat this procedure by using Lemmas 2.6 and 2.7. Then we can get the desired conclusion. The proof of Lemma 2.8 is complete. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. For $k = 1, 2, \dots$, we put

$$\Lambda_k^+ = \{\lambda \in (0,\infty) : x(t;\lambda) \ \text{ has at least } k \text{ zeros in the open interval } (a,\infty)\}.$$

By Lemma 2.8, Λ_k^+ is nonempty. Lemma 2.2 implies that there is $\lambda_* > 0$ such that $\lambda \ge \lambda_*$ for all $\lambda \in \Lambda_k^+$. Let

$$\lambda(k) = \inf \Lambda_k^+, \quad k = 1, 2, \dots$$

Then we have $\lambda(k) \ge \lambda_* > 0$ ($k = 1, 2, \dots$). Since $\Lambda_k^+ \supseteq \Lambda_{k+1}^+$, we also have $\lambda(k) \le \lambda(k+1)$ ($k = 1, 2, \dots$).

For each $k = 1, 2, \dots$, we can take a sequence $\{\lambda(k)_j\}_{j=1}^\infty$ such that $\lambda(k)_j \in \Lambda_k^+$ ($j = 1, 2, \dots$) and $\lim \lambda(k)_j = \lambda(k)$ as $j \to \infty$. The solution $x(t;\lambda(k)_j)$ has at least k zeros in (a,∞). Let

$$(a <)\ z_k(\lambda(k)_j) < z_{k-1}(\lambda(k)_j) < \cdots < z_2(\lambda(k)_j) < z_1(\lambda(k)_j)\ (<\infty)$$

be k zeros of $x(t;\lambda(k)_j)$. It follows from Lemma 2.1 that all the zeros $z_1(\lambda(k)_j)$, $z_2(\lambda(k)_j), \dots, z_k(\lambda(k)_j)$ are in a certain compact interval of the form $[a, T_k]$. Here, T_k does not depend on j while it depends on k. There is a subsequence j' of j such that $\big\{z_1(\lambda(k)_{j'})\big\}, \dots, \big\{z_k(\lambda(k)_{j'})\big\}$ have finite limits $z_1(\lambda(k)), \dots, z_k(\lambda(k))$ as $j' \to \infty$, respectively. Then,

$$a \le z_k\big(\lambda(k)\big) \le z_{k-1}\big(\lambda(k)\big) \le \cdots \le z_2\big(\lambda(k)\big) \le z_1\big(\lambda(k)\big) \le T_k,$$

and the continuity of $x(t;\lambda)$ implies that $z_1(\lambda(k)), \dots, z_k(\lambda(k))$ are zeros of $x(t;\lambda(k))$. Assume that there is $m \in \{1, 2, \dots, k-1\}$ such that $z_m(\lambda(k)) = z_{m+1}(\lambda(k))$. Since we have $x(z_m(\lambda(k)_{j'});\lambda(k)_{j'}) = x(z_{m+1}(\lambda(k)_{j'});\lambda(k)_{j'}) = 0$ and $x'(\xi;\lambda(k)_{j'}) = 0$ for some $\xi \in \big(z_{m+1}(\lambda(k)_{j'}), z_m(\lambda(k)_{j'})\big)$, the continuity of $x(t;\lambda)$ and $x'(t;\lambda)$ implies $x(z_m(\lambda(k));\lambda(k)) = 0$ and $x'(z_m(\lambda(k));\lambda(k)) = 0$. This means that $z_m(\lambda(k))$ is a multiple zero of $x(t;\lambda(k))$, giving a contradiction to Lemma 2.3. Thus we get

$$a \le z_k\big(\lambda(k)\big) < z_{k-1}\big(\lambda(k)\big) < \cdots < z_2\big(\lambda(k)\big) < z_1\big(\lambda(k)\big) < \infty.$$

Assume that $a < z_k\big(\lambda(k)\big)$. Then $x\big(t;\lambda(k)\big)$ has at least k zeros $z_1\big(\lambda(k)\big), \dots, z_k\big(\lambda(k)\big)$ in the open interval (a,∞). The continuity of $x(t;\lambda)$ implies that, for all λ which are sufficiently close to $\lambda(k)$, $x(t;\lambda)$ has at least k zeros in the interval

(a, ∞). This is a contradiction to the property of the infimum $\lambda(k)$ of Λ_k^+. Thus we must have $a = z_k\big(\lambda(k)\big)$, and so

$$a = z_k\big(\lambda(k)\big) < z_{k-1}\big(\lambda(k)\big) < \cdots < z_2\big(\lambda(k)\big) < z_1\big(\lambda(k)\big) < \infty \,.$$

Note that $x\big(t; \lambda(k)\big)$ has at least $k-1$ zeros $z_1\big(\lambda(k)\big), \ldots, z_{k-1}\big(\lambda(k)\big)$ in the open interval (a, ∞). If $x\big(t; \lambda(k)\big)$ has k or more zeros in (a, ∞), then an argument similar to the above yields a contradiction to the property of the infimum $\lambda(k)$ of Λ_k^+. Thus we conclude that $x\big(t; \lambda(k)\big)$ has exactly $k-1$ zeros in the open interval (a, ∞). From this fact it follows that the equality $\lambda(k) = \lambda(k+1)$ does not hold, and consequently,

$$0 < \lambda(1) < \lambda(2) < \cdots < \lambda(k) < \cdots \,.$$

We claim that $\lim \lambda(k) = \infty$ as $k \to \infty$. Assume to the contrary that $\{\lambda(k)\}$ has a finite limit λ_∞ as $k \to \infty$. Then there is $T > 0$ such that, for all λ which are sufficiently close to λ_∞, $x(t; \lambda)$ has no zeros in the interval $[T, \infty)$ (see Lemma 2.1). Let N be an arbitrary positive integer. It is clear that, for all large k, $x(t; \lambda(k))$ has at least N zeros on the compact interval $[a, T]$. Then, in the limiting procedure as the above, we find that $x(t; \lambda_\infty)$ has at least N zeros in the interval $[a, T]$. Since N is arbitrary, this means that $x(t; \lambda_\infty)$ has an infinite number of zeros in the compact interval $[a, T]$. This is a contradiction. Thus we have $\lim \lambda(k) = \infty$ as $k \to \infty$.

By the above discussions it is easily found that the sequence $\big\{\lambda(k)\big\}$ satisfies the properties (i)–(iii) in Theorem 1.2. The proof is complete. $\square$

References

[1] Elias, U., *Eigenvalue problems for the equation $Ly + \lambda p(x)y = 0$*, J. Differential Equations **29** (1978), 28–57.

[2] Elias, U., *Oscillation Theory of Two-Term Differential Equations*, Kluwer, 1997.

[3] Elias, U., *Singular eigenvalue problems for the equation $y^{(n)} + \lambda p(x)y = 0$*, Monatsh. Math. **142** (2004), 205–225.

[4] Elias, U. and Pinkus, A., *Nonlinear eigenvalue problems for a class of ordinary differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **132** (2002), 1333–1359.

[5] Hartman, P., *Ordinary Differential Equations*, Wiley, 1964.

[6] Kiguradze, I. T. and Chanturia, T. A., *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Kluwer, 1993.

[7] Naito, M., *On the number of zeros of nonoscillatory solutions to higher-order linear ordinary differential equations*, Monatsh. Math. **136** (2002), 237–242.

[8] Naito, M. and Naito, Y., *Solutions with prescribed numbers of zeros for nonlinear second order differential equations*, Funkcial. Ekvac. **37** (1994), 505–520.

[9] Naito, Y. and Tanaka, S., *On the existence of multiple solutions of the boundary value problem for nonlinear second-order differential equations*, Nonlinear Anal. **56** (2004), 919–935.

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