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AN ANALOG OF THE FEFFERMAN CONSTRUCTION

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Abstract. The Fefferman construction associates to a manifold carrying a CR–structure a conformal structure on a sphere bundle over the manifold. There are some analogs to this construction, with one giving a Lie contact structure, a refinement of the contact bundle on the bundle of rays in the cotangent bundle of a manifold with a conformal metric. Since these structures are parabolic geometries, these constructions can be dealt with in this setting.

1. Introduction

The Fefferman construction associates to a given manifold equipped with a CR–structure a manifold with conformal structure. Both, CR– and conformal structures are examples of parabolic geometries. The construction of the Fefferman space using the description of these structures as parabolic geometries was done in [2].

In [3] the authors show, that the unit tangent bundle of a conformal manifold inherits a Lie contact structure. Since Lie contact structures are another example of parabolic geometries we will give the construction using the description as parabolic geometries.

A Lie contact structure of an odd dimensional manifold is a refinement of a contact structure. As in [3], we will view Lie contact structures as a structure modeled after the classical Lie sphere geometry of oriented hyperspheres. Let us briefly sketch how to define a Lie contact structure directly.

Let M be a manifold of dimension $2n+1$, with a contact bundle $HM \subset TM$. A contact bundle is a maximally non–integrable subbundle of the tangent bundle of corank 1. We define the Levi bracket for two sections $\xi, \xi' \in \Gamma(HM)$ as the projection of the Lie bracket to the quotient TM/HM, so in a point $x \in M$ the Levi bracket reads like

$$\mathcal{L}_x : \Lambda^2 H_x M \to T_x M / H_x M\,.$$

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Suppose that this subbundle decomposes into $HM = L^*M \otimes RM$, where $\operatorname{rank}(L^*M) = 2$ and $\operatorname{rank}(RM) = n$ and RM is equipped with a metric of signature (p, q). A manifold

$$\big(M, HM = L^*M \otimes RM \subset TM, (RM, g)\big)$$

carries a Lie contact structure of signature (p, q) if the Levi bracket is $O(RM)$–invariant, i.e. for all $\xi_x, \xi'_x \in L(L_xM, R_xM)$ and any $A \in O(R_xM)$ we have

$$\mathcal{L}_x(\xi_x, \xi'_x) = \mathcal{L}_x(A \circ \xi_x, A \circ \xi'_x)\,.$$

In the case considered here, we will use the canonical contact structure on the bundle of rays in the cotangent bundle. Starting with an arbitrary manifold M of $\dim(M) = n + 1$ we define

$$\mathcal{P}^rT^*M_x := \{[\varphi] | \varphi \in T^*M, \varphi \sim \psi \iff \varphi = \lambda\psi, \quad \lambda \in \mathbb{R}_+\}$$

for $x \in M$ and $\mathcal{P}^rT^*M = \sqcup_{x\in M}\mathcal{P}^rT^*_xM$, so $\dim(\mathcal{P}^rT^*M) = 2n+1$. The bundle of rays is a 2–fold cover of the projectivised cotangent bundle and hence carries a natural contact structure which is closely related to the natural symplectic structure of the cotangent bundle. The contact subbundle is given by

$$H_\varphi\mathcal{P}^rT^*M := \{\xi \in T_\varphi\mathcal{P}^rT^*M | \varphi(T_\varphi p \cdot \xi) = 0\}$$

where φ denotes a class in $\mathcal{P}^rT^*M$.

We will stick to the viewpoint of parabolic geometries in the sequel and do the construction in this picture. So a Lie contact structure will be a regular normal parabolic geometry of type $(\tilde{G}, \tilde{P})$ and a conformal structure will be a normal parabolic geometries of type (G, P) for the groups defined below.

2. The homogeneous model

2.1. **The homogeneous model of Lie contact structures.** Let

$$\tilde{V} = (\mathbb{R}^{n+4}, \langle, \rangle_{p+2,q+2})$$

be a real vector space with an inner product of signature $(p + 2, q + 2)$, where $n = p+q$. We will call a vector v in $\tilde{V}$ positive, null or negative if $\langle v, v\rangle$ is positive, zero or negative respectively. When referring to an explicit basis we will use the standard base $\{e_1, \dots, e_{n+4}\}$ with the inner product given by

$$\langle v, w\rangle_{p+2,q+2} := v_1w_{n+4} + w_1v_{n+4} + v_2w_{n+3} + w_2v_{n+3} + \sum_{i=3}^{p+2} v_iw_i - \sum_{i=p+3}^{n+2} v_iw_i\,.$$

So the matrix associated to this inner product reads like

$$\mathbb{J} := \begin{pmatrix} 0 & 0 & \mathbb{J}' \\ 0 & \mathbb{I}_{p,q} & 0 \\ \mathbb{J}' & 0 & 0 \end{pmatrix} \quad \text{where} \quad \mathbb{I}_{p,q} := \begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{pmatrix}, \quad \mathbb{J}' := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Define $\tilde{G} = PSO(p+2, q+2)$, which is just $SO(p+2, q+2)$ in case that n is odd, whereas for n even $SO(p+2, q+2)$ is a 2–fold cover of $PSO(p+2, q+2)$. Take a look at the Lie algebra of $\tilde{G}$, which is $\tilde{\mathfrak{g}} = \mathfrak{so}(\tilde{V})$ and realize $\tilde{\mathfrak{g}}$ as

$$\begin{pmatrix} a & b & R & e & 0 \\ c & d & S & 0 & -e \\ V & W & Z & -\mathbb{J}S^t & -\mathbb{J}R^t \\ f & 0 & -W^t\mathbb{J} & -d & -b \\ 0 & -f & -V^t\mathbb{J} & -c & -a \end{pmatrix}$$

with respect to the standard basis, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{R})$, $Z \in \mathfrak{o}(p, q)$, $V, W \in \mathbb{R}^n$, $R, S \in \mathbb{R}^{n*}$ and $e, f \in \mathbb{R}$.

We look at the space $\mathcal{N}$ of null–planes in $\tilde{V}$. The choice of a fixed null–plane N_0 gives a filtration on $\tilde{V}$, explicitly $\tilde{V}^l = \tilde{V}$ for all $l \leq -1$, $\tilde{V}^0 = N_0^\perp$, $\tilde{V}^1 = N_0$ and $\tilde{V}^l = \{0\}$ for all $l \geq 2$. By $\tilde{P}$ we denote the stabilizer of this null–plane in $\tilde{G}$.

Choosing the fixed null–plane to be $N_0 = \text{span}\{e_1, e_2\}$ the filtration on $\tilde{\mathfrak{g}}$ induced by the filtration of the representation space is easy to see. We get a filtration of $\tilde{\mathfrak{g}}$, with the associated $|2|$–grading, where $\tilde{\mathfrak{g}}_{-2}$ corresponds to f, $\tilde{\mathfrak{g}}_{-1}$ to V, W, $\tilde{\mathfrak{g}}_0$ to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, Z$, $\tilde{\mathfrak{g}}_1$ to X, Y and $\tilde{\mathfrak{g}}_2$ corresponds to e. So $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-2} \oplus \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_2$ with the parabolic subalgebra $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_2$ containing all block upper diagonal matrices, with V, W and f zero.

Note that $\tilde{\mathfrak{g}}_{-2} \oplus \tilde{\mathfrak{g}}_{-1}$ is Heisenberg, i.e.: the Lie bracket $[\ ,\ ] : \tilde{\mathfrak{g}}_{-1} \times \tilde{\mathfrak{g}}_{-1} \to \tilde{\mathfrak{g}}_{-2}$ is non–degenerate. Furthermore $\tilde{\mathfrak{g}}_{-1}$ may be identified with $L(\mathbb{R}^2, \mathbb{R}^n)$, where $\mathbb{R}^n$ carries an inner product of signature (p, q) and the Lie bracket $[\ ,\ ]$ is invariant under the action of $O(p, q)$ on $\mathbb{R}^n$. Using this one concludes that a regular parabolic geometry of type $(\tilde{G}/\tilde{P})$ has an underlying structure as described in the introduction, see [2] for further discussion.

To see that $\tilde{P}$ is the parabolic subgroup characterized by the fact that the adjoint action of the group preserves the filtration of the respective Lie algebra, i.e.

$$\tilde{P} = \{g \in \tilde{G} |\, \text{Ad}(g)(\tilde{\mathfrak{g}}^i) \subset \tilde{\mathfrak{g}}^i\} \quad \text{for all} \quad i\,,$$

we can write a generator of $\tilde{\mathfrak{g}}^2$ as a map, which takes $v \in \tilde{V}$ to $\langle v, e_2\rangle e_1 - \langle v, e_1\rangle e_2$. Applying the adjoint action $\text{Ad}(g)$ we get $v \mapsto \langle g(v), e_2\rangle g(e_1) - \langle g(v), e_1\rangle g(e_2)$ and for this to be in $\tilde{\mathfrak{g}}^2$ we need $g(e_1), g(e_2) \in N_0$. This is equivalent to $g(N_0) \subset N_0$ and hence $\tilde{P}$ is exactly the stabilizer of N_0 in $\tilde{G}$.

We want to identify the homogeneous space $\tilde{G}/\tilde{P}$ with the space of null–planes in $\tilde{V}$. To see that $\tilde{G}$ acts transitively on $\mathcal{N}$ we take an arbitrary null–plane

$$M = \text{span}\{m_1, m_2\} \in \mathcal{N} \quad \text{and} \quad N_0 = \text{span}\{e_1, e_2\} \in \mathcal{N}\,.$$

By non–degeneracy of $\langle\,,\,\rangle$ we find $m_{n+4} \in \tilde{V}$ with $\langle m_{n+4}, m_{n+4}\rangle = 0$ and $\langle m_1, m_{n+4}\rangle = 1$. We find a similar element for m_2 which we denote by m_{n+3} and again by non–degeneracy it is clear, that $\text{span}\{m_1, m_2\} \cap \text{span}\{m_{n+3}, m_{n+4}\}$ is trivial. Choosing an orthonormal basis of the complement of $\text{span}\{m_1, m_2, m_{n+3}, m_{n+4}\}$ and denoting it $\{m_3, \ldots, m_{n+2}\}$ the map A mapping $e_i \mapsto m_i$ for all $i \in \{1, \ldots, n+$

$4\}$ lies in $O(p+2,q+2)$. To find an element $A \in SO(p+2,q+2)$ we may choose to map $e_3 \mapsto -m_3$ and since the class $[A] \in PSO(p+2,q+2)$ is either $\{A\}$ or $\{A,-A\}$ all members of this class map $A(N_0)=M$. Now $\tilde{G}$ acts transitively on $\mathcal{N}$ with $\tilde{P}$ being the stabilizer of $N_0 \in \mathcal{N}$ and we identify the homogeneous space $\tilde{G}/\tilde{P} \cong \mathcal{N}$.

The homogeneous space $(p : \tilde{G} \to \tilde{G}/\tilde{P}, \omega^{MC})$ as a $\tilde{P}$-principle bundle with $\tilde{\omega}^{MC}$ the Maurer–Cartan form on $\tilde{G}$ is the homogeneous model of Lie contact structures of signature (p,q). Since Cartan connections are a generalization of the Maurer–Cartan form, $\tilde{\omega}^{MC}$ is a Cartan connection and since the curvature of a Cartan connection is an obstruction against the Maurer–Cartan equation, it is flat.

2.2. **The construction in the homogeneous model.** We fix the vector $\tilde{v}_0 = \frac{e_2-e_{n+3}}{\sqrt{2}}$ with the property $\langle \tilde{v}_0, \tilde{v}_0\rangle = -1$ and denote the orthogonal complement $V = \tilde{v}_0^{\perp}$ and the orthogonal projection $\sigma : \tilde{V} \to V$. Having chosen the subspace V there are two kinds of null–planes in $\tilde{V}$, the ones lying in V and transversal ones. We denote the set of transversal null–planes by $\mathcal{N}_t$. In case of $q=0$ all null–planes are transversal, if $q>0$ then $\mathcal{N}_t \subset \mathcal{N}$ is an open subset.

The construction starts off with the homogeneous model of conformal structures. The inner product on $\tilde{V}$ restricts to an inner product of signature $(p+2,q+1)$ on V. Take $G = SO(p+2,q+1)$ acting on $V \subset \tilde{V}$ with respect to this inner product. Let $\mathcal{C} := \{v \in V | \langle v,v\rangle = 0, v \neq 0\}$ be the null–cone in V with $\mathcal{PC}$ the quadric – its projectivization. Define P to be stabilizer of the null–line $l_0 := N_0 \cap V$ in G. l_0 is just the line through e_1 with our choices of N_0 and $\tilde{v}_0$. G acts transitively on $\mathcal{C}$, which can be seen by a similar argument as above, so we identify $G/P \cong \mathcal{PC} \cong S^{n+1}$. The tangent space $T_{[v]}S^n$, which is just $v^{\perp}/\mathbb{R}v$ inherits an inner product of signature $(p+1,q)$ from V, which depends on the choice of $v \in [v]$. This gives a conformal class, so S^n is the conformal sphere. The tangent space in the identity $T_{eP}G/P = l_0^{\perp}/l_0 = \mathfrak{g}/\mathfrak{p}$. The homogeneous space $(p : G \to G/P, \omega^{MC})$ with ω^{MC} the Maurer–Cartan form is the homogeneous model of conformal structures.

We embed $G \hookrightarrow SO(p+2,q+2)$ as the subset fixing $\tilde{v}_0$. In a basis of the form $\{\tilde{v}_0,\dots\}$ an element $A \in SO(p+2,q+2)$ has the form $\begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}$. Projecting the image of G to $PSO(p+2,q+2)$ is still injective, since A and $-A$ are in different classes. The inclusion on the level of Lie algebras is easy to see, since $\mathfrak{g} = \mathfrak{so}(p+2,q+1) \subset \tilde{\mathfrak{g}}$ is just the subset annihilating $\tilde{v}_0$:

$$\begin{pmatrix} a & e & R & e & 0 \\ f & 0 & S & 0 & -e \\ V & W & Z & -\mathbb{J}S^t & -\mathbb{J}R^t \\ f & 0 & -W^t\mathbb{J} & 0 & -e \\ 0 & -f & -V^t\mathbb{J} & -f & -a \end{pmatrix}.$$

This Lie algebra admits a $|1|$–grading where $\mathfrak{g}_{-1}$ corresponds to V and f, $\mathfrak{g}_0$ to a, S, W and Z, and $\mathfrak{g}_1$ corresponds to R and e. $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the set up block upper matrices with respect to this grading is a parabolic subalgebra. The parabolic subgroup P is characterized by the fact, that the adjoint action on $\mathfrak{g}$ preserves the respective filtration.

Now we define $Q := \tilde{P} \cap G$ where G is in fact the embedding of G in $\tilde{G}$. $\tilde{P}$ stabilizes N_0 and G stabilizes V, so Q stabilizes $l_0 = N_0 \cap V$ and hence $Q \subset P$. We now consider the action of G on the space of transversal planes $\mathcal{N}_t$, which is an open subset of $\tilde{G}/\tilde{P}$. Take an arbitrary null–plane $N \in \mathcal{N}_t$. We can choose a basis $\{n_1, n_2\}$, such that $n_1 \in V$ and $n_2 = \sigma(n_2) + \tilde{v}_0$ with $\langle \sigma(n_2), \sigma(n_2) \rangle = 1$ and $\langle n_1, \sigma(n_2) \rangle = 0$. We find an element of G mapping n_1 to e_1 and $\sigma(n_2)$ to $\sigma(e_2)$ and hence mapping N to N_0. So G acts transitively on $\mathcal{N}_t$.

The projection $E_0 := \sigma(N_0)$ is a plane in V containing the null–line l_0, with all other directions positive. In the basis of N_0 from above we get $\sigma(e_1) = e_1 \in V$, so $l_0 = \mathbb{R}e_1$ and $\langle \sigma(e_2), \sigma(e_2) \rangle = 1$. By construction Q stabilizes $E_0 \subset V$, $\sigma|_{N_0} : N_0 \to E_0$ defines a linear isomorphism and $\gamma(e) := \langle \sigma^{-1}(e), \tilde{v}_0 \rangle$ is a linear functional on E_0, which gives an orientation on E_0 preserved by the action of Q. But this is equivalent to preserving the class of $\sigma(e_2)$ in $l_0^{\perp}/l_0$ with its orientation. So Q is the orientation preserving stabilizer of the positive line E_0/l_0 in $l_0^{\perp}/l_0$.

Now $\mathfrak{g}/\mathfrak{p}$ carries a conformal class of inner products of signature $(p+1, q)$ with an oriented positive line E_0/l_0 which is preserved by Q, and this characterizes Q. Using the conformal class on $\mathfrak{g}/\mathfrak{p}$ an oriented line in $\mathfrak{g}/\mathfrak{p}$ gives an oriented line in $\mathfrak{p}_+ = (\mathfrak{g}/\mathfrak{p})^*$ as the annihilator of the orthogonal complement. Since via this identification we get a conformal class on $\mathfrak{p}_+$ we have seen, that Q is the stabilizer of a positive line in $\mathfrak{p}_+$.

The action of P on $\mathfrak{p}_+$ has 3 orbits in case of non–trivial signature corresponding to negative, null and positive directions, whereas in definite signature there is only one orbit of positive directions. Since Q is the stabilizer of a positive line in $\mathfrak{p}_+$ preserving its orientation, we identify P/Q with the set of rays of positive directions $\mathcal{P}^r_+(\mathfrak{p}_+)$ in $\mathfrak{p}_+$.

With $Q \subset P$ we get a natural projection $G/Q \to G/P$. Since G/Q is identified with the set $\mathcal{N}_t$, which is an open subset of $\mathcal{N}$ identified with $\tilde{G}/\tilde{P}$, we get an open embedding $G/Q \to \tilde{G}/\tilde{P}$. On the level of Lie algebras this gives an isomorphism $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \to \mathfrak{g}/\mathfrak{q}$ and with the natural projection we get a projection $\psi : \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \to \mathfrak{g}/\mathfrak{p}$. Since G/Q can be equivalently described as the associated bundle $G \times_P P/Q$, G/Q is a fiber bundle over G/P with fiber P/Q. Since G/Q is an open subset of $\tilde{G}/\tilde{P}$ it inherits the geometry of $\tilde{G}/\tilde{P}$.

2.3. **The interplay of the gradings.** $\mathfrak{g}$ is included in $\tilde{\mathfrak{g}}$ as the subalgebra annihilating the line through $\tilde{v}_0$ and we denote this inclusion by $i : \mathfrak{g} \to \tilde{\mathfrak{g}}$. The Killing form B on $\mathfrak{g}$ is just the restriction of $\tilde{B}|_{\mathfrak{g}\times\mathfrak{g}} = B$. Since B is non–degenerate, we get the decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{n}$ with $\mathfrak{n} = \mathfrak{g}^{\perp}$. By invariance of $\tilde{B}$ this is a $\mathbb{Z}^2$–grading of $\tilde{\mathfrak{g}}$, with $\mathfrak{g} = \tilde{\mathfrak{g}}_{\bar{0}}$ and $\mathfrak{n} = \tilde{\mathfrak{g}}_{\bar{1}}$ (i.e. $[\mathfrak{g}, \mathfrak{n}] \subset \mathfrak{n}$ and $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{g}$). We will denote the respective decomposition of an element $\tilde{A} \in \tilde{\mathfrak{g}}$ by $\tilde{A} = \tilde{A}^{\mathfrak{g}} + \tilde{A}^{\mathfrak{n}}$. The Killing form $\tilde{B}$ is non–degenerate on $\tilde{\mathfrak{p}}_+ \times \tilde{\mathfrak{g}}_-$, so we can choose dual bases with nice properties:

Lemma 2.4. There are dual bases $\tilde{Z}_i \in \tilde{\mathfrak{p}}_+$ and $\tilde{X}_j \in \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ with $\tilde{B}(\tilde{Z}_i, \tilde{X}_j) = \delta_{ij}$ for $i, j \in \{1, \dots, 2n+1\}$, with the following properties:

(i) $\tilde{Z}_i \in \tilde{\mathfrak{g}}_1$ for $i \in \{1, \dots, 2n\}$, $\tilde{Z}_{2n+1} \in \tilde{\mathfrak{g}}_2$
(ii) $\tilde{X}_1, \dots, \tilde{X}_n \in \ker(\psi)$
(iii) $B(\tilde{Z}_i^{\mathfrak{g}}, \psi(\tilde{X}_j)) = \delta_{ij}$ for $i, j \in \{n+1, \dots, 2n+1\}$
(iv) $\tilde{Z}_i^{\mathfrak{n}} = 0$ (i.e.: $\tilde{Z}_i \in \mathfrak{g}$) for $i \in \{n+1, \dots, 2n\}$

Remark. By property (ii) $\{\psi(\tilde{X}_{n+1}), \dots, \psi(\tilde{X}_{2n+1})\}$ is a basis of $\mathfrak{g}/\mathfrak{p}$ and (iv) implies that $\dim(\mathfrak{g}_1 \cap \tilde{\mathfrak{g}}_1) = n$.

Such bases are easily chosen explicitly in the matrix representation of $\tilde{\mathfrak{g}}$ from above. We choose

$$\tilde{Z}_i = S_i,\ \tilde{Z}_{i+n} = R_i \quad \text{for} \quad i = 1, \dots, n \quad \text{and} \quad \tilde{Z}_{2n+1} = e$$

where S_i, R_i and e should denote the matrix with all entries trivial, except for the respective entry being 1. The dual basis consists of $\{W_1, \dots, W_n, V_1, \dots, V_n, f\}$, where we do not explicitly mark, that these are in fact just classes in $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. By elementary calculations one proves the properties, which we will use in the sequel to obtain a general result for non–flat geometries.

3. The general case of nontrivial curvature

Let $(M, [g])$ be a $n+1$–dimensional manifold with a conformal structure. By a classical result of Elie Cartan a conformal manifold can be equivalently described by a normal parabolic geometry of type (G, P), with G and P from above. So we have a P–principle bundle $(p : \mathcal{G} \to M, \omega)$ together with a normal Cartan connection ω. Note that regularity is vacuous in this case, since $\mathfrak{g}$ is $|1|$–graded.

Define the Fefferman space of M as $\tilde{M} := \mathcal{G}/Q = \mathcal{G} \times_P P/Q$. This obviously is a Q–principle bundle $(p : \mathcal{G} \to \tilde{M})$. Since the cotangent space of the base manifold of a parabolic geometry is $\mathcal{G} \times_P \mathfrak{p}_+ = T^*M$ we identify the Fefferman space $\tilde{M} = \mathcal{G} \times_P P/Q$ with the space of positive rays $\mathcal{P}_+^r T^*M$ in T^*M.

The properties for a connection to be Cartan (equivariance, reproduction of generators of fundamental vector fields and absolute parallelism) are weaker for $(\mathcal{G} \to \tilde{M})$, so ω is still a Cartan connection. Define $\tilde{\mathcal{G}} := \mathcal{G} \times_Q \tilde{P}$, which by definition is a $\tilde{P}$–principal bundle over $\tilde{M}$. So $\mathcal{G} \subset \tilde{\mathcal{G}}$ and by the defining properties of Cartan connections it is easy to see, that there exists a unique Cartan connection $\tilde{\omega}$ such that $\tilde{\omega}|_{T\mathcal{G}} = \omega$. We get a Lie contact structure underlying the parabolic geometry $(\tilde{p} : \tilde{\mathcal{G}} \to \tilde{M}, \tilde{\omega})$ if $\tilde{\omega}$ is regular. Furthermore, the question arises whether $(\tilde{p} : \tilde{\mathcal{G}} \to \tilde{M}, \tilde{\omega})$ even coincides with the unique regular normal parabolic geometry associated to a Lie contact structure on $\tilde{M}$. So we investigate whether normality of ω implies regularity and normality of $\tilde{\omega}$.

3.1. The curvature of $\tilde{\omega}$.

The curvature form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ is given by

$$K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$$

for vector fields $\xi, \eta \in \mathfrak{X}(\mathcal{G})$. The Maurer–Cartan equation states, that the curvature vanishes for the Maurer–Cartan form ω^{MC} on the homogeneous model. The curvature of a Cartan geometry is a complete obstruction against local flatness, i.e. being locally isomorphic with the homogeneous model.

Since K is horizontal (i.e.: vanishes upon insertion of vertical vectors) the curvature can be equivalently expressed in terms of the curvature function

$$\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g},\ \kappa(u)(X,Y) := K\big(\omega_u^{-1}(X), \omega_u^{-1}(Y)\big)\,.$$

One can characterize regularity, as well as normality for a parabolic geometry in terms of the curvature of the Cartan connection ω as follows. The condition of regularity for a parabolic geometry is, that $\kappa(u)(\mathfrak{g}_i, \mathfrak{g}_j) \subset \mathfrak{g}^{i+j+1}$ for all $u \in \mathcal{G}$ and all i, j. Let $X_j \in \mathfrak{g}$, such that these descend to a basis of $\mathfrak{g}/\mathfrak{p}$ and $Z_j \in \mathfrak{p}_+$ be dual with respect to the Killing form. Then normality reads like

$$\sum_j \big[Z_j, \kappa(u)(X_j, A)\big] + \frac{1}{2}\sum_j \kappa(u)\big([Z_j, A], X_j\big) = 0 \tag{1}$$

for all $u \in \mathcal{G}$ and $A \in \mathfrak{g}_-$.

Since we have an isomorphism $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \cong \mathfrak{g}/\mathfrak{q}$ and we know that $\mathfrak{q} \subset \mathfrak{p}$ we get a natural projection $\psi : \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \to \mathfrak{g}/\mathfrak{p}$. By construction of $\tilde{\omega}$ as equivariant extension of ω we get the curvature function of $\tilde{\omega}$ in a point $u \in \mathcal{G}$ as follows

$$\begin{tikzcd} \Lambda^2\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \arrow[r, dashed, "\tilde{\kappa}(u)"] \arrow[d, "\Lambda^2\psi"'] & \tilde{\mathfrak{g}} \\ \Lambda^2\mathfrak{g}/\mathfrak{p} \arrow[r, "\kappa(u)"] & \mathfrak{g} \arrow[u, hook, "\iota"'] \end{tikzcd} \tag{2}$$

We analyze the relation between ω and $\tilde{\omega}$ to get the following proposition.

Proposition 3.2. Let $(p : \mathcal{G} \to M, \omega)$ be a normal parabolic geometry of type (G, P), then the parabolic geometry $(\tilde{p} : \tilde{\mathcal{G}} \to \tilde{M})$ of type $(\tilde{G}, \tilde{P})$ on the Fefferman space obtained by extension like above is regular and normal.

Proof. Since $\mathfrak{g}$ is $|1|$–graded and κ is horizontal the only condition for regularity is $\kappa(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) \subset \mathfrak{g}_{-1}$, which is trivially fulfilled and so we do not ask for ω to be regular. It is well known that κ has values in $\mathfrak{p}$ and the its $\mathfrak{g}_0$–component is the Weyl curvature and hence totally trace–free. By (2) we know, that for $u \in \mathcal{G}$, $\tilde{\kappa}(u)$ has values in $\mathfrak{g}$ and from the explicit description of $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ one reads off, that $\mathfrak{p} \cap \tilde{\mathfrak{g}}_{-2} = 0$. This shows the only non–trivial condition for regularity of $\tilde{\omega}$ holds, that is $\tilde{\kappa}(u)(\tilde{\mathfrak{g}}_{-1}, \tilde{\mathfrak{g}}_{-1}) \subset \tilde{\mathfrak{g}}_{-1}$ for $u \in \mathcal{G}$. Since $\tilde{\omega}$ is the equivariant extension this holds for all $u \in \tilde{\mathcal{G}}$.

Our first observation concerning normality is, that the second sum of (1) vanishes if $\mathfrak{g}$ is $|1|$–graded, since $[Z_j, X] \in \mathfrak{g}_0 \subset \mathfrak{p}$ and κ is horizontal. So for a connection ω of a parabolic geometry of type (G, P) to be normal means vanishing of both terms of the normality condition on their own. In general this is not true for $|2|$–graded Lie algebras. However the claim is, that both sums vanish for $\tilde{\omega}$ in this situation.

We choose dual bases $\{\tilde{Z}_1, \dots, \tilde{Z}_{2n+1}\}$ for $\tilde{\mathfrak{p}}_+$ and $\{\tilde{X}_1, \dots, \tilde{X}_{2n+1}\}$ for $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ with respect to the Killing form satisfying properties 2.4 from above.

We know use the special nature of $\tilde{\kappa} = i \circ \kappa(u) \circ \Lambda^2\psi$ and the fact that ω is normal to see that huge parts of (1) vanish. From (ii) we know, that all summands which contain $\tilde{\kappa}(u)(\tilde{X}_i, \_)$ for $i = 1, \dots, n$ vanish. The rest $\{\psi(\tilde{X}_{n+1}), \dots, \psi(\tilde{X})_{2n+1}\}$ form a basis of $\mathfrak{g}/\mathfrak{p}$ which is dual to $\{\tilde{Z}^{\mathfrak{g}}_{n+1}, \dots, \tilde{Z}^{\mathfrak{g}}_{2n+1}\}$ in $\mathfrak{g}$ by property (iii), with $\tilde{Z}^{\mathfrak{g}}_i = \tilde{Z}_i$ for all $i = n+1, \dots, 2n$. By normality of ω these terms sum up to 0.

The only non–trivial terms remaining are

$$[\tilde{Z}^{\mathfrak{n}}_{2n+1}, \tilde{\kappa}(u)(\tilde{X}_{2n+1}, \tilde{A})] + \frac{1}{2}\tilde{\kappa}(u)([\tilde{Z}^{\mathfrak{n}}_{2n+1}, \tilde{A}], \tilde{X}_{2n+1})$$

The second of these vanishes, since $\psi([\tilde{Z}^{\mathfrak{n}}_{2n+1}, \tilde{A}]) \in \mathfrak{p}$ for all $\tilde{A} \in \tilde{\mathfrak{g}}_-$. The first one is a little trickier. In the explicit basis from above we can write $\tilde{Z}^{\mathfrak{n}}_{2n+1}$ as

$$\tilde{Z}^{\mathfrak{n}}_{2n+1} = e^{\mathfrak{n}} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We analyze the action of $e^{\mathfrak{n}}$ and $\tilde{\kappa}(u)(\tilde{X}_{2n+1}, \tilde{A})$ on $\tilde{V} = V \oplus \mathbb{R}\tilde{v}_0$. By (2) we know that $\tilde{\kappa}(u)(\tilde{X}_{2n+1}, \tilde{A}) \in \mathfrak{g}$ and therefore restricts to an action on V, vanishing on $\tilde{v}_0$. For the action of $e^{\mathfrak{n}}$ one explicitly calculates that its action vanishes on V, and maps $\tilde{v}_0$ to a multiple of e_1. So $\tilde{\kappa}(u)(\tilde{X}_{2n+1}, \tilde{A}) \circ e^{\mathfrak{n}}$ acts trivially and $e^{\mathfrak{n}} \circ \tilde{\kappa}(u)(\tilde{X}_{2n+1}, \tilde{A})$ acts trivially on V. The part remaining is the image of $\tilde{v}_0$. Now $\tilde{v}_0$ goes to a multiple of e_1 under $e^{\mathfrak{n}}$.

We noted, that the $\mathfrak{g}_0$–part of the curvature of a conformal geometry is exactly the totally trace–free part of the curvature. As one can read off the explicit description of $\mathfrak{g}$ this implies, that the curvature vanishes on multiples of e_1 and we have shown, that $\tilde{\omega}$ is normal.

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