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SPLIT OCTONIONS AND GENERIC RANK TWO DISTRIBUTIONS IN DIMENSION FIVE

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Abstract. In his famous five variables paper Elie Cartan showed that one can canonically associate to a generic rank 2 distribution on a 5 dimensional manifold a Cartan geometry modeled on the homogeneous space $\tilde{G}_2/P$, where P is one of the maximal parabolic subgroups of the exceptional Lie group $\tilde{G}_2$. In this article, we use the algebra of split octonions to give an explicit global description of the distribution corresponding to the homogeneous model.

1. Introduction

The study of generic rank 2 distributions on 5-dimensional manifolds starts in 1910 with Elie Cartan. Cartan was trying to find local normal forms for distributions. This worked well for distributions on manifolds of dimension smaller or equal to 4. In dimension 5 however, he found that generic rank 2 distributions had local invariants, so in particular there could not be any normal form.

Cartan considered rank 2 distributions on 5-dimensional manifolds that are as non-integrable as possible, meaning that local sections of the distribution and their Lie brackets generate a rank 3 distribution and taking triple brackets as well generates the whole tangent bundle. He could show that one can canonically associate to such a distribution what is now called a Cartan geometry of type $(\tilde{G}_2, P)$, where $\tilde{G}_2$ is a Lie group with Lie algebra the split real form of the exceptional complex Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$ and P a certain parabolic subgroup in $\tilde{G}_2$. In modern terminology a Cartan geometry of type $(\tilde{G}_2, P)$ is a P-principal bundle $\mathcal{G} \to M$ together with a certain $\tilde{\mathfrak{g}}_2$-valued 1-form ω which is called Cartan connection. The curvature $K = d\omega + \frac{1}{2}[\omega, \omega]$ of the Cartan connection is a local invariant. Vanishing of the curvature implies that the Cartan geometry is locally equivalent to the so-called homogeneous model, i.e. the principal bundle $\tilde{G}_2 \to \tilde{G}_2/P$ together with the Maurer Cartan form. Cartan gave a local description of the 'flat' distribution in terms of some differential equations. The Lie group $\tilde{G}_2$ can be realized as the symmetry group of this distribution.

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Nowadays, one thinks of the exceptional Lie groups as being related to certain nonassociative algebras. In 1914, Cartan noted that one can realize the compact form G_2 as the automorphism group of the octonions. There is a similar description for the split real form. It can be realized as the automorphism group of the 8-dimensional algebra of split octonions. We will use this realisation to give a global explicit description of the rank two distribution corresponding to the homogeneous model.

2. Split Octonions and $\tilde{G}_2$

We start by collecting some information on split octonions and their relation to the exceptional group $\tilde{G}_2$ that we will need in the sequel. We mean by $\tilde{G}_2$ the split real form of the complex Lie group. The main reference for this section is [5].

A *composition algebra* over $\mathbb{R}$ is by definition a real algebra $\mathcal{A}$ with unit E such that there exists a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathcal{A}$ satisfying

$$\langle XY, XY\rangle = \langle X, X\rangle\langle Y, Y\rangle$$

for all $X, Y \in \mathcal{A}$.

Given a composition algebra, there is a notion of *conjugation*, defined as the reflection with respect to the unit, i.e.

$$\overline{X} = 2\langle X, E\rangle E - X\,.$$

It is not difficult to verify that the bilinear form can be written in terms of conjugation as

$$\langle X, Y\rangle = \frac{1}{2}(\overline{X}Y + \overline{Y}X)\,.$$

Composition algebras are not necessarily associative. However, for all composition algebras a weaker property holds. They are *alternative*, meaning that any subalgebra generated by two elements is associative, or equivalently, that the associator

$$\{X, Y, Z\} := (XY)Z - X(YZ)$$

is skew-symmetric.

The classification of real composition algebras is as follows: In dimension 1 we have the real numbers and the only other possible dimensions for composition algebras are 2, 4 and 8. In each of these dimensions there are up to isomorphism two different real composition algebras. Either the bilinear form is positive definite which leads to the complex numbers in dimension 2, the quaternions in dimension 4 and the octonions in dimension 8, or the bilinear form is indefinite. In case it is indefinite, one can show that the dimension of a maximal totally isotropic subspace is half the dimension of the algebra, so the signature of the bilinear form is either $(1,1)$, $(2,2)$ or $(4,4)$. The corresponding composition algebras are called split complex numbers, split quaternions and split octonions, respectively.

We will now focus on the algebra of *split octonions*, i.e. the unique 8-dimensional composition algebra over $\mathbb{R}$ with indefinite bilinear form. An explicit realisation of the split octonions is given by the set of vector matrices $\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix}$, where $x, y \in \mathbb{R}^3$

and $\xi, \eta \in \mathbb{R}$, with addition defined componentwise and multiplication defined by

$$\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} \begin{pmatrix} \xi' & x' \\ y' & \eta' \end{pmatrix} = \begin{pmatrix} \xi\xi' + \langle x, y' \rangle & \xi x' + \eta' x + y \wedge y' \\ \eta y' + \xi' y - x \wedge x' & \eta\eta' + \langle y, x' \rangle \end{pmatrix},$$

where $\langle \ , \ \rangle$ denotes the Euclidean inner product on $\mathbb{R}^3$ and $\wedge$ the cross product. The bilinear form on the algebra is given by

$$\left\langle \begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix}, \begin{pmatrix} \xi' & x' \\ y' & \eta' \end{pmatrix} \right\rangle = \xi'\eta + \eta'\xi - \langle x, y' \rangle - \langle x', y \rangle \, .$$

The unit is the element $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The orthogonal complement to the unit is called the set of imaginary split octonions and denoted by $\operatorname{Im} \mathbb{O}_S$. It is given by all vector matrices of the form $\begin{pmatrix} \xi & x \\ y & -\xi \end{pmatrix}$. Restricting the bilinear form on the split octonions to $\operatorname{Im} \mathbb{O}_S$ gives a bilinear form of signature $(3, 4)$. With respect to the basis $X_1 = \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix}$, $X_3 = \begin{pmatrix} 0 & e_3 \\ 0 & 0 \end{pmatrix}$, $X_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X_5 = \begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}$, $X_6 = \begin{pmatrix} 0 & 0 \\ -e_2 & 0 \end{pmatrix}$ and $X_7 = \begin{pmatrix} 0 & 0 \\ -e_3 & 0 \end{pmatrix}$, e_1, e_2, e_3 denoting the standard basis of $\mathbb{R}^3$, the bilinear form on $\operatorname{Im} \mathbb{O}_S$ is represented by the matrix $\begin{pmatrix} & & \mathbb{I}_3 \\ & -1 & \\ \mathbb{I}_3 & & \end{pmatrix}$, where $\mathbb{I}_3$ denotes the 3×3 unit matrix.

In the description of the rank 2 distribution corresponding to the homogeneous model we will make use of a certain 3-form $\phi \in \Lambda^3((\operatorname{Im}\mathbb{O}_S)^*)$. Note that

$$\phi(X, Y, Z) = \langle XY, Z \rangle$$

defines a 3-form, i.e. is skew-symmetric. This can either be verified directly using the explicit formulas for multiplication and bilinear form from above, or else one can observe the following:

We obviously have $\overline{X} = -X$ for imaginary split octonions and one can verify that $\overline{XY} = \overline{Y}\,\overline{X}$. It follows that for $X, Y \in \operatorname{Im}\mathbb{O}_S$, the expression $XY + YX = XY + \overline{XY}$ is a real multiple of the unit E and thus orthogonal to $\operatorname{Im} \mathbb{O}_S$. Hence

$$\langle XY, Z \rangle = -\langle YX, Z \rangle$$

for all $X, Y, Z \in \operatorname{Im}(\mathbb{O}_S)$. Next, we have $\langle X, Y \rangle = \frac{1}{2}(X\overline{Y} + Y\overline{X})$. Hence, for $X, Y \in \operatorname{Im}(\mathbb{O}_S)$ we have

$$\langle XY, X \rangle = \frac{1}{2}\big((XY)\overline{X} + X(\overline{XY})\big) = \frac{1}{2}\big(- (XY)X + X(YX)\big) \, .$$

Since split octonions are not associative, it is not immediately clear that this expression vanishes. However, as mentioned above, they are alternative, meaning that any subalgebra generated by two elements is associative. This sufficies to conclude that

$$\langle XY, X \rangle = 0 \, .$$

It follows that ϕ is alternating.

Remark. Let V be a 7-dimensional vector space. One can show that there are exactly two open $GL(V)$-orbits in $\Lambda^3(V^*)$. In case of one of these orbits, the stabilizer of an element is the compact real form G_2. This is well-known and used in the study of manifolds with exceptional holonomy. For the other open orbit, the stabilizer is the split real form of the complex group $\tilde{G}_2$. The above 3-form ϕ is contained in the latter orbit.

Let us denote by $\mathrm{Aut}(\mathbb{O}_S)$ the group of algebra automorphisms of the split octonions. One can prove that given any composition algebra, an algebra automorphism necessarily preserves the corresponding bilinear form on the algebra, see [5]. It follows that $\mathrm{Aut}(\mathbb{O}_S)$ preserves $\mathrm{Im}\,\mathbb{O}_S$ and its restriction to $\mathrm{Im}\,\mathbb{O}_S$ is contained in the orthogonal group $O(3,4)$. Moreover, $\mathrm{Aut}(\mathbb{O}_S)$ preserves the 3-form ϕ.

The Lie algebra corresponding to the automorphism group $\mathrm{Aut}(\mathbb{O}_S)$, the algebra $\mathrm{Der}\,\mathbb{O}_S$ of derivations of the split octonions, can thus be viewed as a subalgebra in $\mathfrak{so}(3,4)$. Writing $\mathfrak{so}(3,4)$ with respect to the quadratic form

$$\begin{pmatrix} & & \mathbb{I}_3 \\ & -1 & \\ \mathbb{I}_3 & & \end{pmatrix},$$

it consists of matrices of the following form

$$\mathfrak{so}(3,4) = \left\{ \begin{pmatrix} A & v & B \\ w & 0 & v^t \\ C & w^t & -A^t \end{pmatrix} : C = -C^t, B = -B^t \right\}.$$

Now we can determine those matrices in $\mathfrak{so}(3,4)$ that act as derivations on $\mathrm{Im}\,\mathbb{O}_S$: Consider for example the equation

$$\begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -e_3 & 0 \end{pmatrix}.$$

Assuming that $M = \begin{pmatrix} A & v & B \\ w & 0 & v^t \\ C & w^t & -A^t \end{pmatrix}$ acts as a derivation, i.e.

$$M \cdot \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix} M \cdot \begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix} = M \cdot \begin{pmatrix} 0 & 0 \\ -e_3 & 0 \end{pmatrix},$$

implies

$$(Ae_1) \wedge e_2 + e_1 \wedge (Ae_2) = -A^t e_3$$

which is equivalent to A being tracefree. Similar computations show that $Ce_i = \frac{1}{\sqrt{2}}(v \wedge e_i)$ and $Be_i = -\frac{1}{\sqrt{2}}(w^t \wedge e_i)$. Thus we get a description of $\mathrm{Der}\,\mathbb{O}_S$ as matrices in $\mathfrak{so}(3,4)$ satisfying the following additional conditions:

$$A \in \mathfrak{sl}(3,\mathbb{R}), \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix}, \quad C = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}.$$

Next, we can consider the subalgebra of diagonal matrices in $\mathrm{Der}\,\mathbb{O}_S$. Via the adjoint action it acts diagonalisable on $\mathrm{Der}\,\mathbb{O}_S$ and we get the corresponding eigenspace decomposition. Once we have this eigenspace decomposition, it is not difficult to see that $\operatorname{Der}\mathbb{O}_S$ cannot contain any solvable ideals. Hence it is semisimple, the subalgebra of diagonal matrices is a Cartan subalgebra and we can write down the corresponding root system. Let us denote by ϕ_i the linear functional that extracts the ith diagonal entry. Choosing as simple roots $\alpha_1 = \phi_1 - \phi_2$ and $\alpha_2 = \phi_2$ we obtain the following set of positive roots:

$$\Delta^+ = \{\, \alpha_1, \alpha_2, \alpha_1 + \alpha_2 = \phi_1, \alpha_1 + 2\alpha_2 = \phi_1 + \phi_2 = -\phi_3,\\ 2\alpha_1 + 3\alpha_2 = \phi_1 - \phi_3, \alpha_1 + 3\alpha_2 = \phi_2 - \phi_3 \,\}\,.$$

¿From the root system we see that the Lie algebra $\operatorname{Der}\mathbb{O}_S$ is indeed the exeptional Lie algebra $\tilde{\mathfrak{g}}_2$.

The description of $\tilde{\mathfrak{g}}_2$ as a subalgebra in $\mathfrak{so}(3,4)$ also enables us to understand the 'standard' representation of $\tilde{\mathfrak{g}}_2 = \operatorname{Der}\mathbb{O}_S$ on $\operatorname{Im}\mathbb{O}_S$ in terms of weights. It turns out that $X_7 = \begin{pmatrix} 0 & 0 \\ -e_3 & 0 \end{pmatrix}$ is annihilated by all positive root spaces and thus a highest weight vector. Its weight $-\phi_3 = \alpha_1 + 2\alpha_2$ is the fundamental weight dual to the shorter simple root α_2. Other weight vectors and their weights are:

$$\begin{array}{ll} X_1 & \alpha_1 + \alpha_2 \\ X_2 & \alpha_2 \\ X_4 & 0 \\ X_6 & -\alpha_2 \\ X_5 & -\alpha_1 - \alpha_2 \\ X_3 & -\alpha_1 - 2\alpha_2\,. \end{array}$$

3. Some background on Cartan geometries

In this section we provide some basic information on Cartan geometries.

Consider a Lie group G, a closed subgroup $P \subset G$ and the corresponding homogeneous space G/P. One can associate to such a homogeneous space the following geometric structure:

- the principal bundle $G \to G/P$,
- the Maurer Cartan form $\omega_{MC} \in \Omega^1(G, \mathfrak{g})$ defined by

  $$\omega(g)(\xi) = T_g\lambda_{g^{-1}}\xi\,,$$

  where $\lambda_{g^{-1}}$ denotes left multiplication by g^{-1}.

It turns out that the automorphisms of this structure, i.e. the principal bundle automorphisms Φ satisfying $\Phi^*\omega_{MC} = \omega_{MC}$, are exactly the left multiplications by elements in the group G.

The notion of a Cartan geometry generalizes this structure, introducing a concept of curvature:

Definition. A *Cartan geometry* of type (G, P) on a manifold M is given by a principal P-bundle $\mathcal{G} \to M$ and a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. A *Cartan connection* is a smooth one-form on $\mathcal{G}$ with values in $\mathfrak{g}$, which is P-equivariant, reproduces generators of fundamental vector fields and satisfies the condition that $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

Since the Maurer Cartan form satisfies the properties required in the definition of a Cartan connection, the principal bundle $G \to G/P$ endowed with the Maurer Cartan form is a Cartan geometry of type (G, P). It is called the *homogeneous model* of a Cartan geometry of type (G, P).

The Maurer Cartan form has a property that general Cartan connections don't have. It satisfies the Maurer-Cartan equation $d\omega(\xi, \eta) + \big[\omega(\xi), \omega(\eta)\big] = 0$. In general, the 2-form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$

$$K = d\omega + \frac{1}{2}[\omega, \omega]$$

is called the curvature of the Cartan geometry. One can show that a Cartan geometry with vanishing curvature is locally equivalent to the homogeneous model.

4. Generic rank 2 distributions on 5 dimensional manifolds and Cartan geometries

Having collected some general backround, we return to the discussion of generic rank two distributions on five dimensional manifolds. In his famous five variables paper [3], Elie Cartan considered distributions $\mathcal{H}$ of rank two on five-dimensional manifolds, satisfying the following generic condition:

$$\mathcal{H}^1 := \mathcal{H} + \big\{[\xi, \eta] : \xi, \eta \in \Gamma(\mathcal{H})\big\}$$

is a distribution of rank three and

$$\mathcal{H}^1 + \big\{[\xi, \eta] : \xi \in \Gamma(\mathcal{H}), \eta \in \Gamma(\mathcal{H}^1)\big\}$$

is equal to the whole tangent bundle TM. Any such distribution corresponds to a filtration $\mathcal{H} \subset \mathcal{H}^1 \subset TM$, such that the Lie bracket of vector fields induces bijective tensorial mappings $\mathcal{L} : \Lambda^2\mathcal{H} \to \mathcal{H}^1/\mathcal{H}$ and $\mathcal{L} : \mathcal{H} \otimes \mathcal{H}^1/\mathcal{H} \to TM/\mathcal{H}^1$. These rank two distributions are related to $\tilde{G}_2$ and a certain parabolic subgroup P.

Recall that for complex semisimple Lie algebras as well as for their split real forms, standard parabolic subalgebras are in bijective correspondence with subsets of the set of simple roots, see [1]. The Lie algebra $\tilde{\mathfrak{g}}_2$ has two simple roots, a longer one α_1 and a shorter one α_2. Correspondingly, there are two maximal standard parabolic subalgebras. We denote by $\mathfrak{p}$ the one consisting of the Cartan subalgebra $\mathfrak{h}$, all positive root spaces and the root space corresponding to $-\alpha_1$. Using the notation for parabolic subalgebras in terms of Dynkin diagrams, see [1], $\mathfrak{p}$ is represented by $\circ\!\Rrightarrow\!\times$. The connected subgroup in $\tilde{G}_2$ with Lie algebra $\mathfrak{p}$ will be denoted by P. Let X_7 be a highest weight vector in the representation of $\tilde{G}_2$ on $\mathrm{Im}\mathbb{O}_S$. Then the parabolic subgroup P can be realized as the connected component of the stabilizer of the real line through X_7, or equivalently, as the stabilizer of the ray through X_7.

Any parabolic subalgebra gives rise to a grading on $\tilde{\mathfrak{g}}_2$ such that the parabolic can be recovered as the sum of all non-negative grading components. In case of the parabolic subalgebra $\mathfrak{p}$, we define $\mathfrak{g}_i$ to be the sum of all root spaces $\mathfrak{g}_\alpha$ such that in the decomposition of α into simple roots α_2 occurs with coefficient i. Thus, we obtain

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3\,,$$

where $\mathfrak{g}_{-3}$ is the direct sum of the root spaces corresponding to $-(3\alpha_2 + 2\alpha_1)$ and $-(3\alpha_2 + \alpha_1)$, $\mathfrak{g}_{-2}$ is the root space corresponding to $-(2\alpha_2 + \alpha_1)$ and $\mathfrak{g}_{-1}$ is the direct sum of the root spaces corresponding to $-(\alpha_2 + \alpha_1)$ and $-\alpha_2$. Note that since $-(\alpha_2 + \alpha_1) - \alpha_2 = -(2\alpha_2 + \alpha_1)$ the Lie bracket $[\ ,\ ] : \Lambda^2\mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is surjective and by equality of dimensions it is an isomorphism. Similarly, we see that $[\ ,\ ] : \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2} \to \mathfrak{g}_{-3}$ is an isomorphism.

It follows, that the vector bundle $\mathrm{gr}(TM) = \mathcal{H} \oplus \mathcal{H}^1/\mathcal{H} \oplus TM/\mathcal{H}^1$ corresponding to a generic rank 2 distribution together with the tensor $\mathcal{L}$ induced by the Lie bracket of vector fields is a bundle of graded Lie algebras such that each fibre is isomorphic to the graded Lie algebra $\mathfrak{g}_- = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$.

Further, a theorem by Elie Cartan states that one can construct a Cartan geometry of type $(\tilde{G}_2, P)$ from a generic rank 2 distribution in dimension 5. In that way one obtains an equivalence of categories between generic distributions of rank two on five-dimensional manifolds and Cartan geometries of type $(\tilde{G}_2, P)$ satisfying a certain normalisation condition. Using modern tools, this result is obtained as a special case of Tanaka theory, see [6] for a treatment of this case, or the theory of parabolic geometries, see [2] and [4].

Note that recovering the generic rank 2 distribution from the Cartan geometry is easy: Given a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ the tangent bundle can be identified with the associated bundle $\tilde{G}_2 \times_P \mathfrak{g}/\mathfrak{p}$, where the P-action on $\mathfrak{g}/\mathfrak{p}$ is induced by the adjoint action. Explicitly, the isomorphism is given by

$$\begin{aligned} \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} &\cong TM \\ \big[(x, X + \mathfrak{p})\big] &\mapsto T_x p \cdot \omega(x)^{-1}(X)\,. \end{aligned}$$

The grading on the Lie algebra $\tilde{\mathfrak{g}}_2$ induces a P-invariant filtration $\mathfrak{g}/\mathfrak{p} \supset \mathfrak{g}^{-2}/\mathfrak{p} \supset \mathfrak{g}^{-1}/\mathfrak{p}$ on $\mathfrak{g}/\mathfrak{p}$, where $\mathfrak{g}^i = \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$. Via the above isomorphism this filtration gives rise to a a filtration $TM \supset T^{-2}M \supset T^{-1}M$ of the tangent bundle. The subbundle $T^{-1}M = \mathcal{G} \times_P \mathfrak{g}^{-1}/\mathfrak{p}$ is the rank 2 distribution corresponding to the Cartan geometry.

Applied to the homogeneous model $(\tilde{G}_2 \to \tilde{G}_2/P, \omega_{MC})$ we obtain the distribution

$$\mathcal{H} = \tilde{G}_2 \times_P \mathfrak{g}^{-1}/\mathfrak{p}$$

on $\tilde{G}_2/P$. In this special case, the equivalence between generic rank 2 distributions and Cartan geometries says that the diffeomorphisms of $\tilde{G}_2/P$ preserving $\mathcal{H}$ are exactly the left multiplications by elements of $\tilde{G}_2$. Our aim is to give a nice description of $\tilde{G}_2/P$ and $\mathcal{H}$.

5. Description of the homogeneous model

Theorem. *Let $\tilde{G}_2$ be the automorphism group of the split octonions and $P \subset G$ the maximal parabolic subgroup introduced in the previous section. We denote by*

$\mathcal{C}$ the cone of nonzero null vectors

$$\mathcal{C} = \left\{ X \in \operatorname{Im}\mathbb{O}_S : \ X \neq 0, \ \langle X, X \rangle = 0 \right\}$$

and by $\mathcal{C}/\mathbb{R}_+$ the space of rays in $\mathcal{C}$, i.e. the quotient where X and Y are identified if and only if $X = \lambda Y$ for $\lambda \in \mathbb{R}_+$. Then the action of $\tilde{G}_2$ on $\mathcal{C}/\mathbb{R}_+$ induces a diffeomorphism between $\tilde{G}_2/P$ and $\mathcal{C}/\mathbb{R}_+$. The tangent map of this diffeomorphism maps the distribution $\tilde{G}_2 \times_P \mathfrak{g}^{-1}/\mathfrak{p}$ on $\tilde{G}_2/P$ onto the distribution $\mathcal{H}$ on $\mathcal{C}/\mathbb{R}_+$ defined by

$$\mathcal{H}_{[X]} = \left\{ Y \in \operatorname{Im}(\mathbb{O}_S) : \phi(X, Y, Z) = 0 \ : \forall\, Z \in \operatorname{Im}(\mathbb{O}_S) \right\} / \ell\,,$$

where $X \in \mathcal{C}$, where $[X]$ denotes the ray through X and ℓ the real line through X.

Proof. The first step is to get the identification between the homogeneous space $\tilde{G}_2/P$ and the space of rays in the cone $\mathcal{C}$. The 'standard' representation of $\tilde{G}_2 = \operatorname{Aut}\mathbb{O}_S$ on $\operatorname{Im}\mathbb{O}_S$ preserves the bilinear form $\langle\ ,\ \rangle$. Hence, it preserves the cone $\mathcal{C}$ and we get an action of $\tilde{G}_2$ on the quotient $\mathcal{C}/\mathbb{R}_+$. Next note that the highest weight vector X_7 is a null vector. Thus, the ray $[X_7]$ through X_7 is contained in $\mathcal{C}/\mathbb{R}_+$ and we can consider its orbit under the action of $\tilde{G}_2$. The stabilizer of $[X_7]$ in $\tilde{G}_2$ is the parabolic subgroup P. This parabolic is 9-dimensional. We know that $\tilde{G}_2$ has dimension 14. Thus the homogeneous space $\tilde{G}_2/P$ has dimension 5. The space of rays $\mathcal{C}/\mathbb{R}_+$ has also dimension 5 and it follows that the $\tilde{G}_2$-orbit of $[X_7]$ is open in $\mathcal{C}/\mathbb{R}_+$. Moreover, we know that every quotient of a semisimple Lie group modulo a parabolic subgroup is compact. Hence the $\tilde{G}_2$-orbit of $[X_7]$ and the space of rays $\mathcal{C}/\mathbb{R}_+$ coincide and we get an identification

$$\tilde{G}_2/P \cong \tilde{G}_2 \cdot [X_7] = \mathcal{C}/\mathbb{R}_+\,.$$

Next, we come to the description of the rank two distribution $\tilde{G}_2 \times \mathfrak{g}^{-1}/\mathfrak{p}$ in this picture. Let us describe the tangent space of $\mathcal{C}/\mathbb{R}_+$ first. The tangent space of the cone $\mathcal{C}$ in a point X is given by the orthogonal complement to X, i.e. $T_X\mathcal{C} = \{Y \in \operatorname{Im}\mathbb{O}_S : \langle X, Y \rangle = 0\}$. We denote by p the projection $p : \mathcal{C} \to \mathcal{C}/\mathbb{R}_+$. Then the tangent space of $\mathcal{C}/\mathbb{R}_+$ in the point $[X]$ is given by the tangent space of $\mathcal{C}$ in X modulo the kernel of $T_X p$ which is the real line ℓ through X. Thus, we have

$$T_{[X]}(\mathcal{C}/\mathbb{R}_+) = \ell^{\perp}/\ell\,.$$

Now we consider the subspace $\mathbb{W}/\ell_7$ in $T_{[X_7]}(\mathcal{C}/\mathbb{R}_+) = \ell_7^{\perp}/\ell_7$, where

$$\mathbb{W} = \left\{ Y \in \operatorname{Im}(\mathbb{O}_S) : \phi(X_7, Y, Z) = 0 \ \forall\, Z \in \operatorname{Im}(\mathbb{O}_S) \right\},$$

and ϕ denotes the generic 3-form $\phi(X, Y, Z) = \langle XY, Z \rangle$. We claim that this subspace describes the rank 2 distribution $\mathcal{H}$ in the point $[X_7]$.

To verify the claim take an arbitrary imaginary split octonion $Y = \begin{pmatrix} \xi & x \\ y & -\xi \end{pmatrix}$. Note that $\langle X_7 Y, Z \rangle = 0$ for all $Z \in \operatorname{Im}\mathbb{O}_S$ if and only if the product $X_7 Y$ is a real multiple of the identity E. We have

$$\begin{pmatrix} 0 & 0 \\ -e_3 & 0 \end{pmatrix} \begin{pmatrix} \xi & x \\ y & -\xi \end{pmatrix} = \begin{pmatrix} 0 & -e_3 \wedge y \\ -\xi e_3 & -\langle e_3, x \rangle \end{pmatrix}\,.$$

The right hand side is a multiple of the identity if and only if it vanishes and this is the case if and only if Y has zeros in the diagonal, the entry y is a real multiple of e_3 and x is orthogonal to e_3. Comparing with the weight decomposition of the standard representation on $\operatorname{Im}\mathbb{O}_S$ at the end of section 2, we conclude that $\mathbb{W}$ is the sum of the following weight spaces

$$\mathbb{W} = \mathbb{V}_{\alpha_1+2\alpha_2} \oplus \mathbb{V}_{\alpha_1+\alpha_2} \oplus \mathbb{V}_{\alpha_2}\,.$$

The isomorphim between the tangent spaces $T_{eP}(\tilde{G}_2/P)$ and $T_{[X_7]}(\mathcal{C}/\mathbb{R}_+)$ is given by

$$\begin{aligned}\tilde{\mathfrak{g}}_2/\mathfrak{p} &\cong \ell_7^{\perp}/\ell_7 \\ A+\mathfrak{p} &\mapsto A\cdot\ell_7+\ell_7\,.\end{aligned}$$

The filtration component $\mathfrak{g}^{-1}$ consists of the parabolic subalgebra $\mathfrak{p}$ and the root spaces corresponding to the roots $-\alpha_2$ and $-\alpha_1-\alpha_2$. The parabolic subalgebra $\mathfrak{p}$ stabilizes the line ℓ_7 and the remaining two root spaces map ℓ_7 onto the weight spaces $\mathbb{V}_{\alpha_1+\alpha_2}$ respectively $\mathbb{V}_{\alpha_2}$. It follows that the isomorphism maps $\mathfrak{g}^{-1}/\mathfrak{p}$ onto $\mathbb{W}/\ell_7$.

Finally, note that since $\tilde{G}_2$ preserves the 3-form ϕ, it follows that the distribution $\mathcal{H}$ is given in any point $[X]\in\mathcal{C}/\mathbb{R}_+$ by

$$\mathcal{H}_{[X]} = \big\{\,Y\in\operatorname{Im}(\mathbb{O}_S) : \phi(X,Y,Z)=0\;\forall Z\in\operatorname{Im}(\mathbb{O}_S)\big\}/\ell\,.$$

□

The theorem gives a description of the distribution corresponding to the homogeneous model $(\tilde{G}_2\to\tilde{G}_2/P,\omega_{MC})$ on the space $\mathcal{C}/\mathbb{R}_+$ of rays in the cone $\mathcal{C}$. Now $\mathcal{C}/\mathbb{R}_+$ is diffeomorphic to $S^2\times S^3$, an explicit diffeomorphism is given by

$$\begin{aligned}S^2\times S^3 &\to \mathcal{C}/\mathbb{R}_+ \\ \big(x,(\alpha,y)\big) &\mapsto \mathbb{R}_+\cdot\begin{pmatrix}\alpha & y-x\\ y+x & -\alpha\end{pmatrix}\,.\end{aligned}$$

Hence we can try to find a description of the distribution on $S^2\times S^3$ as well, which leads to the following result.

Corollary. *Let $\mathcal{H}$ be the distribution on $S^2\times S^3$ given by*

$$\mathcal{H}_{(x,(\alpha,y))} = \left\{\begin{matrix}\big(v,(\beta,w)\big)\subset\mathbb{R}^3\times\mathbb{R}^4: & \langle v,x\rangle=0,\ \beta=\langle v\wedge y,x\rangle\,, \\ & w=\langle v,y\rangle\,x+\alpha(v\wedge x)-\langle y,x\rangle\,v\end{matrix}\right\},$$

for any $(x,(\alpha,y))\in S^2\times S^3$. Then the group of diffeomorphisms of $S^2\times S^3$ such that the tangent maps preserve the distribution $\mathcal{H}$ is a Lie group with Lie algebra $\tilde{\mathfrak{g}}_2$.

Proof. In view of section 4 and the theorem, it remains to understand that via the tangent map of the diffeomorphism $S^2\times S^3\cong\mathcal{C}/\mathbb{R}_+$, the distribution described in the corollary is mapped onto the distribution described in the theorem.

We start with an element $\big(x,(\alpha,y)\big)$ in $S^2\times S^3$, that is $x\in\mathbb{R}^3$,$(\alpha,y)\in\mathbb{R}\times\mathbb{R}^3$ such that $|x|^2=1$ and $\alpha^2+|y|^2=1$. The tangent space of $S^2\times S^3$ in $(x,(\alpha,y))$ is given by

$$T_{(x,(\alpha,y))}(S^2\times S^3)=x^\perp\oplus(\alpha,y)^\perp,$$

i.e. $\big(v,(\beta,w)\big)\in T_{(x,(\alpha,y))}(S^2\times S^3)$ satisfies $\langle x,v\rangle=0$ and $\alpha\beta+\langle w,y\rangle=0$. The inclusion $\rho:S^2\times S^3\to\mathcal{C}$ maps $\big(x,(\alpha,y)\big)$ to $X=\begin{pmatrix}\alpha & y-x\\ y+x & -\alpha\end{pmatrix}$ and its tangent map is given by

$$T_{(x,(\alpha,y))}\rho\cdot\big(v,(\beta,w)\big)=\begin{pmatrix}\beta & w-v\\ w+v & -\beta\end{pmatrix}.$$

Now we consider the space $\mathcal{H}_{(x,(\alpha,y))}$ of all $(v,(\beta,w))$ where v is any vector orthogonal to x, $\beta=\langle v\wedge y,x\rangle$ and $w=\langle v,y\rangle\, x+\alpha(v\wedge x)-\langle y,x\rangle$. This is a subspace of $T_{(x,(\alpha,y))}(S^2\times S^3)$ since one easily verifies that $\alpha\beta+\langle w,y\rangle=0$ and obviously it is two dimensional. We need to verify that it coincides with the subspace of all elements in $T_{(x,(\alpha,y))}(S^2\times S^3)$ that are mapped via $T_{(x,(\alpha,y))}\rho\cdot\big(v,(\beta,w)\big)$ into the set of imaginary split octonions Y such that $\phi(X,Y,Z)=0$ for all $Z\in\operatorname{Im}(\mathbb{O}_S)$. Note that by dimensional reasons we are done, if we can show that $\mathcal{H}_{(x,(\alpha,y))}$ contains all elements mapped via $T_{(x,(\alpha,y))}\rho$ into the set of imaginary split octonions Y such that $\phi(X,Y,Z)=0$ for all $Z\in\operatorname{Im}(\mathbb{O}_S)$. This will be done in the following:

The set of imaginary split octonions Y such that $\phi(X,Y,Z)=0$ for all $Z\in\operatorname{Im}(\mathbb{O}_S)$ can equivalently be described as the set of all imaginary split octonions Y such that XY is a real multiple of the identity. Demanding that

$$\begin{pmatrix}\alpha & y-x\\ y+x & -\alpha\end{pmatrix}\begin{pmatrix}\beta & w-v\\ w+v & -\beta\end{pmatrix}=\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$$

leads to the following equations

$$\begin{aligned}&\alpha\beta+\langle y-x,v+w\rangle=\lambda\\&\alpha\beta+\langle y+x,w-v\rangle=\lambda\\&\alpha(w-v)-\beta(y-x)+(x+y)\wedge(v+w)=0\\&-\alpha(v+w)+\beta(y+x)-(y-x)\wedge(w-v)=0\,.\end{aligned}$$

Expanding these equations an subtracting the second from the first implies

$$\langle v,y\rangle=\langle w,x\rangle$$

and adding the fourth to the third gives

$$-\alpha v+\beta x+y\wedge v+x\wedge w=0\,.$$

Now we consider the inner product

$$\langle-\alpha v+\beta x+y\wedge v+x\wedge w,x\rangle=\beta+\langle y\wedge v,x\rangle$$

from which we derive

$$\beta=\langle v\wedge y,x\rangle\ .$$

Next

$$(w \wedge x) \wedge x = \langle w, x\rangle\, x - \langle x, x\rangle\, w = \langle v, y\rangle\, x - w$$

and

$$(w \wedge x) \wedge x = (-\alpha v + \beta x + y \wedge v) \wedge x = -\alpha(v \wedge x) + \langle y, x\rangle\, v$$

which gives

$$w = \langle v, y\rangle\, x + \alpha(v \wedge x) - \langle y, x\rangle\, v\,.$$

This concludes the proof of the corollary. □

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