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CLASSIFICATION OF RINGS SATISFYING SOME CONSTRAINTS ON SUBSETS

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Abstract. Let R be an associative ring with identity 1 and $J(R)$ the Jacobson radical of R. Suppose that $m \geq 1$ is a fixed positive integer and R an m-torsion-free ring with 1. In the present paper, it is shown that R is commutative if R satisfies both the conditions (i) $[x^m, y^m] = 0$ for all $x, y \in R \backslash J(R)$ and (ii) $[x, [x, y^m]] = 0$, for all $x, y \in R \backslash J(R)$. This result is also valid if (ii) is replaced by (ii)' $[(yx)^m x^m - x^m (xy)^m, x] = 0$, for all $x, y \in R \backslash N(R)$. Our results generalize many well-known commutativity theorems (cf. [1], [2], [3], [4], [5], [6], [9], [10], [11] and [14]).

1. Introduction

Throughout, R represents an associative ring with identity 1, $Z(R)$ the centre of $R, U(R)$ denotes the group of units of R, $J(R)$ the Jacobson radical of $R, N(R)$ the set of nilpotent elements of R, and $C(R)$ the commutator ideal of R. As usual, for any $x, y \in R$, the symbol $[x, y]$ will stand for the commutator $xy - yx$. Let $m \geq 1$ be a fixed positive integer and a non-empty subset S of R. We consider the following ring properties.

$C_1(m, S)$ $[x^m, y^m] = 0$ for all $x, y \in S$.

$C_2(m, S)$ $[x, [x, y^m]] = 0$ for all $x, y \in S$.

$C_3(m, S)$ $(xy)^m = x^m y^m$ for all $x, y \in S$.

$C_4(m, S)$ $(xy)^m - x^m y^m \in Z(R)$ for all $x, y \in S$.

$C_5(m, S)$ $(xy)^m - y^m x^m \in Z(R)$ for all $x, y \in S$.

$C_6(m, S)$ $[(xy)^m \pm y^m x^m, x] = 0 = [(yx)^m \pm x^m y^m, x]$ for all $x, y \in S$.

$C_7(m, S)$ $[(yx)^m x^m - x^m (xy)^m, x] = 0$ for all $x, y \in S$.

$Q(m)$ For any $x, y \in R$, $m[x, y] = 0$ implies $[x, y] = 0$.

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A celebrated theorem of Herstein [9] states that a ring R which possesses the property $C_3(m, R)$ must have a nil commutator ideal. Many authors have extended Herstein's result in several ways (see [1], for references). One of the interesting generalisations of this result is due to Abu-Khuzam et. al. [4]. They established commutativity of m-torsion-free ring R satisfying $C_1(m, R)$ and $C_3(m+1, R)$. Motivated by these observations, a natural question in this context is: What can we say about the commutativity of R if the property $C_3(m+1, R)$ in the above result is replaced by $C_2(m+1, R)$?

The aim of the present paper, in Section 2, is to establish that an m-torsion-free ring R satisfy $C_1\big(m, R\backslash J(R)\big)$ and $C_2(m, R\backslash J(R))$ must be commutative. Also commutativity of rings satisfies $C_7(m, R\backslash J(R))$ has been investigated. Finally, in Section 3, commutativity of a periodic ring satisfies $C_7\big(m, R\backslash N(R)\big)$ has been studied.

2. Commutativity of rings with 1

Theorem 2.1. *Let R satisfy $C_1\big(m, R\backslash J(R)\big)$, $C_2\big(m, R\backslash J(R)\big)$ and $Q(m)$. Then R is commutative.*

We begin with

Lemma 2.2 ([12, p.221])**.** *If $[x, y]$ commutes with x, then $[x^n, y] = nx^{n-1}[x, y]$ for all positive integers $n \geq 1$.*

Lemma 2.3 ([13, Theorem 1])**.** *Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, x_3, \ldots, x_n$ with integer coefficients. Then the following statements are equivalent:*

(i) *For any ring R satisfying the polynomial identity $f = 0$, $C(R)$ is nil.*

(ii) *For every prime p, $(G(F(p))_2$ fails to satisfy $f = 0$.*

Lemma 2.4 ([8, Theorem])**.** *Let R be a ring in which for given $x, y \in R$ there exist integers $m = m(x, y) \geq 1$, $n = n(x, y) \geq 1$ such that $[x^m, y^n] = 0$. Then the commutator ideal of R is nil.*

Lemma 2.5 ([6, Lemma 4])**.** *Let R be an m-torsion-free ring with unity 1 satisfying $C_1(m, R)$. Then*

(i) *$a \in N(R)$, $x \in R$ imply $[a, x^m] = 0$.*

(ii) *$a \in N(R)$, $b \in N(R)$ imply $[a, b] = 0$.*

Lemma 2.6 ([15, Lemma])**.** *Let R be a ring with unity 1. If $kx^m[x, y] = 0$ and $k(x+1)^m[x, y] = 0$ for some integers $m \geq 1$ and $k \geq 1$, then $k[x, y] = 0$ for all $x, y \in R$.*

Lemma 2.7 ([11, Theorem 1])**.** *Let R be a ring without non-zero nil right ideal. Suppose that, given $x, y \in R$, there exist positive integers $s = s(x, y) \geq 1$, $m = m(x, y) \geq 1$ and $t = t(x, y) \geq 1$ such that $\big[x^s, [x^t, y^m]\big] = 0$. Then R is commutative.*

Lemma 2.8 ([14, Step 2.2])**.** *Let R be a ring. Suppose that $N(R)$ is commutative and assume that $a^2 = 0$ and $r \in R$ imply that $ra \in N(R)$. Then $N(R)$ is an ideal.*

Now we establish the following results.

Lemma 2.9. *Let R satisfy $C_1(m,R)$, $C_7(m,R)$ and $Q(m)$. Then R is commutative.*

Proof. First, we claim that $[a, x^m] = 0$ for all $x \in R$ and $a \in N(R)$. Since a is nilpotent, there exists a minimal positive integer t such that $[a^k, x^m] = 0$ for all integers $k \geq t$. Let $m > 2$. Then

$$0 = \big[(1 + a^{t-1})^m, x^m\big] = \big[1 + ma^{t-1} + \ldots + a^{(t-1)m}, x^m\big] = m\big[a^{t-1}, x^m\big]\,.$$

By an application of $Q(m)$, this gives $[a^{t-1}, x^m] = 0$, which contradicts minimality of m. Hence $t = 1$, and $[a, x^m] = 0$. In view of [10, Lemma 10], there exists a positive integer s, such that $s[x^m, y] = 0$. Since $C(R) \subseteq N(R)$ by a special case of [8, Theorem], it follows from what is just shown above that $[x^m, [x^m, y]] = 0$. Thus by Lemma 2.2, we have

$$[x^{ms}, y] = sx^{m(s-1)}[x^m, y] = 0\,.$$

Further, let c, d be arbitrary elements of R. Then replacing x by c and y by $c^{ms-1}d$ in $C_7(m,R)$, and combining the above result, we get

$$\big[(c^{ms-1}dc)^m c^m - c^m(c^{ms}d)^m, c\big] = 0$$

or

$$\big[(c^{ms-1+ms(m-1)}d^m c)c^m - c^m(c^{m^2s}d^m), c\big] = 0$$

that is

$$\big[(c^{m^2s-1}d^m c)c^m - c^m(c^{m^2s}d^m), c\big] = 0\,.$$

After simplification, this gives

$$c^{ms-1}\big[c, [c^{m+1}, d^m]\big] = 0\,.$$

Using the commutator identity; $[xy, z] = x[y, z] + [x, z]y$, for all $x, y, z \in R$ and $C_1(m,R)$, we have

$$c^{m^2s-1}\big[c, c^m[c, d^m]\big] = 0$$

or

$$c^{m^2s-1+m}\big[c, [c, d^m]\big] = 0\,.$$

Therefore, by Lemma 2.6, $\big[c, [c, d^m]\big] = 0$, and in view of Lemma 2.2 we obtain $0 = [c^m, d^m] = mc^{m-1}[c, d^m]$. Also by Lemma 2.6 $m[c, d^m] = 0$. Using the property $Q(m)$, we conclude that $[c, d^m] = 0$. Hence commutativity of R follows by [8, Theorem]. □

Lemma 2.10. *Let R satisfy $C_1(m,R)$, $C_2(m,R)$ and $Q(m)$. Then R is commutative.*

Proof. By hypothesis, we have

$$[x,[x,y^m]]=0\,,\quad \text{for all}\quad x,y\in R\,. \tag{2.1}$$

From the hypothesis $C_1(m,R)$, and by Lemma 2.4, the commutator ideal is nil. It follows that $N(R)$ forms an ideal. In view of Lemma 2.5 (ii), $N(R)$ is a commutative ideal. This implies that $(N(R))^2\subseteq Z(R)$. Next, for any $a\in N(R)$, replace y by $1+a$ in (2.1) and use $Q(m)$ to get

$$[x,[x,a]]=0\,,\quad \text{for all}\quad x\in R\quad\text{and}\quad a\in N(R)\,. \tag{2.2}$$

From Lemma 2.5 (i), we have

$$[a,x^m]=0\quad \text{for all}\quad x\in R\quad\text{and}\quad a\in N(R)\,. \tag{2.3}$$

Using (2.2) and Lemma 2.2 together with (2.3), we get

$$mx^{m-1}[a,x]=0\,.$$

Replacing x by $x+1$ and using Lemma 2.6 together with $Q(m)$, we get $[a,x]=0$ for all $x\in R$ and $a\in N(R)$. But, then $C(R)\subseteq N(R)$, and thus

$$C(R)\subseteq N(R)\subseteq Z(R)\,. \tag{2.4}$$

Next, Lemma 2.2 and $C_1(m,R)$ yield that $mx^{m-1}[x,y^m]=[x^m,y^m]=0$ for all $x,y\in R$. Again, using Lemma 2.6 and $Q(m)$, we get $[x,y^m]=0$ for all $x,y\in R$. Similarly, we have $my^{m-1}[x,y]=[x,y^m]=0$ and also $[x,y]=0$ for all $x,y\in R$. Hence R is commutative. $\square$

Lemma 2.11. *Let R satisfy $C_1(m,R\backslash J(R))$ and $Q(m)$. If $[u,[u,v^m]]=0$ for all units u,v, then $J(R)^2\subseteq Z(R)$.*

Proof. Suppose that u,v are units in R. By hypothesis, we have

$$[u,[u,v^m]]=0\,,\quad \text{for all}\quad u,v\in U(R)\,. \tag{2.5}$$

Using the property $C_1(m,R\backslash J(R))$, we obtain $[u^m,v^m]=0$. In view of (2.5) and Lemma 2.2, we get $mu^{m-1}[u,v^m]=0$. This implies that

$$[u,v^m]=0\,,\quad \text{for all}\quad u,v\in U(R)\,. \tag{2.6}$$

Let $a\in N(R)$. Then there exists a minimal positive integer l such that

$$[u,a^n]=0\,,\quad \text{for all}\quad n\geq l\quad\text{and}\quad u\in U(R)\,. \tag{2.7}$$

Let $l>1$. Then $1+a^{l-1}\in U(R)$, and (2.6) yields that $[u,(1+a^{l-1})^m]=0$. Next by (2.7), one gets $m[u,a^{l-1}]=0$, and by the property $Q(m)$, we get $[u,a^{l-1}]=0$, and hence contradicts the minimality of l in (2.7); thus $l=1$. In view of (2.7), we get

$$[u,a]=0\,,\quad \text{for all}\quad u\in U(R)\quad\text{and}\quad a\in N(R)\,. \tag{2.8}$$

Let $j_1,j_2\in J(R)$. Then, by (2.6), we have

$$[1+j_1,(1+j_2)^m]=0\,,\quad \text{for all}\quad j_1,j_2\in J(R)\,. \tag{2.9}$$

Since semisimple rings satisfying $C_1(m,R)$ are commutative and hence by our assumption $R\backslash J(R)$ is commutative, so $C(R)\subseteq J(R)$. Further, we claim that $C(R)\subseteq N(R)$. Choose arbitrary elements x_1,y_1,x_2,y_2,x_3,y_3 of R, and let $c_1=$

$[x_1, y_1]$, $c_2 = [x_2, y_2]$ and $c_3 = [x_3, y_3]$ be any commutators. In view of (2.9), c_1, c_2, c_3 are all in $J(R)$, so $(1 + c_1 + c_2 + c_1c_2)$ and $(1 + c_3)$ are in $U(R)$ and hence are not in $J(R)$. By hypothesis, we can write

$$\big[1 + c_3, (1 + c_1 + c_2 + c_1c_2)^m\big] = 0\,. \tag{2.10}$$

One observes that (2.10) is a polynomial identity which is satisfied by all elements of R. But (2.10) is not satisfied by any 2×2 matrix ring over $GF(p)$, a prime p, if we take $c_1 = [e_{11}, e_{11} + e_{12}]$, $c_2 = [e_{11} + e_{12}, e_{21}]$ and $c_3 = c_1$. Hence by Lemma 2.3, $C(R) \subseteq N(R)$ and by (2.8) we obtain

$$\big[1 + j_2, [1 + j_1, 1 + j_2]\big] = 0\,, \quad \text{for all} \quad j_1, j_2 \in J(R)\,. \tag{2.11}$$

From (2.9) and (2.11), Lemma 2.2 gives that $m(1 + j_2)^{m-1}[1 + j_1, 1 + j_2] = 0$. This implies that $m[j_1, j_2] = 0$. By the property $Q(m)$, one gets $[j_1, j_2] = 0$ for all $j_1, j_2 \in J(R)$. This implies that $J(R)$ is commutative and $\big(J(R)\big)^2 \subseteq Z(R)$. □

Lemma 2.12. *Let R satisfy $C_1\big(m, R\backslash N(R)\big)$ and $Q(m)$. Then $N(R)$ is an ideal.*

Proof. Lemma 2.5 (ii) holds for R with 1 satisfying $C_1\big(m, R\backslash N(R)\big)$ and $Q(m)$, hence $N(R)$ is commutative. If $a^2 = 0$ and $ra \notin N(R)$, then

$$\big[(ra)^m, (1 + a)^m\big] = 0 = \big[(ra)^m, 1 + ma\big] = m\big[(ra)^m, a\big] = \big[(ra)^m, a\big]\,.$$

Therefore $a(ra)^m = 0$ and $(ra)^{(m+1)} = 0$, a contradiction. This implies that $ra \in N(R)$ for all $r \in R$ and $N(R)$ is an ideal by Lemma 2.8. □

Proof of Theorem 2.1. The case $m = 1$ is trivial, so we assume $m > 1$. By Lemma 2.11, $J(R)^2 \subseteq Z(R)$, so $\big[x, [x, y^m]\big] = 0$ for all $x \in R\backslash J(R)$ and $y \in R$. If $x \in J(R)$, then $\big[1 + x, [1 + x, y^m]\big] = 0$. for all $y \in R$; thus $\big[x, [x, y^m]\big] = 0$ for all $x, y \in R$. Moreover, if either x or y is in $J(R)$, $[x^m, y^m] = 0$, so R satisfies $C_1(m, R)$. Thus R is commutative by Lemma 2.10. □

The following are the immediate consequences of the above theorem (see [14] for details).

Corollary 2.13. *Let R satisfy $C_1\big(m, R\backslash J(R)\big)$, $C_6\big(m, R\backslash J(R)\big)$ and $Q(m)$. Then R is commutative.*

Proof. By hypothesis, we have $\big[(xy)^m \pm y^m x^m, x\big] = 0$ and $\big[(yx)^m \pm x^m y^m, x\big] = 0$ for all $x, y \in R/J(R)$. The first property can be written as

$$x\big\{(xy)^m - (yx)^m\big\} = \pm(y^m x^{m+1} - xy^m x^m)\,, \quad \text{for all} \quad x, y \in R\backslash J(R)\,. \tag{2.12}$$

And the second property gives that

$$\big\{(xy)^m - (yx)^m\big\}x = \pm(x^m y^m x - x^{m+1} y^m)\,, \quad \text{for all} \quad x, y \in R\backslash J(R)\,. \tag{2.13}$$

Multiplying (2.12) by x on the right, and (2.13) by x on the left, and then after subtracting we get

$$\big[x, [x^{m+1}, y^m]\big] = 0\,, \quad \text{for all} \quad x, y \in R/J(R)\,. \tag{2.14}$$

But $[x^{m+1}, y^m] = x^m[x, y^m] + [x^m, y^m]x$, in view of the property $C_1(m, R)$ and (2.13) yields that

$$x^m\big[x, [x, y^m]\big] = 0 \tag{2.15}$$

for all $x, y \in R\backslash J(R)$. If $x \in J(R)$ replacing x by $1+x$ in (2.15) yields $\big[x, [x, y^m]\big] = 0$ for all $x \in J(R)$ and $y \notin J(R)$; hence (2.15) holds for all $x \in R$ and all $y \notin J(R)$, and by Lemma 2.6 $\big[x, [x, y^m]\big] = 0$ for all $x \in R$ and all $y \notin J(R)$. In particular, $\big[x, [x, y^m]\big] = 0$ for all $x, y \in R\backslash J(R)$. Thus R is commutative by Theorem 2.1. □

Corollary 2.14. *Let R satisfy $C_1\big(m, R\backslash N(R)\big)$, $C_6\big(m, R\backslash N(R)\big)$ and $Q(m)$. Then R is commutative.*

Proof. Immediate from Corollary 2.13 and Lemma 2.11.

Theorem 2.15. *If R satisfy $C_1\big(m, R\backslash J(R)\big)$, $C_7\big(m, R\backslash J(R)\big)$ and $Q(m)$, then R is commutative.*

Proof. Let u, v be units in R. Then by hypothesis $C_7\big(m, R\backslash J(R)\big)$, we have

$$\big[(u^{-1}vu)^m - u^m(uu^{-1}v)^m, u\big] = 0$$

or

$$\big[u, [u^{m+1}, v^m]\big] = 0\,.$$

This implies that $\big[u, [u, v^m]\big] = 0$, for all u, $v \in U(R)$; therefore, $\big(J(R)\big)^2 \subseteq Z(R)$. By Lemma 2.11.

If $m = 1$, then nothing to prove.

Let $m > 1$. Clearly by inductive hypothesis, we have $[x^n, y^n] = 0$ and $\big[(yx)^n(x)^n - x^n(xy)^n, x\big] = 0)$, for all $n \geq 2$, provided $x \in J(R)$ or $y \in J(R)$. Hence by $C_1\big(m, R\backslash J(R)\big)$ and $C_7\big(m, R\backslash J(R)\big)$, we observe that R satisfies the properties $C_1(m, R)$ and $C_7(m, R)$ for $m > 1$. Now, by Lemma 2.9, R is commutative. □

Corollary 2.16. *Let R satisfy $C_1\big(m, R\backslash N(R)\big)$, $C_7\big(m, R\backslash N(R)\big)$ and $Q(m)$. Then R is commutative.*

Proof. Follows from Theorem 2.2 and Lemma 2.10.

3. Commutativity of periodic rings

In this section, a ring R is called periodic if for each $x \in R$, there exist distinct positive integers r, s such that $x^r = x^s$. Recently Abu-Khuzam and Yaqub [4, Theorem 3] proved that a periodic ring R is commutative if R satisfies the property $C_5\big(m, R\backslash N(R)\big)$. Also they established that if $N(R)$ is commutative in a periodic ring R and R is an $m(m+1)$-torsion-free ring satisfying the property $C_5\big(m, R\backslash N(R)\big)$, then R is commutative. It is natural to ask a question: Is the above result valid if the property $C_5\big(m, R\backslash N(R)\big)$ is replaced by $C_7\big(m, R\backslash N(R)\big)$?

We settle this question affirmatively here.

Theorem 3.1. *Let $m \geq 1$ be a fixed positive integer and let R be a periodic ring satisfying the properties $Q\big(m(m+1)\big)$ and $C_7\big(m, R\backslash N(R)\big)$. Suppose, further, that $N(R)$ is commutative. Then R is commutative.*

We state the following known results.

Theorem 3.2 ([2, Theorem 1])**.** *Let R be a periodic ring such that $N(R)$ is commutative. If for each $a \in N(R)$ and $x \in R$ there exists an integer $m = m(x,a) \geq 1$ such that $\big[x^m,[x^m,a]\big] = 0$ and $\big[x^{m+1},[x^{m+1},a]\big] = 0$, then R is commutative. In particular: If R is a periodic ring such that $N(R)$ is commutative and $\big[x,[x,a]\big] = 0$ for all $a \in N(R)$, $x \in R$, then R is commutative.*

Theorem 3.3 ([5, Theorem 1])**.** *Let R be a periodic ring such that $N(R)$ is commutative. Then the commutator ideal of R is nil, and $N(R)$ forms an ideal.*

Lemma 3.4 ([4, Lemma 4])**.** *Let R be a periodic ring and let $f : R \to S$ be a homomorphism of R onto S. Then the nilpotents of S coincide with $f\big(N(R)\big)$, where $N(R)$ is the set of nilpotents of R.*

Proof of Theorem 3.1. Since R is periodic and $N(R)$ is commutative, Lemma 3.3 yields that the commutator ideal $C(R)$ of R is nil; that is $C(R) \subseteq N(R)$ and $N(R)$ forms an ideal of R. But $N(R)$ is commutative, and also $\big(N(R)\big)^2 \subseteq Z(R)$.

First we claim that the idempotents of R are central: Let $e^2 = e \in R$ and $r \in R$. Replacing x by e and y by $e + er - ere$ in the hypothesis $C_7\big(m, Z(R)\big)$, we get

$$\big((e+er-ere)e\big)^m e^m - e^m\big(e(e+er-ere)\big)^m \in Z(R)\,.$$

This implies that $ere - er \in Z(R)$. Thus

$$ere - er = e(ere-er) = (ere-er)e = 0$$

or

$$ere = er\,.$$

Similarly, if $x = e$ and $y = e + re - ere$, we obtain

$$ere = re\,.$$

Thus $er = re$ for all $r \in R$ and the result follows immediately.

Secondly, we shall prove the theorem for R with identity 1: Suppose that $a \in N(R)$ and $b \in R\backslash N(R)$. Then by hypothesis $C_7\big(m, R\backslash N(R)\big)$, we can write

$$\big[b^m(1+a)^m - (1+a)^{m+1}b^m(1+a)^{-1}, 1+a\big] = 0 \tag{3.1}$$

for all $a \in N(R)$, $b \in R\backslash N(R)$. This implies that

$$\begin{aligned}\big\{b^m(1+a)^m - (1+a)^{m+1}b^m(1+a)^{-1}\big\}(1+a)\\ = (1+a)\big\{b^m(1+a)^m - (1+a)^{m+1}b^m(1+a)^{-1}\big\}\end{aligned}$$

or

$$\begin{aligned}&\left\{b^m(1+a)^{m+1}-(1+a)^{m+1}b^m\right\}\\&\qquad=(1+a)\left\{b^m(1+a)^m-(1+a)^{m+1}b^m(1+a)^{-1}\right\}.\end{aligned}$$

Using the binomial expansion and the condition $\big(N(R)\big)^2 \subseteq Z(R)$, one gets

$$(m+1)(b^m a - ab^m) = (1+a)\left\{b^m(1+a)^m-(1+a)^{m+1}b^m(1+a)^{-1}\right\}. \tag{3.2}$$

But $N(R)$ is a commutative ideal, $(1+a)(b^m a - ab^m) = b^m a - ab^m$, and also by (3.2), we have

$$(1+a)(m+1)(b^m a - ab^m) = (1+a)\left\{(b)^m(1+a)^m-(1+a)^{m+1}b^m(1+a)^{-1}\right\}.$$

Since $a \in N(R)$, $1+a \in U(R)$ and by (3.1), this gives that

$$(m+1)(b^m a - ab^m) = \left\{b^m(1+a)^m-(1+a)^{m+1}b^m(1+a)^{-1}\right\} \in Z(R).$$

This implies that $(m+1)[b^m, a] \in Z(R)$. Using the property $Q\big(m(m+1)\big)$, we get

$$[b^m, a] \in Z(R), \quad \text{for all} \quad a \in N(R), \quad b \in R\backslash N(R). \tag{3.3}$$

Now since $N(R)$ is commutative, (3.3) implies that

$$[b^m, a] \in Z(R), \quad \text{for all} \quad a \in N(R), \quad b \in R. \tag{3.4}$$

Next, let $x_1, x_2, \ldots, x_n \in R$. Then $R\backslash C(R)$ is commutative; so, by Lemma 3.3, $(x_1 \ldots x_n)^m - x_1^m \ldots x_n^m \in C(R) \subseteq N(R)$. Therefore $N(R)$ is commutative yields that

$$\big[(x_1 \ldots x_n)^m, a\big] = [x_1^m \ldots x_n^m], \quad \text{for all} \quad a \in N(R). \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$[x_1^m \ldots x_n^m, a] \in Z(R) \quad \text{for all} \quad a \in N(R), \quad x_1 \ldots x_n \in R \quad \text{and} \quad n \geq 1. \tag{3.6}$$

Let S be the subring generated by the m-th powers of the elements of R. Then by (3.6) we have

$$[x, a] \in Z(S) \quad \text{for all} \quad a \in N(S), \quad x \in S, \tag{3.7}$$

where $Z(S)$ and $N(S)$ represent the centre of S and the set of nilpotent elements of S respectively. Combining the facts that S is periodic, $N(S)$ is commutative, and (3.7), Lemma 3.2 shows that S is commutative, and hence $[x^m, y^m] = 0$ for all $x, y \in R$. This implies that R satisfies $C_1(m, R)$. But R also satisfies $Q(m)$ and $C_7\big(m, R\backslash N(R)\big)$, by Corollary 2.16, we get the required result.

To complete the proof of Theorem 3.1: Note first that idempotents are central, and then prove the theorem for R with 1. It follows that for every nonzero idempotent e, eR is commutative, and hence $e[x, y] = 0$ for all $x, y \in R$. Thus if $a \in R$ is potent with $a^n = a$, $n > 1$, $a^{n-1}[a, b] = 0 = [a, b]$ for all $b \in R$. Since every element in a periodic ring is the sum of a potent element and nilpotent element, this gives $N(R) \subseteq Z(R)$ and R is commutative by a well-known theorem of Herstein [7]. □

4. Counter Examples

Example 4.1. The ring of 3×3, without unit, of strictly upper triangular matrices over Q, the ring of rational numbers shows that the hypotheses of Theorems 2.1 and 2.15 alone without additional condition 1, does not guarantee commutativity.

Next, we provide an example to show that the property $Q(m)$ in the hypotheses of Theorems 2.1 and 2.15 is not superfluous even if the properties $[x^m, y^m] = 0$ and $\big[(xy)^m \pm y^m x^m, x\big] = 0 = \big[(yx)^m \pm x^m y^m, x\big]$ hold for all $x, y \in R$.

Example 4.2. Let $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix} \;\Big|\; \alpha, \beta, \gamma, \delta \in GF(3) \right\}$.

Clearly, R satisfies $[x^3, y^3] = 0$ and $(xy)^3 = y^3x^3$ for all $x, y \in R$. Hence R satisfies all the hypotheses except $Q(3)$.

Example 4.3. Consider the ring $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix} \;\Big|\; \alpha, \beta, \gamma, \delta \in GF(3) \right\}$.

Clearly, R satisfies $(xy)^2 - y^2x^2 = 0$ and $(yx)^2 - x^2y^2 = 0$. We observe that for $n = 2$, R satisfies the conditions $C_6(m, S)$ and $Q(m)$. This indicates that the property $C_1(m, R\backslash J)$ is essential in the hypothesis of Theorem 2.1.

Example 4.4. Let $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix} \;\Big|\; \alpha, \beta, \gamma, \delta \in GF(2) \right\}$.

It is trivial to check that R satisfies $[x^2, y^2] = 0$ and $\big[(yx)^2x^2 - x^2(xy)^2, x\big] = 0$ for all $x, y \in R$. This shows that, for $n = 2$ the property $Q(m)$ can not be omitted from that hypothesis of Theorem 2.15.

Example 4.5. Let $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix} \;\Big|\; \alpha, \beta, \gamma, \delta \in GF(3) \right\}$.

The ring R has property $Q(m)$ and the properties $\big[(yx)^m x^m - x^m(xy)^m, x\big] = 0$. Hence for $n = 4$, R satisfied all the hypothesis of Theorem 2.15 except $C_1(m, R\backslash J)$.

The following example shows that a ring R with unity 1 satisfying the properties $C_1(m, S)$ and $Q(m)$ need not be commutative.

Example 4.6. Let $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \;\Big|\; \alpha, \beta, \gamma \in GF(4) \right\}$.

In the non-commutative ring R and it satisfies the properties $C_1(m, S)$ and $Q(m)$ for $m = 3$. This shows that the existence of the property $C_6(m, R\backslash J)$ (resp. $C_7(m, R/\backslash J)$ in Theorem 2.1 (resp. Theorem 2.15).

Example 4.7. Let $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \;\middle|\; \alpha, \beta, \gamma \in GF(3) \right\}$.

The ring R satisfies all the hypothesis of Theorem 3.1 except the hypothesis "$N(R)$ is commutative". This shows that commutativity of $N(R)$ is essential in Theorem 3.1.

Example 4.8. Let $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \;\middle|\; \alpha, \beta, \gamma \in GF(5) \right\}$.

The non-commutative ring R satisfies all the hypothesis of Theorem 4.1 except $C_7(m, R\backslash N)$, for $n = 2$. This shows that the condition $C_7(m, R\backslash N)$ is essential in Theorem 3.1.

Example 4.9. Let $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \;\middle|\; \alpha, \beta, \gamma \in GF(3) \right\}$.

For $m = 5$, R satisfies all the hypothesis of Theorem 3.1 except $Q(m(m+1))$. But R is not commutative. This strengthens the existence of the property $Q(m(m+1))$ in the hypothesis of Theorem 3.1 (see also [14] for details).

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