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A CHARACTERIZATION PROPERTY OF THE SIMPLE GROUP $PSL_4(5)$ BY THE SET OF ITS ELEMENT ORDERS

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Abstract. Let $\omega(G)$ denote the set of element orders of a finite group G. If H is a finite non-abelian simple group and $\omega(H) = \omega(G)$ implies G contains a unique non-abelian composition factor isomorphic to H, then G is called quasirecognizable by the set of its element orders. In this paper we will prove that the group $PSL_4(5)$ is quasirecognizable.

1. Introduction

Given a finite group G, we denote by $\omega(G)$ the set of orders of elements of G. This set is closed and partially ordered by divisibility relation, and hence is uniquely determined by the set $\mu(G)$ of elements in $\omega(G)$ which are maximal under the divisibility relation. Let $h(G)$ denote the number of non-isomorphic finite groups G having $\omega(G)$ as the set of their element orders. A group G is said to be characterizable or recognizable by $\omega(G)$ if $h(G) = 1$, the group G is called $k-$recognizable if $h(G) = k$ and is called irrecognizable if $h(G) = \infty$. A finite simple non-abelian group P is said to be quasirecognizable if any finite group G with $\omega(G) = \omega(P)$ has a composition factor isomorphic to P.

The set $\omega(G)$ of a finite group G defines a graph whose vertices are prime divisors of the order of G and two primes p and q are adjacent if G contains an element of order pq. This graph is defined by Gruenberg and Kegel and hence it is denoted by $GK(G)$ and is called the Gruenberg-Kegel graph of G. We also call $GK(G)$ the prime graph of G. The connected components of the graph $GK(G)$ are denoted by $\pi_i, 1 \le i \le t(G)$, where $t(G)$ is the number of connected components of the graph. We define π_1 the component containing the prime 2 for a group of even order.

In [2] and [15-18], it has been proved that the groups $L_2(q)$, $q > 3$, $q \neq 9$ are characterizable. The groups $L_3(q)$, $q = 7$, 2^m are recognizable by [12]. Concerning the groups $G = PSL_3(q)$, q odd, it is shown in [4] that $h(G) = 1$ for $q = 11, 13, 19,$

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23, 25 and 27; $h(G) = 2$ for $q = 17$ and 29. The group $PSL_4(3)$ is characterizable by [11].

The goal of this article is to study the recognizability property of the simple group $PSL_4(5)$ by its set of element orders. In particular we prove that the simple group $PSL_4(5)$ is quasirecognizable. This will imply that a conjecture of W. Shi and J. Bi holds for $PSL_4(5)$. That is to say if $\omega(G) = \omega\big(PSL_4(5)\big)$ and $|G| = |PSL_4(5)|$, then $G \cong PSL_4(5)$.

2. Preliminary results

First we quote some results which are used to deduce the main result of this paper.

Lemma 1 ([8])**.** *If G is a finite solvable group all of whose elements are of prime power order, then $|\pi(G)| \leq 2$.*

In the following we list some properties of the Frobenius groups whose proofs can be found in [14].

Lemma 2. *Let G be a Frobenius group with kernel F and complement C. Then the following assertions hold.*

(a) *F is a nilpotent group; in particular, the prime graph of F is complete.*

(b) $|F| \equiv 1 \pmod{|C|}$.

(c) *Every subgroup of C of order pq, with p and q (not necessarily distinct) primes, is cyclic. In particular, every Sylow subgroup of C of odd order is cyclic and a Sylow $2-$subgroup of C is either cyclic or a generalized quaternion group. If C is non-solvable then C has a subgroup of index at most 2 isomorphic to $SL_2(5) \times M$, where M has cyclic Sylow p-subgroups and order coprime to 2,3 and 5.*

Definition 1. A 2-Frobenius group is a group G having a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and $\frac{G}{H}$ are Frobenius groups with kernels H and $\frac{K}{H}$ respectively.

Lemma 3. *Let G be a $2-$Frobenius group, then G is a solvable group.*

Proof. By definition, there exists a normal series, $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K and $\frac{G}{H}$ are Frobenius groups with kernels H and $\frac{K}{H}$ respectively. Then $\frac{K}{H}$ is isomorphic to kernel of a Frobenius group and complement of another Frobenius group, therefore $\frac{K}{H}$ is nilpotent, hence K is solvable. Now $\frac{G}{K}$ is isomorphic to a subgroup of the automorphism group of a cyclic group, hence $\frac{G}{K}$ is abelian. Since both K and $\frac{G}{K}$ are solvable, then G is a solvable group. $\square$

For the groups with disconnected prime graph the following result is a useful tool.

Lemma 4 ([20])**.** *If G is a group such that $t(G) \geq 2$, then G has one of the following structures.*

(a) *A Frobenius or a $2-$Frobenius group.*

(b) *G has a normal series $1 \trianglelefteq N \triangleleft G_1 \trianglelefteq G$, such that $\pi(N) \cup \pi(\frac{G}{G_1}) \subseteq \pi_1$ and $\overline{G}_1 = \frac{G_1}{N}$ is a non-abelian simple group.*

Lemma 5 ([13])**.** *Let G be a finite group, $N \triangleleft G$ and $\frac{G}{N}$ be a Frobenius group with kernel F and cyclic complement C. If $(|F|, |N|) = 1$ and F is not contained in $\frac{NC_G(N)}{N}$, then $p|C| \in \omega(G)$ for some prime divisor p of $|N|$.*

Definition 2. Let $A \in GL_n(q)$. Then $\delta_A : SL_n(q) \to SL_n(q)$ defined by $B \mapsto A^{-1}BA, B \in SL_n(q)$, is an automorphism of $SL_n(q)$ and it is called a diagonal automorphism of $SL_n(q)$. It is possible to choose A so that δ_A induces an outer automorphism of order $(n, q-1)$ of the group $PSL_n(q)$ if $n \neq 2$.

Definition 3. Let $\theta : GL_n(q) \to GL_n(q)$ be the mapping sending A to $(A^t)^{-1}$ where A^t denotes the transpose of A. Then θ is an involuntary outer automorphism of $G = GL_n(q)$ if $(n, q) \neq (2, 2)$. This automorphism is called a graph automorphism of G. It also induces an outer automorphism of the group $PSL_n(q)$ if $(n, q) \neq (2, 2)$.

Definition 4. Let $q = p^f$ be a power of the prime p. Then $\sigma_p : GF(q) \to GF(q)$ defined by $\sigma_p(a) = a^p$ is an automorphism of the Galois field $GF(q)$, called the Frobenius automorphism. If for $A = (a_{ij})_{1 \leq i,j \leq n} \in GL_n(q)$ we define $\sigma_p(A) = (a_{ij}^p)_{1 \leq i,j \leq n}$, then σ_p induces an automorphism of the group $GL_n(q)$ which is called a field automorphism of $GL_n(q)$ and it is denoted by σ_p again. σ_p induces an automorphism of the group $PSL_n(q)$ in the natural way.

Now in the following we give the structure of the group of outer automorphisms of the group $PSL_n(q)$.

Lemma 6 ([9])**.** *Let $n \geq 2$, and $q = p^f$. Then*
(a) Out $(PSL_n(q) \cong Z_{(n,q-1)} : Z_f : Z_2$; *if $n \geq 3$.*
(b) Out $\big(PSL_2(q)\big) \cong Z_{(2,q-1)} \times Z_f$.

Suppose δ, σ_p and θ are diagonal, field and graph outomorphisms of $PSL_n(q)$, $q = p^f$, respectively. Then we have $O(\delta) = (n, q-1)$, $O(\sigma_p) = f$, $O(\theta) = 2$, and furthermore $[\sigma_p, \theta] = 1$, $\delta^{\sigma_p} = \delta^p$ and $\delta^\theta = \delta^{-1}$.

According to [10] and [20] the prime graph of the group $PSL_p(5)$, where p is a prime number, has two components. The first component is $\pi_1 = \pi\big(5 \prod_{i=1}^{p-1}(q^i - 1)\big)$ and the second component is $\pi_2 = \pi(\frac{5^p - 1}{4})$.

Now for the group $PSL_4(5)$ we have $|PSL_4(5)| = 2^7 \cdot 3^2 \cdot 5^6 \cdot 13 \cdot 31$. Therefore the components of the prime graph of this group are as follows: $\pi_1 = \{2, 3, 5, 13\}$ and $\pi_2 = \{31\}$.

By [6] we have $\mu\big(PSL_4(5)\big) = \{20,\ 24,\ 30,\ 31,\ 39\}$. Therefore $\omega\big(PSL_4(5)\big) = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12,\ 13,\ 15,\ 20,\ 24,\ 30,\ 31,\ 39\}$ and the prime graph of the group $PSL_4(5)$ is as in Figure 1.

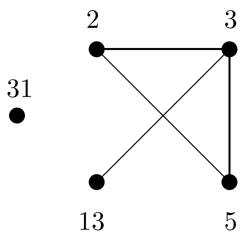

Figure 1. The prime graph of the group $PSL_4(5)$

Lemma 7. *Let G be a simple group of Lie type. If* $\{31\} \subseteq \pi(G) \subseteq \{2, 3, 5, 13, 31\}$, *then G is isomorphic to* $A_1(31) \cong PSL_2(31)$, $A_2(5) \cong PSL_3(5)$ *or* $A_3(5) \cong PSL_4(5)$.

Proof. Suppose $G = L(q)$ is a simple group of Lie type over the finite field of order $q = p^s$, where p is a prime number and s is a natural number. The orders of these groups are given in [3] and are multiples of numbers of the form $p^k \pm 1$, where $k \in \mathbb{N}$. Since p divides $|G|$, therefore p must be one of the numbers $2,\ 3,\ 5,\ 13$ or 31.

If $p = 2$, then it is clear that the order of 2 modulo 31 is 5. But $7 \mid 2^3 - 1$ and $7 \nmid |G|$. Hence by [3] no candidates for G will arise.

If $p = 3$, then the least integer k for which $3^k + 1 \equiv 0(mod\ 31)$ is 15. But $7 \mid 3^3 + 1$ and $7 \nmid |G|$.We don't obtain a possibility for G on this case.

If $p = 5$, then the order of 5 modulo 31 is 3. Since $11 \mid 5^5 - 1$ and $7 \mid 5^6 - 1$, hence by [3] the only candidates are the groups $A_2(5)$ and $A_3(5)$.

If $p = 13$, then the least integer k for which $13^k + 1 \equiv 0(mod\ 31)$ is 15. But $7 \mid 13 + 1$ and $7 \nmid |G|$. Then by [3] no candidate for G will arise.

If $p = 31$, then since $37 \mid 31^2 + 1$ and $331 \mid 31^3 - 1$,we don't get a possibility for a finite simple group G except $A_1(31)$. $\square$

3. Proof of the main theorem

In this section we prove that the simple group $PSL_4(5)$ is quasirecognizable by the set of its element orders.

Theorem 1. *Let G be a finite group . If* $\omega(G) = \omega\big(PSL_4(5)\big)$, *then G has a normal* $5-$*subgroup N such that* $\frac{G}{N} \cong PSL_4(5)$. *In particular G is quasirecognizable by its set of element orders.*

Proof. We have $\mu\big(PSL_4(5)\big) = \{20, 21, 30, 31, 39\}$. Let G be a finite group such that $\mu(G) = \mu\big(PSL_4(5)\big)$. Then components of prime graph of G are $\pi_1 = \{2, 3, 5, 13\}$ and $\pi_2 = \{31\}$. Since G has a disconnected Gruenberg-Kegel graph, we can use Lemma 4 for the structure of G. But by [1] only Case (b) of the Lemma 4 may hold (we also could use Lemmas 1,2 and 3 to prove that a group with the given set of element orders is not Frobenius or 2-Frobenius group).

Therefore there exists a normal series $1 \trianglelefteq N \lhd G_1 \trianglelefteq G$, such that $\frac{G}{G_1}$ and N are π_1-groups, $\overline{G}_1 := \frac{G_1}{N}$ is a non-abelian simple $\pi_1(G)$-group and, $t(\overline{G}_1) \geq 2$. We may assume that $\frac{G}{N} \leq \mathrm{Aut}\,(\overline{G}_1)$. Note that one of the components of the prime graph of $\overline{G}_1$ must be $\{31\}$, hence $31 \mid |\overline{G}_1|$.

Now according to the classification of finite non-abelian simple groups we know that the possibilities for $\overline{G}_1$ are the alternating groups $\mathbb{A}_n$, $n \geq 5$, one of the 26 sporadic simple groups and finite simple groups of Lie type. We deal with the above cases separately.

Case (1). Suppose $\overline{G}_1$ is an alternating group $\mathbb{A}_n$, $n \geq 5$. Since $31 \in \omega(\overline{G}_1)$, then $n \geq 31$, which implies that for example $7 \in \omega(G)$, a contradiction.

Case (2). By [3] it is easy to see that $\overline{G}_1$ can not be isomorphic to a sporadic simple group.

Case (3). Finally suppose that $\overline{G}_1$ is a simple group of Lie type . From Lemma 7, $\overline{G}_1$ may be isomorphic to one of the following groups $A_1(31)$, $A_2(5)$ or $A_3(5)$.

Since $16 \in \omega\big(A_1(31)\big)$ but $16 \notin \omega(G)$, then $\overline{G}_1$ is not isomorphic to $A_1(31)$. Suppose $\overline{G}_1 \cong A_2(5)$ and $\overline{G}_1 = \frac{G_1}{N}$. If $N \neq 1$, we may assume that N is an elementary abelian $p-$group, where $p \in \{2,3,5,13\}$. Since $\pi\big(A_2(5)\big) = \{2,3,5,31\}$ and $\frac{G}{N} \leq \mathrm{Aut}\,(\overline{G}_1) = A_2(5) : 2$, hence $13 \mid |N|$. Therefore N is an elementary abelian $13-$group . Now $\frac{G_1}{N} = \overline{G}_1 \cong A_2(5)$ and $A_2(5) \cong PSL_3(5)$ contains a Frobenius subgroup of the shape $5^2 : 24$. Now it is easy to verify that all conditions of Lemma 5 are fulfilled, hence G_1 must contain an element of order 13×24, which is a contradiction.

Finally assume $\overline{G}_1 \cong A_3(5)$. Our aim is to show that G has a normal $5-$subgroup N such that $\frac{G}{N} \cong A_3(5) \cong PSL_4(5)$. Suppose $N \neq 1$. By the prime graph of G, Figure 1, an element of order 31 of G acts fixed-point-freely on N, hence by ([7], page 337) N is a nilpotent $\pi_1(G)$-group. Therefore N is the product of p-groups for $p \in \pi_1 = \{2,3,5,13\}$. Then we may assume that N is a $p-$group for some prime $p \in \pi_1 = \{2,3,5,13\}$. $\overline{G}_1$ contains a Frobenius group of the shape $5^3 : 31$. First assume $p \neq 5$. We let $\frac{H}{N} = 5^3 : 31 = F : C$ be the Frobenius subgroup of $\overline{G}_1$. Since $\frac{NC_H(N)}{N} \cong \frac{C_H(N)}{N \cap C_H(N)}$ and $C_H(N) \leq C_G(N) = N$, we deduce that F is not contained in $\frac{NC_H(N)}{N}$. Therefore by Lemma 5 we obtain an element of order $31 \times p$ in G, a contradiction. Therefore $p = 5$ and G has a normal $5-$subgroup N (possibly $N = 1$) such that $\overline{G}_1 = \frac{G_1}{N} \cong PSL_4(5)$. But then $\frac{G_1}{N} \trianglelefteq \frac{G}{N}$ and hence $\overline{G}_1 \leq \frac{G}{N} \leq \mathrm{Aut}\,(\overline{G}_1)$. By Lemma 6 we have $\mathrm{Out}\,(\overline{G}_1) \cong D_8$, the dihedral group of order 8, which can be given by $\mathrm{Out}\,(\overline{G}_1) = \langle \theta, \delta : \delta^4 = \theta^2 = 1, \theta^{-1}\delta\theta = \delta^{-1} \rangle$. We assume $\delta = \mathrm{diag}\,(2,1,1,1)$. Let T be a subgroup of $\mathrm{Out}\,(\overline{G}_1)$, then T may be one of the following groups:
$T_1 = \{1,\theta\}$, $T_2 = \{1,\delta\theta\}$, $T_3 = \{1,\delta^2\theta\}$, $T_4 = \{1,\delta^3\theta\}$, $T_5 = \{1,\delta^2\}$, $T_6 = \{1,\delta,\delta^2,\delta^3\}$, $T_7 = \{1,\delta^2,\theta,\delta^2\theta\}$, $T_8 = \{1,\delta^2,\delta\theta,\delta^3\theta\}$, $T_9 = \mathrm{Out}\,(\overline{G}_1)$, $T_{10} = \{1\}$. Therefore $\frac{G}{N} \cong \overline{G}_1 : T_i$, for some i, $i = 1,\ldots,10$.

Let $\overline{G}_1^+ = \overline{G}_1 : \langle\theta\rangle$. Then by [5], $13 \mid \left|C_{\overline{G}_1^+}(\theta)\right|$, therefore $26 \in \omega(\frac{G}{N})$, a contradiction. Therefore $\frac{G}{N} \cong \overline{G}_1 : T_i,\ i = 1,\ 7,\ 9$ are impossible.

If $\frac{G}{N} \cong \overline{G}_1 : T_6 = \overline{G}_1 : \langle\delta\rangle$, then by [9], $\frac{G}{N} \cong PGL_4(5)$, therefore by [6] $26 \in \omega(\frac{G}{N})$, a contradiction.

If $\frac{G}{N} = \overline{G}_1 : T_i,\ i = 5, 8$, then we have $C_{SL_4(5)}(\delta^2) = \{A \in SL_4(5) \mid A\delta^2 = \delta^2 A\} = \left\{ \left[\begin{array}{c|c} (\det X)^{-1} & 0 \\ \hline 0 & X \end{array} \right] \mid X \in GL_3(5) \right\} \cong GL_3(5)$, then $C_{PSL_4(5)}(\delta^2) = PGL_3(5)$, therefore by [6], $62 \in \omega(\frac{G}{N})$, a contradiction.

If $\frac{G}{N} \cong \overline{G}_1 : T_2$, we have $C_{SL_4(5)}(\delta\theta) \cong SO_4^-(5)$. By [3], $52 \in \omega(\frac{G}{N})$, contradicting $\omega(\frac{G}{N})$.

If $\frac{G}{N} \cong \overline{G}_1 : T_3$, we have $C_{SL_4(5)}(\theta\delta^2) \cong SO_4^+(5) \cong SL_2(5) \times SL_2(5)$, therefore $60 \in \omega(\frac{G}{N})$, that is a contradiction.

If $\frac{G}{N} \cong \overline{G}_1 : T_4$, we have $C_{SL_4(5)}(\theta\delta^3) = SO_4^-(5)$, then by [3], $52 \in \omega(\frac{G}{N})$, which is a contradiction. Therefore we only have $\frac{G}{N} \cong \overline{G}_1 \cong PSL_4(5)$, and the theorem is proved. □

Corollary 1. *Let G be a finite group with $\omega(G) = \omega\big(PSL_4(5)\big)$ and $|G| = |PSL_4(5)|$. Then $G \cong PSL_4(5)$.*

Proof. By the main theorem G has a normal subgroup N such that $\frac{G}{N} = PSL_4(5)$. Now $|G| = |PSL_4(5)|$ implies $N = 1$ and $G \cong PSL_4(5)$. □

There is a conjecture due to W. Shi and H. Bi [19], which states:

Conjecture 1. *Let G be a group and M a finite simple group. Then $G \cong M$ if and only if:*
(a) $|G| = |M|$ *and*
(b) $\omega(G) = \omega(M)$.

Therefore according to Corollary 1, the conjecture of Shi and Bi holds for the simple group $PSL_4(5)$.

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