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REMARKS ON SYMMETRIES OF PARABOLIC GEOMETRIES

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Abstract. We consider symmetries on filtered manifolds and we study the |1|-graded parabolic geometries in more details. We discuss the existence of symmetries on the homogeneous models and we conclude some simple observations on the general curved geometries.

This paper follows the lecture at the Winter School 'Geometry and Physics' in Srní, January 2006. The aim is to introduce and discuss the symmetries for the so called parabolic geometries. The strategy is to view the Cartan geometries as straightforward generalizations of the classical affine structures and to extend the notion of the affine symmetries on manifolds in this way. The affine structures can be understood as reductive Cartan geometries of the first order, the classical results are expressed naturally in the language of the Cartan geometries, and the classical approach generalizes easily.

The parabolic geometries represent another special case of the general Cartan geometries. They are of second order and never reductive. The symmetries will no more be unique in this case and we may have more than one symmetry at each point.

The parabolic geometries live mostly on filtered manifolds, but we shall deal with the |1|-graded geometries, where the filtration is trivial and the whole structure is given by a specific G-structure in the classical sense. For this special class of geometries, the symmetries are defined in the same intuitive way as in the affine geometry. At the same time, much of the classical theory of affine symmetries extends. In particular, the existence of a symmetry at a point kills the torsion of the geometry at this point. In view of the nice general theory of parabolic geometries, this already proves the local flatness of the symmetric geometries for most types of them.

1. Basic facts

Throughout the paper we use the notation and concepts of [2] and [7]. Nevertheless we recall some of the basic facts and notation on the parabolic geometries below.

The paper is in final form and no version of it will be submitted elsewhere.

Cartan geometries. Let G be a Lie group, $P \subset G$ a Lie subgroup, and write $\mathfrak{g}$ and $\mathfrak{p}$ for their Lie algebras. A *Cartan geometry* of type (G, P) on a smooth manifold M is a principal fiber bundle $p : \mathcal{G} \to M$ with structure group P, together with a 1-form $\omega \in \Omega(\mathcal{G}; \mathfrak{g})$ called a *Cartan connection* such that:

- $(r^h)^*\omega = \mathrm{Ad}_{h^{-1}} \circ \omega$ for each $h \in P$
- $\omega(\zeta_X(u)) = X$ for each $X \in \mathfrak{p}$
- $\omega(u) : T_u\mathcal{G} \longrightarrow \mathfrak{g}$ is a linear isomorphism for each $u \in \mathcal{G}$.

The *homogeneous model* for Cartan geometries of type (G, P) is the canonical P-bundle $p : G \longrightarrow G/P$ endowed with the left Maurer-Cartan form $\omega_G \in \Omega(G, \mathfrak{g})$.

The third property of ω defines the horizontal fields $\omega^{-1}(X)$ for every element $X \in \mathfrak{g}$. The curvature K of a Cartan geometry is defined by the structure equation and K is also given explicitly by the *curvature function* $\kappa : \mathcal{G} \to \wedge^2\mathfrak{g}^* \otimes \mathfrak{g}$,

$$\kappa(u)(X, Y) = [X, Y] - \omega\big([\omega^{-1}(X), \omega^{-1}(Y)]\big).$$

If at least one of the arguments is from $\mathfrak{p}$, then the curvature vanishes and the curvature function may be viewed as $\kappa : \mathcal{G} \to \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$. This function is also right-invariant, i.e.

$$\kappa \circ r^g = g^{-1} \cdot \kappa$$

for all $g \in P$, where $\cdot$ is the tensor product of the adjoint actions $\underline{\mathrm{Ad}}$ on $(\mathfrak{g}/\mathfrak{p})^*$ and Ad on $\mathfrak{g}$.

A *morphism of Cartan geometries* from $(\mathcal{G} \to M, \omega)$ to $(\mathcal{G}' \to M', \omega')$ is a principal bundle morphism $\varphi : \mathcal{G} \to \mathcal{G}'$ such that $\varphi^*\omega' = \omega$. In this case their curvature functions κ and κ' satisfy $\kappa = \kappa' \circ \varphi$. We shall deal with the automorphisms of Cartan geometries. In the homogeneous case, there is the famous Liouville theorem, see [6]:

Theorem 1.1. *The automorphisms of the homogeneous model $(G \to G/P, \omega_G)$ of a Cartan geometry are exactly the left multiplications by elements of G.*

The curvature is the complete local invariant of a Cartan geometry, see e.g. [6]:

Theorem 1.2. *If the curvature of a Cartan geometry vanishes then the geometry is locally isomorphic with the homogeneous model of the same type.*

Cartan geometry is called *locally flat* if the curvature κ vanishes. The *torsion* T of the Cartan geometry is defined by the composition of the values of the curvature function with the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$. If the torsion is zero, i.e. the values of κ are in $\wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$, we call the Cartan geometry *torsion free*. Obviously, the homogeneous models are Cartan geometries with zero curvature.

Parabolic geometries. Let $\mathfrak{g}$ be a semisimple Lie algebra. The $|k|$*-grading on* $\mathfrak{g}$ is the vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all i and j (we understand $\mathfrak{g}_r = 0$ for $|r| > k$) and such that the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$. We will suppose that there is no simple ideal of $\mathfrak{g}$ contained in $\mathfrak{g}_0$. Each gradation of $\mathfrak{g}$ defines the *filtration* $\mathfrak{g} = \mathfrak{g}^{-k} \supset \mathfrak{g}^{-k+1} \supset \cdots \supset \mathfrak{g}^{k} = \mathfrak{g}_k$, where $\mathfrak{g}^i = \oplus_{j\geq i}\mathfrak{g}_j$. In particular $\mathfrak{g}_0$ and $\mathfrak{g}^0 =: \mathfrak{p}$ are subalgebras of $\mathfrak{g}$ and $\mathfrak{g}^1 =: \mathfrak{p}_+$ is a nilpotent ideal in $\mathfrak{p}$.

Let G be a semisimple Lie group with the Lie algebra $\mathfrak{g}$. Not only the choice of the group G but also the choice of the subgroups $G_0 \subset P \subset G$ with the prescribed subalgebras $\mathfrak{p}$ a $\mathfrak{g}_0$ impacts the properties of the resulting geometries. The obvious maximal choice is this one:

$$\begin{aligned} G_0 &:= \left\{g \in G \mid \mathrm{Ad}_g(\mathfrak{g}_i) \subset \mathfrak{g}_i,\ \forall i = -k,\dots,k\right\} \\ P &:= \left\{g \in G \mid \mathrm{Ad}_g(\mathfrak{g}^i) \subset \mathfrak{g}^i,\ \forall i = -k,\dots,k\right\}, \end{aligned}$$

but we may also take the connected component of the unit in these subgroups or anything between these two extremes. It is not difficult to show for all such subgroups (see [8]):

Proposition 1.3. *Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra and G be a Lie group with Lie algebra $\mathfrak{g}$.*

(1) *$G_0 \subset P \subset G$ are closed subgroups with Lie algebras $\mathfrak{g}_0$ and $\mathfrak{p}$, respectively.*
(2) *The map $(g_0, Z) \mapsto g_0 \exp Z$ defines a diffeomorphism $G_0 \times \mathfrak{p}_+ \to P$.*

A *parabolic geometry* is a Cartan geometry of type (G, P), where G and P are as above. If the length of the gradation of $\mathfrak{g}$ is k, then the geometry is called *$|k|$-graded*.

The curvature of parabolic geometries. The curvature function $\kappa : \mathcal{G} \to \wedge^2\mathfrak{g}_-^* \otimes \mathfrak{g}$ is valued in the cochains for the second cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$. This group can be also computed as the homology of the codifferential $\partial^* : \wedge^{k+1}\mathfrak{g}_-^* \otimes \mathfrak{g} \to \wedge^k\mathfrak{g}_-^* \otimes \mathfrak{g}$, where

$$\begin{aligned} \partial^*(Z_0 \wedge \cdots \wedge Z_k \otimes W) = &\sum_{i=0}^{k} (-1)^{i+1} Z_0 \wedge \cdots \hat{i} \cdots \wedge Z_k \otimes Z_i \cdot W \\ &+ \sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge \cdots \hat{i} \cdots \hat{j} \cdots \wedge Z_k \otimes W \end{aligned}$$

for all $Z_0, \dots, Z_k \in \mathfrak{g}_-^* \simeq \mathfrak{p}_+$ and $W \in \mathfrak{g}$.

The parabolic geometry is called *normal* if the curvature satisfies $\partial^* \circ \kappa = 0$. If the geometry is normal, we can define the *harmonic part of curvature*, $\kappa_H : \mathcal{G} \to H^2(\mathfrak{g}_-, \mathfrak{g})$, as the composition of the curvature and the projection to the second cohomology group.

Thanks to the gradation of $\mathfrak{g}$, there are several decompositions of the curvature of the parabolic geometry. One of the possibilities is the decomposition into *homogeneous components*, which is of the form

$$\kappa = \sum_{i=-k+2}^{3k} \kappa^{(i)},$$

where $\kappa^{(i)}(u)(X, Y) \in \mathfrak{g}_{p+q+i}$ for all $X \in \mathfrak{g}_p, Y \in \mathfrak{g}_q$ and $u \in \mathcal{G}$. The parabolic geometry is called *regular* if the curvature function κ satisfies $\kappa^{(r)} = 0$ for all

$r \leq 0$. The crucial structural description of the curvature is provided by the following theorem (see [8]):

Theorem 1.4. *The curvature κ of regular normal geometry vanishes if and only if its harmonic part κ_H vanishes. Moreover, if all homogeneous components of κ of degrees less than j vanish identically and there is no cohomology $H^2_j(\mathfrak{g}_-, \mathfrak{g})$, then also the curvature component of degree j vanishes.*

Another possibility is the decomposition of the curvature according to the values:

$$\kappa = \sum_{j=-k}^{k} \kappa_j$$

and in an arbitrary frame u we have $\kappa_j(u) \in \mathfrak{g}_- \wedge \mathfrak{g}_- \to \mathfrak{g}_j$. The component κ_- valued in $\mathfrak{g}_-$ corresponds to the torsion.

In the case of $|1|$-graded geometries the decomposition by the homogeneity corresponds to the decomposition according to the values. The homogeneous component of degree 1 corresponds to the torsion while the homogeneous components of degrees 2 and 3 coincide with κ_0 and κ_1.

Underlying structures. Let $(\mathcal{G} \to M, \omega)$ be a regular parabolic geometry. We obtain the filtration of the tangent bundle on M by subbundles $T^iM := \mathcal{G} \times_P \mathfrak{g}^i$ such that for all sections ξ of T^iM and η of T^jM the Lie bracket $[\xi, \eta]$ is a section of $T^{i+j}M$. We get directly the *associated graded bundle*

$$\mathrm{gr}(TM) = \mathrm{gr}_{-k}(TM) \oplus \cdots \oplus \mathrm{gr}_{-1}(TM)$$

where $\mathrm{gr}_i(TM) = T^iM/T^{i+1}M$. In this case we have that $\mathrm{gr}(T_xM) \simeq \mathfrak{g}_-$ as graded Lie algebras for all $x \in M$ and we obtain the frame bundle $\mathcal{G}_0 := \mathcal{G}/\exp \mathfrak{p}_+$ as the frame bundle of $\mathrm{gr}(TM)$. In particular, $\mathrm{gr}_i(TM) \simeq \mathcal{G}_0 \times_{G_0} \mathfrak{g}_i$.

More generally, we define a *filtered manifold* as a manifold M together with a filtration $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ such that for sections ξ of T^iM and η of T^jM the Lie bracket $[\xi, \eta]$ is a section of $T^{i+j}M$ and we again obtain an associated graded bundle. For each $x \in M$, the graded algebra $\mathrm{gr}(T_xM)$ is called the *symbol algebra*.

If the symbol algebras are all isomorphic to our fixed graded algebra $\mathfrak{g}_-$, then there is the frame bundle for $\mathrm{gr}(TM)$ with the structure group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$. As discussed thoroughly in [3], our group G_0 may equal to the latter graded automorphism group (in the case when all the first cohomology lives in negative homogeneities), or it is a proper subgroup and the geometry is given by the reduction of the graded frame bundle to the structure group $G_0 \subset \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$.

If we start with a regular parabolic geometry, we get exactly these data on the base manifold and we call them the *infinitesimal flag structure*.

Conversely, the Cartan's equivalence method has lead to the equivalence between such infinitesimal flag structures and regular and normal parabolic geometries, see [8, 1],

Theorem 1.5. *Let M be a filtered manifold such that each symbol algebra is isomorphic to $\mathfrak{g}_-$ and let $\mathcal{G}_0 \to M$ be a reduction of the frame bundle of* $\mathrm{gr}(TM)$

to the structure group G_0. Then there is a regular normal parabolic geometry $(p : \mathcal{G} \to M, \omega)$ inducing the given data. If $H^1_\ell(\mathfrak{g}_-, \mathfrak{g})$ are trivial for all $\ell > 0$, then the normal regular geometry is unique up to isomorphism.

The construction is functorial and the latter theorem describes an equivalence of categories.

|1|-graded geometries. In this case the filtration of the tangent bundle is trivial. We need only the reduction of $\mathrm{gr}(TM)$ to the structure group G_0. The |1|-graded geometries are automatically regular and we get the correspondence between normal |1|-graded parabolic geometries and first order G-structures whose structure groups G_0 appear as the reductive part of a parabolic subgroup $P \subset G$ and $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

We give here survey on all |1|-graded geometries. The classification of semisimple Lie algebras in terms of simple roots is well known and for a given $\mathfrak{g}$ there is a complete description of all parabolic subalgebras, see [8, 2] for more details. The latter description allows to classify all corresponding geometries. We list them all here together with their non-zero components of the harmonic curvature (notice some overlaps in low dimensions).

- A_ℓ **:** the split form, $\ell \geq 2$, the almost Grassmannian structures, $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{R})$, $\mathfrak{g}_0 = \mathfrak{s}(\mathfrak{gl}(p, \mathbb{R}) \times \mathfrak{gl}(q, \mathbb{R}))$, $p + q = \ell + 1$. Moreover
  - $p = 1$**,** $q = 2$ or $p = 2$**,** $q = 1$ **:** the projective structures dim $= 2$, one curvature of homogeneity 3
  - $p = 1$**,** $q > 2$ or $p > 2$**,** $q = 1$ **:** the projective structures dim > 2, one curvature of homogeneity 2
  - $p = 2$**,** $q = 2$ **:** dim $= 4$, two curvatures of homogeneity 2
  - $p = 2$**,** $q > 2$ or $p > 2$**,** $q = 2$ **:** dim $= pq$, one torsion, one curvature of homogeneity 2
  - $p > 2$**,** $q > 2$ **:** dim $= pq$, two torsions
- A_ℓ **:** the quaternionic form, $\ell = 2p + 1 > 2$, $\mathfrak{g} = \mathfrak{sl}(p + 1, \mathbb{H})$. We have:
  - $p = 1$**:** the almost quaternionic geometries, dim $= 4$, two curvatures of homogeneity 2
  - $p > 1$**:** the almost quaternionic geometries, dim $= 4p$, one torsion, one curvature of homogeneity 2
  - the geometries modeled on quaternionic Grassmannians: two torsions
- A_ℓ **:** $\ell = 2p - 1$ one type geometry for the algebra $\mathfrak{su}(p, p)$. We have:
  - $p = 2$ **:** two curvatures of homogeneity 2
  - $p > 2$ **:** two torsions
- B_ℓ **:** the pseudo-conformal geometries in odd dimension ≥ 3. We have:
  - $\ell = 2$ **:** dim $= 3$, one curvature of homogeneity 3
  - $\ell > 2$ **:** dim $= 2\ell - 1$, one curvature of homogeneity 2
- C_ℓ **:** the split form, $\ell > 2$, the almost Lagrangian geometries - one torsion
- C_ℓ **:** one type of geometry corresponding to $\mathfrak{sp}(p, p)$, $2p = \ell$ - one torsion
- D_ℓ **:** the pseudo-conformal geometries in all even dimensions ≥ 4
  - $\ell = 3$ **:** dim $= 4$, two curvatures of homogeneity 2
  - $\ell > 3$ **:** dim ≥ 6, one curvature of homogeneity 2

D_ℓ **:** the real almost spinorial geometries $\mathfrak{g} = \mathfrak{so}(l,l)$

$\ell = 4$ **:** one curvature of homogeneity 2

$\ell \geq 5$ **:** one torsion

D_ℓ **:** the quaternionic almost spinorial geometries, $\mathfrak{g} = \mathfrak{u}^*(\ell, \mathbb{H})$, $l = 2m$ - one torsion

E_6 **:** two exotic geometries with $\mathfrak{g}_0 = \mathfrak{so}(5,5) \oplus \mathbb{R}$ and $\mathfrak{g}_0 = \mathfrak{so}(1,9) \oplus \mathbb{R}$ - one torsion

E_7 **:** another two exotic geometries - one torsion

Let us remark, that in the low dimensional cases some of the algebras are isomorphic and the corresponding geometries are in fact equal. In particular, $\mathfrak{sl}(4,\mathbb{R}) \simeq \mathfrak{so}(3,3)$, $\mathfrak{so}(2,4) \simeq \mathfrak{su}(2,2)$, $\mathfrak{so}(1,5) \simeq \mathfrak{sl}(2,\mathbb{H})$ and so all the four-dimensional conformal pseudo-Riemannian geometries are covered by the corresponding A_4–cases. Moreover, the spinorial geometries for D_4 are isomorphic to the conformal Riemannian geometries.

The symmetries of parabolic geometries. Let us first consider an affine locally symmetric space. This is a manifold M with a linear connection ∇ such that at each point $x \in M$, there is a locally defined affine transformation s_x satisfying $s_x(x) = x$ and $T_x s_x = -\operatorname{id}_{T_x M}$. The classical result on affine locally symmetric spaces says that a manifold with an affine connection is locally symmetric if and only if the torsion vanishes and the curvature is covariantly constant, see e.g. [5].

A manifold with a linear connection can be viewed equivalently as the first order $Gl(n,\mathbb{R})$-structure, i.e. a principal bundle $p : \mathcal{P}^1 M \longrightarrow M$ equipped by the canonical form $\theta \in \Omega^1(\mathcal{P}, \mathbb{R}^n)$, together with a fixed principal connection form $\gamma \in \Omega^1(\mathcal{P}^1 M, \mathfrak{gl}(n))$. Clearly the form $\omega := \theta + \gamma \in \Omega(\mathcal{P}^1 M, \mathbb{R}^n + \mathfrak{gl}(n,\mathbb{R}))$ enjoys the properties of a Cartan connection of the type $(A(n,\mathbb{R}), Gl(n,\mathbb{R}))$, with the affine group $A(n,\mathbb{R})$.

Now, a symmetry on M can be viewed as a morphism of this Cartan geometry, such that its underlying smooth map $\underline{\varphi}$ on M (base morphism) has the properties of the classical affine symmetries, i.e. $\underline{\varphi}(x) = x$ and $T_x\underline{\varphi} = -\operatorname{id}_{T_x M}$.

We might try to extend this definition to a general Cartan geometry, but clearly this cannot work in general. In order to see the point, let us consider any contact parabolic geometry, a CR–manifold for example. Thus we deal with a $2n+1$–dimensional manifold M equipped with the contact distribution HM of codimension one, such that the algebraic map $\mathcal{L} : \wedge^2 HM \to TM/HM$ induced by the Lie bracket is non–degenerate. If there were a symmetry $\underline{\varphi}$, then for any ξ, $\eta \in H_x M \subset T_x M$ the action of $\underline{\varphi}$ on $\mathcal{L}(\xi,\eta)$ must be

$$\underline{\varphi}(\mathcal{L}(\xi,\eta)) = \mathcal{L}(\underline{\varphi}(\xi), \underline{\varphi}(\eta)) = \mathcal{L}(\xi,\eta)$$

and thus we cannot require the differential to be minus identity everywhere.

At the same time, the above analysis indicates how to proceed for all parabolic geometries. Let us recall that each $|k|$–graded parabolic geometry comes with the tangent bundle equipped by a filtration $TM = T^{-k}M \supset T^{-k+1}M \supset \cdots \supset T^{-1}M$.

Definition 1.6. Let $(\mathcal{G} \to M, \omega)$ be a parabolic geometry. The *symmetry at the point x* is a locally defined automorphism φ such that its base morphism satisfies

$\underline{\varphi}(x) = x$ and $T_x\underline{\varphi}|_{T_x^{-1}M} = -\operatorname{id}_{T^{-1}M}$. The geometry is called (*locally*) *symmetric* if there is a symmetry at each point $x \in M$.

In other words, symmetries revert by the sign change only the smallest subspace in the filtration, while their actions on the rest are completely determined by the algebraic structure of the symbol algebra.

In the case of $|1|$-graded geometries, however, the filtration is trivial and so we have $T^{-1}M = TM$. Thus the definition of the symmetries of $|1|$-graded geometries follow completely the classical intuitive idea.

2. Symmetries on homogeneous models of $|1|$-graded geometries

The following lemma works in all situations where the symmetry is defined as reflecting the whole tangent space by the sign change. In particular, the lemma is very useful for homogeneous models of such geometries.

Lemma 2.1. *If there is a symmetry in x on a Cartan geometry of type (G, P), then there exists an element $g \in P$ such that $\underline{\operatorname{Ad}}_g(X + \mathfrak{p}) = -X + \mathfrak{p}$ for all $X + \mathfrak{p} \in \mathfrak{g}/\mathfrak{p}$.*

In the case of a $|1|$-graded geometry, all such elements g are of the form $g = g_0 \exp Z$, where $g_0 \in G_0$ such that $\operatorname{Ad}_{g_0}(X) = -X$ for all $X \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{g}_1$ is arbitrary.

Proof. Suppose, that there is a symmetry in a point $x \in M$. We know, that the base morphism $\underline{\varphi}$ preserves the point x. Hence the morphism φ preserves the fiber over x. Let u be an arbitrary fixed point in the fiber over x. There is an element $g \in P$ such that $\varphi(u) = u \cdot g$. Let $\xi \in \mathfrak{X}(M)$ be a vector field on M. It holds that

$$T\underline{\varphi}.\xi(x) = -\operatorname{id}_{T_xM}(\xi(x)) = -\xi(x)\,.$$

Using the identification $TM = \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ we have $\xi(x) = [\![u, X + \mathfrak{p}]\!]$ for some $X \in \mathfrak{g}$.

In the chosen frame u, we then have

$$T\underline{\varphi}([\![u, X + \mathfrak{p}]\!]) = [\![\varphi(u), X + \mathfrak{p}]\!] = [\![ug, X + \mathfrak{p}]\!] = [\![u, \underline{\operatorname{Ad}}_{g^{-1}}(X + \mathfrak{p})]\!]\,.$$

The definition of a symmetry implies

$$T\underline{\varphi}([\![u, X + \mathfrak{p}]\!]) = [\![u, -(X + \mathfrak{p})]\!]$$

and the two vectors are the same. Comparing the coordinates in the frame u gives us an element $g \in P$ such that $\underline{\operatorname{Ad}}_g(-X + \mathfrak{p}) = X + \mathfrak{p}$ and we take its inverse.

In the case of $|1|$-graded parabolic geometry, we have $g = g_0 \exp Z$ for some $g_0 \in G_0$ and $Z \in \mathfrak{g}_1$, because $g \in P$. We get $\underline{\operatorname{Ad}}_{g_0 \exp Z}(X) = -X$ for all $X \in \mathfrak{g}_{-1}$. But the action of the component $\exp Z$ is trivial while the action of g_0 preserves the gradation, i.e. $\underline{\operatorname{Ad}}_{g_0} = \operatorname{Ad}_{g_0}$, and so we get $\operatorname{Ad}_{g_0}(X) = -X$. □

Suppose that there is such an element g. If some element differs from g by a conjugation by an element from P, then it has the same property. If g corresponds to the frame $u \in p^{-1}(x)$, then the element $h^{-1}gh,\ h \in P$ corresponds to the frame $uh \in p^{-1}(x)$.

Proposition 2.2. *All symmetries of the homogeneous model* $(G \to G/P, \omega_G)$ *at the origin* o *are exactly the left multiplications by elements* $g \in P$ *satisfying the condition in Lemma* 2.1. *Moreover, if there is a symmetry in the origin* o*, then the homogeneous model is symmetric.*

Proof. We have seen that all automorphisms of G/P are exactly left multiplications by elements from G. We have to look for elements from G which give symmetries. Let us denote λ_g the left multiplication by element g. Because λ_g should be symmetry, element g must be in P. We make the same computation as in the proof of Lemma 2.1. Let $\xi \in \mathfrak{X}(G/P)$ be a vector field. In the origin of the homogeneous model we have $\xi(o) = [\![e, X + \mathfrak{p}]\!]$ for some $X + \mathfrak{p} \in \mathfrak{g}/\mathfrak{p}$ and we get

$$T\underline{\lambda}_g\big([\![e, X+\mathfrak{p}]\!]\big) = [\![ge, X+\mathfrak{p}]\!] = [\![g, X+\mathfrak{p}]\!] = [\![e, \underline{\mathrm{Ad}}_{g^{-1}}(X+\mathfrak{p})]\!]$$

and the element g must satisfy the equation in Lemma 2.1.

If there is a symmetry in the origin, we can use the conjugation to get symmetry in each point $hP \in G/P$. If λ_g is symmetry in the origin for some $g \in P$, then $\lambda_{hgh^{-1}}$ is symmetry in the point hP. □

Now we present some examples. We start with the affine geometry. This is not a parabolic geometry but our definition of the symmetry recovers the classical one and the latter proposition holds. Finally we show some $|1|$-graded examples.

Example. *Affine geometry.* We have $G = A(n,\mathbb{R}) = \big\{ \left(\begin{smallmatrix} 1 & 0 \\ v & A \end{smallmatrix}\right) \mid v \in \mathbb{R}^n, A \in Gl(n,\mathbb{R}) \big\}$ the affine group and the elements from $P = Gl(n,\mathbb{R})$ are of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & A \end{smallmatrix}\right) \in A(n,\mathbb{R})$. The Lie algebra of G looks like $\mathfrak{aff}(n,\mathbb{R}) = \big\{ \left(\begin{smallmatrix} 0 & 0 \\ w & B \end{smallmatrix}\right) \mid w \in \mathbb{R}^n, B \in \mathfrak{gl}(n,\mathbb{R}) \big\}$ and $\mathfrak{gl}(n,\mathbb{R}) = \big\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & B \end{smallmatrix}\right) \mid B \in \mathfrak{gl}(n,\mathbb{R}) \big\}$.

We have to look for elements $\left(\begin{smallmatrix} 1 & 0 \\ v & A \end{smallmatrix}\right) \in A(n,\mathbb{R})$ satisfying $\left(\begin{smallmatrix} 1 & 0 \\ x & A \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ w & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ x & A \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 0 & 0 \\ -w & 0 \end{smallmatrix}\right)$ for all $\left(\begin{smallmatrix} 0 & 0 \\ w & 0 \end{smallmatrix}\right) \in \mathfrak{g}_- \simeq \mathbb{R}^n$. Consequently, $\left(\begin{smallmatrix} 1 & 0 \\ x & A \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ w & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ -w & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ x & A \end{smallmatrix}\right)$ and thus $\left(\begin{smallmatrix} 0 & 0 \\ Aw & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ -w & 0 \end{smallmatrix}\right)$. We can see that there is only one element satisfying this equality and this is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -E \end{smallmatrix}\right)$, the conjugation by another element of P gives trivial change.

This matches well the known facts on the classical symmetric spaces. There can exist only one symmetry in each point on the affine (locally) symmetric space. The symmetry corresponds to the element we found above. The homogeneous model is the affine plane $\mathbb{R}^n \simeq A(n,\mathbb{R})/Gl(n,\mathbb{R}))$ and this clearly is a symmetric space and the symmetry in the origin is the left multiplication by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -E \end{smallmatrix}\right)$.

Example. *Projective structures.* We can make two reasonable choices of the Lie group G with the given Lie algebra and grading. We can consider $G = Sl(m+1,\mathbb{R})$. Then the maximal P is the subgroup of all matrices of the form $\left(\begin{smallmatrix} d & W \\ 0 & D \end{smallmatrix}\right)$ such that $\frac{1}{d} = \det D$ and $W \in \mathbb{R}^m$, but we take the connected component of the unit only. Clearly with this choice G/P is diffeomorphic to the m–dimensional sphere and P is the stabilizer of the ray spanned by the first basis vector in $\mathbb{R}^{m+1}$. The subgroup G_0 contains exactly elements of P such that $W = 0$, and this subgroup is isomorphic to $Gl^+(m,\mathbb{R})$.

The second reasonable choice is $G = PGl(m+1,\mathbb{R})$, the quotient of $Gl(m,\mathbb{R})$ by the subgroup of all multiples of the identity. Here P is the stabilizer of the line generated by the first basis vector too and the subgroup G_0 is isomorphic to $Gl(m,\mathbb{R})$, because each class in G_0 has exactly one representant of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & D \end{smallmatrix}\right)$. We can make the computation simultaneously and then discuss both cases separately.

We have $\mathfrak{g}_{-1} = \{\left(\begin{smallmatrix} 0 & 0 \\ X & 0 \end{smallmatrix}\right) \mid X \in \mathbb{R}^m\}$ and the adjoint action of $a = \left(\begin{smallmatrix} b & 0 \\ 0 & B \end{smallmatrix}\right)$ on $V = \left(\begin{smallmatrix} 0 & 0 \\ X & 0 \end{smallmatrix}\right)$ is $\operatorname{Ad}_a V = b^{-1}BX$. We look for elements $\left(\begin{smallmatrix} b & 0 \\ 0 & B \end{smallmatrix}\right)$ such that $BX = -bX$ for each $X \in \mathfrak{g}_{-1}$. It is easy to see that B is a diagonal matrix and that all elements on the diagonal are equal to $-b$. Thus we may represent the prospective solution as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -E \end{smallmatrix}\right)$.

Now we discuss the choice $G = Sl(m+1,\mathbb{R})$ with $G/P \simeq S^m$. The element has the determinant ± 1 and the sign depends on the dimension on the geometry. If m is even, then the element gives a symmetry but if m is odd, then there is no symmetry on this model. The reason is obvious — our choice of the groups has lead to the oriented sphere with the canonical projective structure (represented e.g. by the metric connection of the round sphere metric) and the obvious symmetries are orientation preserving in the even dimensions only.

In the case of $G = PGl(m+1,\mathbb{R})$, the above element always represents the class in G_0 and thus yields the symmetry. In both cases, all elements giving symmetry are of the form $\left(\begin{smallmatrix} 1 & W \\ 0 & -E \end{smallmatrix}\right)$ for all $W \in \mathbb{R}^m$.

These two choices of the group G correspond to projective structures on oriented and not oriented manifolds and we get that the projective space is always a symmetric homogeneous model. The existence of a symmetry on the oriented projective geometry depends on its dimension. Only the even-dimensional geometries can be symmetric.

Example. *Almost quaternionic structures.* Now we take almost quaternionic structures. There are again two interesting choices of the groups. We can choose $G = Sl(m+1,\mathbb{H})$ with the canonical action on $\mathbb{H}^{m+1}$. The parabolic subgroup P is the stabilizer of the quaternionic line spanned by the first basis vector in $\mathbb{H}^{m+1}$. Then $G_0 = \{\left(\begin{smallmatrix} a & 0 \\ 0 & A \end{smallmatrix}\right) \mid |a|^4 \det_{\mathbb{R}} A = 1\}$.

Next we can take $G = PGl(m,\mathbb{H})$, the quotient of all invertible quaternionic linear endomorphisms by the subgroup of real multiples of identity. Let P be again the stabilizer of the quaternionic line spanned by the first basis vector. The subgroup G_0 consists of all elements of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & A \end{smallmatrix}\right)$ such that $0 \neq a \in \mathbb{H}$ and $A \in Gl(m,\mathbb{H})$.

We have $\mathfrak{g}_{-1} = \{\left(\begin{smallmatrix} 0 & 0 \\ X & 0 \end{smallmatrix}\right) \mid X \in \mathbb{H}^m\}$ and we look for elements $\left(\begin{smallmatrix} q & 0 \\ 0 & B \end{smallmatrix}\right)$ such that $BX = -Xq$ for each $X \in \mathfrak{g}_{-1}$. Again, such an element must be diagonal and the elements on the diagonal of B are equal to $-q$. Suppose, that $q = a + bi + cj + dk$. If we choose $X = \left(\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right)$ we get $(-a - bi - cj - dk)i = -i(a + bi + cj + dk)$, thus $-ai + b + ck - dj = -ai + b - ck + dj$ and so $c = d = 0$. Then the choice $X = \left(\begin{smallmatrix} j \\ 0 \end{smallmatrix}\right)$ gives that q has to be real. We again get the element $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -E \end{smallmatrix}\right)$.

In the case of $PGl(m+1,\mathbb{H})$, this element clearly represents the class giving symmetry. In the case of $Sl(m,\mathbb{H})$ it should again depend on the dimension of the manifold. But the real dimension equals to $4m$ and thus also in this case the symmetry is well defined. All elements giving symmetries look like $\begin{pmatrix} 1 & W \\ 0 & -E \end{pmatrix}$ for all $W \in \mathbb{H}^m$.

Let us point out some observations coming from these examples. The existence of symmetry on the homogeneous model depends on the choice of the groups. We know, that if there is no element satisfying the condition from lemma 2.1, then the homogeneous model of the corresponding type is not symmetric. In addition, none of the Cartan geometry of the same type is symmetric. Let us mention the oriented projective structures in odd dimension. These Cartan geometries cannot be symmetric. If we forget the orientation (i.e. if we cosinder different groups) then we get geometries which can be symmetric.

3. Symmetric |1|-graded geometries

The following theorem plays a cruical role for us.

Theorem 3.1. *Symmetric* $|1|$*-graded parabolic geometries are torsion free.*

Proof. Let us choose an arbitrary $x \in M$ and let us denote φ the symmetry in x. We have that $\underline{\varphi}$ fixes x and thus φ preserves the fiber over x. The curvature function satisfies $\kappa = \kappa \circ \varphi$ and for appropriate $g \in P$ we have

$$\kappa(u) = \kappa(\varphi(u)) = \kappa(u \cdot g) = g^{-1} \cdot \kappa(u)\,.$$

The torsion is identified with the component κ_{-1} and we have the same equation for this correctly defined component (we have just to keep in mind the proper action of P on the quotient space $\mathfrak{g}_{-1} \simeq \mathfrak{g}/\mathfrak{p}$). We compare κ_{-1} in the frames u and $\varphi(u)$, where $u, \varphi(u) \in p^{-1}(x)$. We arrive at

$$\begin{aligned}\kappa_{-1}\big(\varphi(u)\big)(X,Y) &= \kappa_{-1}(u \cdot g)(X,Y) = g^{-1} \cdot \kappa_{-1}(u)(X,Y)\\ &= \underline{\mathrm{Ad}}_{g^{-1}}\big(\kappa_{-1}(u)(\underline{\mathrm{Ad}}_g X, \underline{\mathrm{Ad}}_g Y)\big)\\ &= -\kappa_{-1}(u)(-X,-Y) = -\kappa_{-1}(u)(X,Y)\end{aligned}$$

and this should be equal to $\kappa_{-1}(u)(X,Y)$ for all $X, Y \in \mathfrak{g}_{-1}$. Thus, we obtain $\kappa_{-1}(u)(X,Y) = -\kappa_{-1}(u)(X,Y)$ and so $\kappa_{-1}(u)$ vanishes. This is true for all frames $u \in \mathcal{G},\ p(u) = x$ and so this part of curvature vanishes at the point x.

If the geometry is symmetric then there is symmetry in all $x \in M$. Then κ_{-1} vanishes in all $x \in M$ and the geometry is torsion free. □

Corollary 3.2. *Let* $(\mathcal{G}, \omega)$ *be a normal* $|1|$*-graded parabolic geometry on a manifold* M *such that its homogeneous components of the harmonic curvature are only of degree* 1*. If there is a local symmetry at a point* $x \in M$*, then the whole curvature vanishes in this point. In particular, if the geometry is symmetric, than it is locally isomorphic with the homogeneous model.*

Proof. The existence of a symmetry forces κ_{-1} to vanish. If all harmonic curvature is concentrated to this homogeneity, then the whole curvature κ has to vanish, see Theorem 1.4. Of course, then the geometry is locally flat, see Theorem 1.2. □

It is not difficult to name all geometries satisfying the condition on the curvature in the latter Corollary browsing through the description after Theorem 1.5.

In particular, the following symmetric normal $|1|$-graded geometries have always to be locally flat: almost Grassmannian geometries such that $p > 2$ and $q > 2$, geometries modeled on quaternionic Grassmannians (but not the almost quaternionic ones), geometries for the algebra $\mathfrak{sp}(p,p)$ where $p > 2$, all geometries coming from the algebras of types C_ℓ, spinorial geometries in the D_ℓ types with $\ell > 4$, and all exotic geometries.

The crucial point in the above considerations was the odd homogeneity degree of the components in harmonic cuvatures. Thus, a similar argument applies for all geometries where the only available homogeneity is three:

Proposition 3.3. *Symmetric conformal geometries of dimension* 3 *and symmetric projective geometries of dimension* 2 *are locally flat.*

Proof. We prove that κ_1 is zero. Suppose that there is a symmetry φ in $x \in M$. In arbitrary frame u over x we have

$$\begin{aligned}\kappa_1\big(\varphi(u)\big)(X,Y) &= \kappa_1(u\cdot g)(X,Y) = g^{-1}\cdot\kappa_1(u)(X,Y)\\ &= \underline{\mathrm{Ad}}_{g^{-1}}\big(\kappa_1(u)(\underline{\mathrm{Ad}}_g X, \underline{\mathrm{Ad}}_g Y)\big)\\ &= \underline{\mathrm{Ad}}_{g^{-1}}\big(\kappa_1(u)(-X,-Y)\big) = \underline{\mathrm{Ad}}_{g^{-1}}\big(\kappa_1(u)(X,Y)\big)\end{aligned}$$

and this should be equal to $\kappa_1(u)(X,Y)$.

The action of g on $\mathfrak{g}_{-1}$ is $-\mathrm{id}$ from the definition. It holds that $\mathfrak{g}_1$ is dual to $\mathfrak{g}_{-1}$ with respect to the Killing form. The adjoint action of P on $\mathfrak{g}_1$ is the dual action of the adjoint action on $\mathfrak{g}_{-1}$. The action of the element g on $\mathfrak{g}_1$ is then the dual action of $-\mathrm{id}$ and it is again $-\mathrm{id}$. We have $\kappa_1(u)(X,Y) = -\kappa_1(u)(X,Y)$ for all u over x and therefore κ_1 vanish in x. If we have a symmetry in each point then κ_1 vanishes.

By Theorem 1.4 and our list of the features of all $|1|$–graded geometries, the geometries in question have no homogeneous parts of curvature of degree 1 and 2. They have only one homogeneous harmonic part of degree 3. This component belongs to κ_1 and therefore has to vanish. But then the harmonic part of curvature κ_H vanishes and so the curvature κ vanishes and the geometries are locally flat. □

The curvature of a symmetric $|1|$-graded geometry looks like $\kappa = \kappa^0$ and its lowest part $\kappa_0 : \mathcal{G} \to \mathfrak{g}^* \wedge \mathfrak{g}^* \otimes \mathfrak{g}_0$ is a correctly defined (quotient) object. Unfortunately, comparing $\kappa_0(u)$ with $\kappa_0(\varphi(u))$ does not give us any new information. Indeed,

$$\begin{aligned}\kappa_0\big(\varphi(u)\big)(X,Y) &= \kappa_0(u\cdot g)(X,Y) = g^{-1}\cdot\kappa_0(u)(X,Y)\\ &= \underline{\mathrm{Ad}}_{g^{-1}}(\kappa_0(u)(-X,-Y)) = \underline{\mathrm{Ad}}_{g^{-1}}\big(\kappa_0(u)(X,Y)\big)\end{aligned}$$

is equal to $\kappa_0(u)(X,Y)$ for all $X, Y \in \mathfrak{g}_{-1}$. Since $\mathfrak{g}_0 \subseteq \mathfrak{gl}(\mathfrak{g}_{-1}) \simeq \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}$, and the action on $\mathfrak{g}_{-1}$ and $\mathfrak{g}^*_{-1}$ is $-\mathrm{id}$, the action of g on the tensor is obviously trivial.

Again, we can easy find all remaining $|1|$–graded normal parabolic geometries allowing some homogeneous component of curvature of degree 2. These are just four lines of examples: projective geometries, conformal Riemannian geometries, almost quaternionic geometries and almost Grassmannian structures such that $p = 2$ or $q = 2$. We should like to remark that all the other geometries allowing curvature in our list above are in fact isomorphic with some of these four types. We have commented on that already.

The best known example of such geometries are the conformal Riemannian ones and the rich theory of Riemannian symmetric spaces indicates that there will indeed be examples of symmetric geometries which will not be locally flat. The study of all these most interesting geometries has to rely on more of the general theory of parabolic geometries and we shall treat this elsewhere.

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