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ALGEBRAIC THEORY OF AFFINE CURVATURE TENSORS

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Abstract. We use curvature decompositions to construct generating sets for the space of algebraic curvature tensors and for the space of tensors with the same symmetries as those of a torsion free, Ricci symmetric connection; the latter naturally appear in relative hypersurface theory.

1. Introduction

Let V be a real vector space of dimension m; to simplify the discussion, we shall assume that $m \geq 4$ henceforth; similar results hold in dimensions $m = 2$ and $m = 3$. In Section 2, we discuss the space of curvature operators $\mathfrak{R}(V) \subset \otimes^2 V^* \otimes \operatorname{End}(V)$. These are operators with the same symmetries as those of the curvature operator of a torsion free connection on the tangent bundle of a smooth manifold. One has that $\mathcal{R} \in \mathfrak{R}(V)$ if and only if for all $x, y, z \in V$,

$$\mathcal{R}(x,y)z = -\mathcal{R}(y,x)z \quad \text{and} \tag{1.a}$$

$$\mathcal{R}(x,y)z + \mathcal{R}(y,z)x + \mathcal{R}(z,x)y = 0\,. \tag{1.b}$$

Equation (1.b) is called the *first Bianchi identity*. We have, see for example Strichartz [7], that

$$\dim \mathcal{R}(V) = \tfrac{1}{3} m^2 (m^2 - 1)\,.$$

In Section 3, we discuss the space of algebraic curvature tensors $\mathfrak{a}(V) \subset \otimes^4 V^*$. This is the space of tensors with the same symmetries as that of the curvature tensor defined by the Levi-Civita connection of a pseudo-Riemannian metric; $A \in \mathfrak{a}(V)$ if and only if for all $x, y, z, w \in V$,

$$A(x,y,z,w) = -A(y,x,z,w), \tag{1.c}$$

$$A(x,y,z,w) + A(y,z,x,w) + A(z,x,y,w) = 0, \tag{1.d}$$

$$A(x,y,z,w) = A(z,w,x,y), \tag{1.e}$$

$$A(x,y,z,w) = -A(x,y,w,z)\,. \tag{1.f}$$

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The paper is in final form and no version of it will be submitted elsewhere.

We shall show in Theorem 3.2 that identities (1.e) and (1.f) are equivalent in the presence of identities (1.c) and (1.d). One has, see for example Strichartz [7], that:

$$\dim\{\mathfrak{a}(V)\} = \tfrac{m^2(m^2-1)}{12}\,.$$

If $\mathcal{R} \in \mathfrak{R}(V)$, it is natural to consider the traces:

$$\begin{aligned}
\rho_{14}(\mathcal{R})(x,y) &:= \operatorname{Tr}\left\{z \to \mathcal{R}(z,x)y\right\},\\
\rho_{24}(\mathcal{R})(x,y) &:= \operatorname{Tr}\left\{z \to \mathcal{R}(x,z)y\right\},\\
\rho_{34}(\mathcal{R})(x,y) &:= \operatorname{Tr}\left\{z \to \mathcal{R}(x,y)z\right\}.
\end{aligned} \tag{1.g}$$

The identities of Equations (1.a) and (1.b) show that:

$$\begin{aligned}
&\rho_{24}(\mathcal{R}) = -\rho_{14}(\mathcal{R}) \quad \text{and}\\
&\rho_{34}(\mathcal{R})(x,y) = -\rho_{14}(\mathcal{R})(x,y) + \rho_{14}(\mathcal{R})(y,x)\,.
\end{aligned} \tag{1.h}$$

In Section 4, we discuss the affine curvature operators $\mathfrak{F}(V) \subset \mathfrak{R}(V)$. These are the operators with the same symmetries as those of an affine connection without torsion; $\mathcal{F} \in \mathfrak{F}(V)$ if and only if for all $x, y, z \in V$,

$$\mathcal{F}(x,y)z = -\mathcal{F}(y,x)z, \tag{1.i}$$

$$\mathcal{F}(x,y)z + \mathcal{F}(y,z)x + \mathcal{F}(z,x)y = 0, \tag{1.j}$$

$$\rho_{14}(\mathcal{F})(x,y) = \rho_{14}(\mathcal{F})(y,x), \tag{1.k}$$

$$\rho_{34}(\mathcal{F}) = 0\,. \tag{1.l}$$

By Equation (1.h), Equations (1.k) and (1.l) are equivalent in the presence of Equations (1.i) and (1.j); thus these are the symmetries of the curvature operator of a torsion free, Ricci symmetric connection on the tangent bundle of a smooth manifold. Such curvature operators appear naturally as curvature operators of the induced and of the conormal connections in relative hypersurface theory; see [6].

The natural structure group of the spaces $\mathfrak{R}(V)$, $\mathfrak{a}(V)$, and $\mathfrak{F}(V)$ is the general linear group $GL(V)$. Let $O(V, \langle\cdot,\cdot\rangle)$ be the orthogonal group associated to a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle \in S^2(V^*)$ of signature (p,q) on V. We can use $\langle\cdot,\cdot\rangle$ to raise and lower indices and define an $O(V,\langle\cdot,\cdot\rangle)$ equivariant identification between $\otimes^4 V^*$ and $\otimes^2 V^* \otimes \operatorname{End}(V)$ by means of the identity:

$$R(x,y,z,w) = \langle \mathcal{R}(x,y)z, w\rangle\,. \tag{1.m}$$

We let

$$\mathfrak{r}(V) \subset \otimes^4 V^*, \quad \mathfrak{A}(V,\langle\cdot,\cdot\rangle) \subset \otimes^2 V^* \otimes \operatorname{End}(V), \quad \mathfrak{f}(V,\langle\cdot,\cdot\rangle) \subset \otimes^4 V^*$$

be the subspaces associated to $\mathfrak{R}(V)$, $\mathfrak{a}(V)$, and $\mathfrak{F}(V)$, respectively; $R \in \mathfrak{r}(V)$ if and only if for all $x, y, z, w \in V$, one has

$$\begin{aligned}
&R(x,y,z,w) = -R(y,x,z,w),\\
&R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0\,.
\end{aligned}$$

We have $\mathcal{A}\in\mathfrak{A}(V,\langle\cdot,\cdot\rangle)$ if and only if for all $x,y,z,w\in V$ one has:

$$\begin{aligned}
&\mathcal{A}(x,y)=-\mathcal{A}(y,x),\\
&\mathcal{A}(x,y)z+\mathcal{A}(y,z)x+\mathcal{A}(z,x)y=0,\\
&\langle\mathcal{A}(x,y)z,w\rangle=\langle\mathcal{A}(z,w)x,y\rangle,\\
&\langle\mathcal{A}(x,y)z,w\rangle=-\langle\mathcal{A}(x,y)w,z\rangle,
\end{aligned}$$

the last two identities being equivalent in the presence of the first two. Finally $F\in\mathfrak{f}(V,\langle\cdot,\cdot\rangle)$ if and only if for all $x,y,z,w\in V$ one has:

$$F(x,y,z,w)=-F(y,x,z,w), \tag{1.n}$$

$$F(x,y,z,w)+F(y,z,x,w)+F(z,x,y,w)=0, \tag{1.o}$$

$$\rho_{14}(F)(x,y)=\rho_{14}(F)(y,x), \tag{1.p}$$

$$(\mathrm{id}\otimes\mathrm{Tr})F=0\,. \tag{1.q}$$

Again, identities (1.p) and (1.q) are equivalent given the identities of Equations (1.n) and (1.o).

The spaces $\mathfrak{A}(V,\langle\cdot,\cdot\rangle)$ and $\mathfrak{f}(V,\langle\cdot,\cdot\rangle)$ depend upon the choice of the inner product; the space $\mathfrak{r}(V)$ does not. Thus it is convenient to keep the distinction between subspaces of $\otimes^2V^*\otimes\mathrm{End}(V)$ and $\otimes^4V^*$; this will play a crucial role in the proof of Theorem 4.2. The spaces $\mathfrak{R}(V)$, $\mathfrak{A}(V)$, and $\mathfrak{F}(V)$ are subspaces of $\otimes^2V^*\otimes\mathrm{End}(V)$; elements of these spaces will be denoted by $\mathcal{R}$, $\mathcal{A}$, and $\mathcal{F}$, respectively, and are operator valued bilinear forms. The spaces $\mathfrak{r}(V)$, $\mathfrak{a}(V)$, and $\mathfrak{f}(V)$ are subspaces of $\otimes^4V^*$; elements of these spaces will be denoted by R, A, and F, respectively, and are quadralinear forms. We have the inclusions:

$$\begin{array}{lllll}
\mathfrak{A}(V,\langle\cdot,\cdot\rangle) & \subset & \mathfrak{F}(V) & \subset & \mathfrak{R}(V)\,,\\
\mathfrak{a}(V) & \subset & \mathfrak{f}(V,\langle\cdot,\cdot\rangle) & \subset & \mathfrak{r}(V)\,.
\end{array}$$

Let $\{e_i\}$ be a basis for V. If $\psi\in\otimes^2V^*$ and if $\Theta\in\otimes^4V^*$, set

$$\psi_{ij}:=\psi(e_i,e_j)\quad\text{and}\quad\Theta_{ijkl}:=\Theta(e_i,e_j,e_k,e_l)\,.$$

Let $\{e^i\}$ be the associated dual basis for V^*. Then

$$\psi=\textstyle\sum_{ij}\psi_{ij}e^i\otimes e^j\quad\text{and}\quad\Theta=\textstyle\sum_{ijkl}\psi_{ijkl}e^i\otimes e^j\otimes e^k\otimes e^l\,.$$

If $\langle\cdot,\cdot\rangle$ is a non-degenerate inner product on V, let

$$\Xi_{ij}:=\langle e_i,e_j\rangle\quad\text{and}\quad\textstyle\sum_j\Xi^{ij}\Xi_{jk}=\delta^i_k \tag{1.r}$$

where δ is the Kronecker symbol. One then has:

$$\textstyle\sum_{ij}\Xi^{ij}\langle x,e_i\rangle e_j=x\quad\text{and}\quad\mathrm{Tr}\{\psi\}=\textstyle\sum_{ij}\Xi^{ij}\psi_{ij}\,.$$

We shall decompose the natural action of $GL(V)$ and of $O(V,\langle\cdot,\cdot\rangle)$ on the spaces $\mathfrak{R}(V)$, $\mathfrak{a}(V)$, and $\mathfrak{F}(V)$ as the direct sum of irreducible modules and use these decompositions to exhibit generating sets for these spaces and to derive other natural geometric properties.

Our motivation in this paper is to study affine curvature operators; as already stated above, these are the curvature operators which naturally appear as curvature operators of the induced and of the conormal connections in relative hypersurface theory. Moreover, in this situation, there naturally appears a metric, the so called relative metric, which permits us to raise and lower indices. Our aim is a characterization of the affine curvature operators, arising from torsion free and Ricci symmetric connections, in the space of all curvature operators arising from torsion free connections. Via the decomposition results of Section 4, these are characterized by the vanishing of the component W_3. We will study the geometric meaning of the various components in this decomposition, at least in the case of relative hypersurfaces, in a subsequent paper.

2. Curvature Operators

In this section, we study operators with the same symmetries as those of a torsion free connection on the tangent bundle of a smooth manifold.

2.1. **Geometric representability of curvature operators.** If ∇ is a torsion free connection on the tangent bundle of a smooth manifold M, let $\mathcal{R}^\nabla$ be the associated curvature operator; if $P \in M$ and if $x, y, z \in T_PM$, then

$$\mathcal{R}_P^\nabla(x,y)z := \left\{\nabla_x\nabla_y - \nabla_y\nabla_x - \nabla_{[x,y]}\right\} z\,.$$

One then has $\mathcal{R}_P^\nabla \in \mathfrak{R}(T_PM)$ since the symmetries of Equations (1.a) and (1.b) hold. Conversely, every curvature operator is geometrically representable by an torsion free connection:

Theorem 2.1. *Let $\mathcal{R} \in \mathfrak{R}(V)$ be given. Regard V as a smooth manifold in its own right. Let 0 be the origin of V and identify $T_0V = V$. Then there exists a torsion free connection ∇ on V so that $\mathcal{R}_0^\nabla = \mathcal{R}$.*

Proof. Let $\mathcal{R} \in \mathfrak{R}(V)$. Expand $\mathcal{R}(e_i, e_j)e_k = \sum_l R_{ijk}{}^l e_l$ relative to some basis $\{e_i\}$ for V. Let $\{x_i\}$ be the associated dual coordinates; if $e \in V$, then $e = \sum_i x_i(e)e_i$. Define a connection ∇ on TV by setting

$$\nabla_{\partial_{x_a}}\partial_{x_b} := \textstyle\sum_d \Gamma_{ab}{}^d\partial_{x_d} \quad \text{for} \quad \Gamma_{ab}{}^d := -\frac{1}{3}\sum_c x_c\{\mathcal{R}_{acb}{}^d + \mathcal{R}_{bca}{}^d\}\,.$$

Since $\nabla_{\partial_{x_a}}\partial_{x_b} = \nabla_{\partial_{x_b}}\partial_{x_a}$, ∇ is torsion free. As $\Gamma(0) = 0$,

$$\begin{aligned}
\mathcal{R}_0^\nabla(\partial_{x_i}, \partial_{x_j})\partial_{x_k} &= \textstyle\sum_l(\partial_{x_i}\Gamma_{jk}{}^l - \partial_{x_j}\Gamma_{ik}{}^l)\partial_{x_l}\\
&= -\tfrac{1}{3}\textstyle\sum_l\{\mathcal{R}_{jik}{}^l + \mathcal{R}_{kij}{}^\ell - \mathcal{R}_{ijk}{}^\ell - \mathcal{R}_{kji}{}^l\}\partial_{x_l}\\
&= -\tfrac{1}{3}\textstyle\sum_l\{-2\mathcal{R}_{ijk}{}^l + \mathcal{R}_{kij}{}^l + \mathcal{R}_{jki}{}^l\}\partial_{x_l} = \mathcal{R}_{ijk}{}^l\partial_{x_l}\,.
\end{aligned}$$

This completes the proof of the desired result. □

2.2. **The Jacobi operator.** This operator is defined by setting:

$$\mathcal{J}_\mathcal{R}(x)y := \mathcal{R}(y,x)x\,.$$

It plays a central role in the study of geodesic sprays. The following result is known in the context of Riemannian geometry; it extends easily to the more general setting.

Lemma 2.2. *Let $\mathcal{R} \in \mathfrak{R}(V)$. If $\mathcal{J}_{\mathcal{R}} = 0$, then $\mathcal{R} = 0$.*

Proof. $\mathcal{J}_{\mathcal{R}}(x)$ is quadratic in x. The associated bilinear form is given by

$$\mathcal{J}_{\mathcal{R}}(x,y) : z \to \tfrac{1}{2}\big\{\partial_\varepsilon \mathcal{J}_{\mathcal{R}}(x+\varepsilon y)\big\} z|_{\varepsilon=0} = \tfrac{1}{2}\big\{\mathcal{R}(z,x)y + \mathcal{R}(z,y)x\big\}\,.$$

If $\mathcal{J}_{\mathcal{R}}(x) = 0$ for all $x \in V$, one has the additional curvature symmetry

$$\mathcal{R}(z,x)y + \mathcal{R}(z,y)x = 0$$

for all $x, y, z \in V$. We compute:

$$\begin{aligned} 0 &= \mathcal{R}(x,y)z + \mathcal{R}(y,z)x + \mathcal{R}(z,x)y \\ &= \mathcal{R}(x,y)z - \mathcal{R}(y,x)z - \mathcal{R}(x,z)y \\ &= \mathcal{R}(x,y)z + \mathcal{R}(x,y)z + \mathcal{R}(x,y)z\,. \end{aligned}$$

The Lemma now follows. □

2.3. **The action of the general linear group on $\mathfrak{R}(V)$.** This action is not irreducible, but decomposes as the direct sum of irreducible modules. Let

$$\mathfrak{U}(V) := \ker\{\rho_{14}\} \cap \mathfrak{R}(V)\,. \tag{2.a}$$

The decomposition $V^* \otimes V^* = \Lambda^2(V^*) \oplus S^2(V^*)$ is a $GL(V)$ equivariant decomposition of $V^* \otimes V^*$ into irreducible modules; we let π_a and π_s be the appropriate projections where

$$\pi_a(\psi)_{ij} := \tfrac{1}{2}(\psi_{ij} - \psi_{ji}) \quad \text{and} \quad \pi_s(\psi)_{ij} := \tfrac{1}{2}(\psi_{ij} + \psi_{ji})\,. \tag{2.b}$$

We may therefore decompose $\rho_{14} = \pi_a \circ \rho_{14} \oplus \pi_s \circ \rho_{14}$ where ρ_{14} is as defined in Equation (1.g). One has the following result of Strichartz [7]:

Theorem 2.3. *The map ρ_{14} defines a $GL(V)$ equivariant short exact sequence*

$$0 \to \mathfrak{U}(V) \to \mathfrak{R}(V) \xrightarrow{\rho_{14}} \Lambda^2(V^*) \oplus S^2(V^*) \to 0$$

which is equivariantly split by the map $\sigma_{\pi_a\circ\rho_{14}} \oplus \sigma_{\pi_s\circ\rho_{14}}$ where

$$\begin{aligned} \sigma_{\pi_a\circ\rho_{14}}(\omega)(x,y)z &= \tfrac{-1}{1+m}\{2\omega(x,y)z + \omega(x,z)y - \omega(y,z)x\} \quad \textit{for} \quad \omega \in \Lambda^2(V^*)\,, \\ \sigma_{\pi_s\circ\rho_{14}}(\psi)(x,y)z &= \tfrac{1}{1-m}\{\psi(x,z)y - \psi(y,z)x\} \quad \textit{for} \quad \psi \in S^2(V^*)\,. \end{aligned}$$

This gives a $GL(V)$ equivariant decomposition of

$$\mathfrak{R}(V) = \mathfrak{U}(V) \oplus \Lambda^2(V^*) \oplus S^2(V^*)$$

as the direct sum of irreducible $GL(V)$ modules. We have

$$\begin{aligned} &\dim\{\mathfrak{U}(V)\} = \tfrac{1}{3}m^2(m^2-4)\,, \quad \dim\{\Lambda^2(V^*)\} = \tfrac{1}{2}m(m-1)\,, \\ &\dim\{S^2(V^*)\} = \tfrac{1}{2}m(m+1)\,, \quad \dim\{\mathfrak{R}(V)\} = \tfrac{1}{3}m^2(m^2-1)\,. \end{aligned}$$

Proof. We check the splitting as follows. If $\omega \in \Lambda^2(V^*)$, let $\mathcal{R}_\omega := \sigma_{\pi_a\circ\rho_{14}}(\omega)$. Then $\mathcal{R}_\omega(x,y) = -\mathcal{R}_\omega(y,x)$. We check the Bianchi identity by computing:

$$\begin{aligned} \mathcal{R}_\omega(x,y)z + \mathcal{R}_\omega(y,z)x + \mathcal{R}_\omega(z,x)y &= \tfrac{-1}{1+m}\{2\omega(x,y)z + \omega(x,z)y - \omega(y,z)x \\ &\quad + 2\omega(y,z)x + \omega(y,x)z - \omega(z,x)y + 2\omega(z,x)y + \omega(z,y)x - \omega(x,y)z\} \\ &= 0\,. \end{aligned}$$

Thus $\mathcal{R}_\omega \in \mathfrak{R}(V)$. One also has that:

$$\begin{aligned}\rho_{14}(\mathcal{R}_\omega)(y,z) &= \tfrac{-1}{1+m}\sum_i e^i\{2\omega(e_i,y)z+\omega(e_i,z)y-\omega(y,z)e_i\}\\ &= \tfrac{-1}{1+m}\{2\omega(z,y)+\omega(y,z)-m\omega(y,z)\}=\omega(y,z)\,.\end{aligned}$$

Let $\psi \in S^2(V^*)$ and let $\mathcal{R}_\psi = \sigma_{\pi_s\circ\rho_{14}}(\psi)$. Again, $\mathcal{R}_\psi(x,y) = -\mathcal{R}_\psi(y,x)$. We verify the Bianchi identity by computing:

$$\begin{aligned}&\mathcal{R}_\psi(x,y)z+\mathcal{R}_\psi(y,z)x+\mathcal{R}_\psi(z,x)y\\ &\quad=\frac{1}{1-m}\{\psi(x,z)y-\psi(y,z)x+\psi(y,x)z-\psi(z,x)y+\psi(z,y)x-\psi(x,y)z\}\\ &\quad=0\,.\end{aligned}$$

This shows that $\mathcal{R}_\psi \in \mathfrak{R}(V)$. Furthermore:

$$\begin{aligned}\rho_{14}(\mathcal{R}_\psi)(y,z) &= \tfrac{1}{1-m}\sum_i e^i\{\psi(e_i,z)y-\psi(y,z)e_i\}\\ &= \tfrac{1}{1-m}\{\psi(y,z)-m\psi(y,z)\}=\psi(y,z)\,.\end{aligned}$$

Consequently one has an equivariant decomposition of $\mathfrak{R}(V)$ into $GL(V)$ modules:

$$\mathfrak{R}(V)=\mathfrak{U}(V)\oplus\Lambda^2(V^*)\oplus S^2(V^*)\,.$$

We refer to [7] for the proof of the remaining assertions of the Theorem. □

We say that two torsion free connections ∇ and $\bar\nabla$ on a differentiable manifold M are *projectively equivalent* if and only if every every geodesic for ∇ can be reparametrized to be a geodesic for $\bar\nabla$, or equivalently if there exists a smooth 1-form ω so

$$\nabla_x y-\bar\nabla_x y=\omega(x)y+\omega(y)x\,.$$

The summand $\mathfrak{U}(V)$ plays the role of the Weyl projective tensor; it also plays a role in the affine setting as we shall see presently in Theorem 4.1. Let $\pi_{\mathfrak{U}}$ be the associated projection on this summand in the decomposition of Theorem 2.3. One has [6, 7, 8]:

Lemma 2.4. *Let ∇ and $\bar\nabla$ be torsion free connections on M.*

1. *If ∇ and $\bar\nabla$ are projectively equivalent, then $\pi_{\mathfrak{U}}\mathcal{R}=\pi_{\mathfrak{U}}\bar{\mathcal{R}}$.*
2. *The connection ∇ is projectively flat if and only if $\pi_{\mathfrak{U}}\mathcal{R}=0$.*

2.4. **The action of the orthogonal group on $\mathfrak{R}(V)$.** The associated orthogonal group $O(V,\langle\cdot,\cdot\rangle)$ acts on $\mathfrak{R}(V)$ and on $\mathfrak{r}(V)$; the natural map from $\mathfrak{R}(V)$ to $\mathfrak{r}(V)$ given by Equation (1.m) is an equivariant isomorphism. Let Ξ be as in Equation

(1.r). We define:

$$\begin{aligned}
&\Lambda^2(V^*) := \{\omega \in \otimes^2 V^* : \omega_{ij} = -\omega_{ji}\}\,,\\
&S_0^2(V^*, \langle\cdot,\cdot\rangle) := \{\psi \in \otimes^2 V^* : \psi_{ij} = \psi_{ji}, \quad \textstyle\sum_{ij} \Xi^{ij}\psi_{ij} = 0\}\,,\\
&\mathfrak{w}(V, \langle\cdot,\cdot\rangle) := \{\Theta \in \otimes^4 V^* : \Theta_{ijkl} + \Theta_{jkil} + \Theta_{kijl} = 0\,,\\
&\qquad\qquad \Theta_{ijkl} = -\Theta_{jikl} = \Theta_{klij}, \textstyle\sum_{il} \Xi^{il}\Theta_{ijkl} = 0\}\,,\\
&\Lambda^2(\Lambda^2(V^*)) := \{\Theta \in \otimes^4 V^* : \Theta_{ijkl} = -\Theta_{jikl} = -\Theta_{ijlk} = -\Theta_{klij}\}\,,\\
&\Lambda_0^2(\Lambda^2(V^*)) := \{\Theta \in \Lambda^2(\Lambda^2(V^*)) : \textstyle\sum_{il} \Xi^{il}\Theta_{ijkl} = 0\}\,,\\
&\mathfrak{S}(V, \langle\cdot,\cdot\rangle) := \{\Theta \in \otimes^4 V^* : \Theta_{ijkl} = -\Theta_{jikl} = \Theta_{ijlk}, \textstyle\sum_{il} \Xi^{il}\Theta_{ijkl} = 0\,,\\
&\qquad\qquad \Theta_{kjil} + \Theta_{ikjl} - \Theta_{ljik} - \Theta_{iljk} = 0\}\,.
\end{aligned}$$

Note that $\Lambda^2(\Lambda^2(V^*))$, $\Lambda_0^2(\Lambda^2(V^*))$, and $\mathfrak{S}(V, \langle\cdot,\cdot\rangle)$ are not subsets of $\mathfrak{a}(V)$.

Theorem 2.5.

(1) *There is an $O(V, \langle\cdot,\cdot\rangle)$ equivariant orthogonal decomposition of*

$$\mathcal{R}(V) \approx \mathfrak{r}(V) = W_1 \oplus \cdots \oplus W_8$$

as the direct sum of irreducible $O(V, \langle\cdot,\cdot\rangle)$ modules where:

$$\begin{array}{ll}
\dim\{W_1\} = 1\,, & \dim\{W_2\} = \dim\{W_5\} = \frac{(m-1)(m+2)}{2}\,,\\
\dim\{W_3\} = \dim\{W_4\} = \frac{m(m-1)}{2}\,, & \dim\{W_6\} = \frac{m(m+1)(m-3)(m+2)}{12}\,,\\
\dim\{W_7\} = \frac{(m-1)(m-2)(m+1)(m+4)}{8}\,, & \dim\{W_8\} = \frac{m(m-1)(m-3)(m+2)}{8}\,.
\end{array}$$

(2) *There are the following isomorphisms as $O(\langle\cdot,\cdot\rangle)$ modules:*
  (a) *$W_1 \approx \mathbb{R}$, $W_2 \approx W_5 \approx S_0^2(V^*, \langle\cdot,\cdot\rangle)$, and $W_3 \approx W_4 \approx \Lambda^2(V^*)$.*
  (b) *$W_6 \approx \mathfrak{w}(V, \langle\cdot,\cdot\rangle)$ is the space of Weyl conformal curvature tensors.*
  (c) *$W_7 \approx \mathfrak{S}(V, \langle\cdot,\cdot\rangle)$ and $W_8 \approx \Lambda_0^2(\Lambda^2(V^*))$.*

We refer to Bokan [1] for the proof of Assertion (1) in the context of a positive definite inner product; it extends immediately to the indefinite inner products. We will prove Assertion (2a) later in this section. We will prove Assertion (2b) in Section 3. We will prove Assertion (2c) in Section 4.

Remark 2.6. Since W_2 and W_5 are isomorphic as $O(V, \langle\cdot,\cdot\rangle)$ modules and since W_3 and W_4 are isomorphic as $O(V, \langle\cdot,\cdot\rangle)$ modules, the decomposition of $\mathfrak{R}(V)$ into irreducible module summands is not unique; this fact plays an important role in the analysis of Bokan [1].

We shall need a technical result before proving Theorem 2.5 (2). We use Equation (1.m) to lower indices and to define a curvature tensor R associated to a curvature operator $\mathcal{R}$. Let Ξ be as in Equation (1.r). Then:

$$\begin{aligned}
&\rho_{14}(R)(x,y) := \textstyle\sum_{ij} \Xi^{ij} R(e_i, x, y, e_j), \quad \rho_{23}(R)(x,y) := \sum_{ij} \Xi^{ij} R(x, e_i, e_j, y),\\
&\rho_{24}(R)(x,y) := \textstyle\sum_{ij} \Xi^{ij} R(x, e_i, y, e_j), \quad \rho_{13}(R)(x,y) := \sum_{ij} \Xi^{ij} R(e_i, x, e_j, y),\\
&\rho_{34}(R)(x,y) := \textstyle\sum_{ij} \Xi^{ij} R(x, y, e_i, e_j) = -\rho_{14}(R)(x,y) + \rho_{14}(R)(y,x)\,.
\end{aligned}$$

There is an $O(V,\langle\cdot,\cdot\rangle)$ equivariant decomposition:

$$V^*\otimes V^*=\Lambda^2(V^*)\oplus S_0^2(V^*,\langle\cdot,\cdot\rangle)\oplus\mathbb{R}$$

where $S_0^2(V^*,\langle\cdot,\cdot\rangle)$ is the space of trace free symmetric bilinear forms, and where $\mathbb{R}$ is the trivial $O(V,\langle\cdot,\cdot\rangle)$ module. If π_a, π_0, and τ are the associated orthogonal projections, then

$$\begin{aligned}&\pi_a(\psi)(x,y):=\tfrac{1}{2}\{\psi(x,y)-\psi(y,x)\},\\&\pi_s(\psi)(x,y):=\tfrac{1}{2}\{\psi(x,y)+\psi(y,x)\},\\&\tau(\psi):=\textstyle\sum_{ij}\Xi^{ij}\psi(e_i,e_j),\\&\pi_0(\psi)(x,y):=\pi_s(\psi)(x,y)-\tfrac{1}{m}\tau(\psi)\langle\cdot,\cdot\rangle\,.\end{aligned}\tag{2.c}$$

There is only one non-trivial scalar curvature arising from $R\in\mathfrak{r}(V)$ since

$$\begin{aligned}&\tau(\rho_{14}(R))=\textstyle\sum_{ijkl}\Xi^{il}\Xi^{jk}R(e_i,e_j,e_k,e_l)=\tau(\rho_{23}(R))=-\tau(\rho_{24}(R)),\\&\tau(\rho_{34}(R))=\textstyle\sum_{ijkl}\Xi^{ij}\Xi^{kl}R(e_i,e_j,e_k,e_l)=0\,.\end{aligned}$$

If $\psi\in S_0^2(V^*,\langle\cdot,\cdot\rangle)$ and if $\omega\in\Lambda^2(V^*)$, let:

$$\begin{aligned}&\sigma_1(\psi)(x,y,z,w):=\psi(x,w)\langle y,z\rangle-\psi(y,w)\langle x,z\rangle,\\&\sigma_2(\psi)(x,y,z,w):=\langle x,w\rangle\psi(y,z)-\langle y,w\rangle\psi(x,z),\\&\sigma_3(\omega)(x,y,z,w):=2\omega(x,y)\langle z,w\rangle+\omega(x,z)\langle y,w\rangle-\omega(y,z)\langle x,w\rangle,\\&\sigma_4(\omega)(x,y,z,w):=\omega(x,w)\langle y,z\rangle-\omega(y,w)\langle x,z\rangle\,.\end{aligned}$$

Lemma 2.7.

(1) *σ_1 and σ_2 are $O(V,\langle\cdot,\cdot\rangle)$ equivariant maps from $S_0^2(V^*,\langle\cdot,\cdot\rangle)$ to $\mathfrak{r}(V)$, σ_3 and σ_4 are $O(V,\langle\cdot,\cdot\rangle)$ equivariant maps from $\Lambda^2(V^*)$ to $\mathfrak{r}(V)$, and*

$$\begin{pmatrix}\rho_{14}\circ\sigma_1&\rho_{23}\circ\sigma_1\\\rho_{14}\circ\sigma_2&\rho_{23}\circ\sigma_2\end{pmatrix}=\begin{pmatrix}-\operatorname{id}&(m-1)\operatorname{id}\\(m-1)\operatorname{id}&-\operatorname{id}\end{pmatrix},$$
$$\begin{pmatrix}\rho_{13}\circ\sigma_3&\rho_{34}\circ\sigma_3\\\rho_{13}\circ\sigma_4&\rho_{34}\circ\sigma_4\end{pmatrix}=\begin{pmatrix}-3\operatorname{id}&2(m+1)\operatorname{id}\\(1-m)\operatorname{id}&2\operatorname{id}\end{pmatrix}.$$

(2) *We have $O(V,\langle\cdot,\cdot\rangle)$ equivariant sequences which are equivariantly split:*

$$\begin{aligned}&\tau\circ\rho_{14}:\mathfrak{r}(V)\to\mathbb{R}\to 0,\\&\pi_0\circ\rho_{14}\oplus\pi_0\circ\rho_{13}:\mathfrak{r}(V)\to S_0^2(V^*,\langle\cdot,\cdot\rangle)\oplus S_0^2(V^*,\langle\cdot,\cdot\rangle)\to 0,\\&\pi_a\circ\rho_{13}\oplus\pi_a\circ\rho_{34}:\mathfrak{r}(V)\to\Lambda^2(V^*)\oplus\Lambda^2(V^*)\to 0\,.\end{aligned}$$

Proof. Let $\psi\in S_0^2(V^*,\langle\cdot,\cdot\rangle)$ and let $\omega\in\Lambda^2(V^*)$. Set $R_1:=\sigma_1(\psi)$, $R_2:=\sigma_2(\psi)$, $R_3:=\sigma_3(\omega)$, and $R_4:=\sigma_4(\omega)$. It is immediate $R_i(x,y,z,w)=-R_i(y,x,z,w)$.

To show that $R_i \in \mathfrak{r}(V)$, we must verify the first Bianchi identity is satisfied:

$$\begin{aligned}
&R_1(x,y,z,w)+R_1(y,z,x,w)+R_1(z,x,y,w)\\
&\qquad=\psi(x,w)\langle y,z\rangle-\psi(y,w)\langle x,z\rangle\\
&\qquad\quad+\psi(y,w)\langle z,x\rangle-\psi(z,w)\langle y,x\rangle\\
&\qquad\quad+\psi(z,w)\langle x,y\rangle-\psi(x,w)\langle z,y\rangle=0\,,\\
&R_2(x,y,z,w)+R_2(y,z,x,w)+R_2(z,x,y,w)\\
&\qquad=\langle x,w\rangle\psi(y,z)-\langle y,w\rangle\psi(x,z)\\
&\qquad\quad+\langle y,w\rangle\psi(z,x)-\langle z,w\rangle\psi(y,x)\\
&\qquad\quad+\langle z,w\rangle\psi(x,y)-\langle x,w\rangle\psi(z,y)=0\,,\\
&R_3(x,y,z,w)+R_3(y,z,x,w)+R_3(z,x,y,w)\\
&\qquad=2\omega(x,y)\langle z,w\rangle+\omega(x,z)\langle y,w\rangle-\omega(y,z)\langle x,w\rangle\\
&\qquad\quad+2\omega(y,z)\langle x,w\rangle+\omega(y,x)\langle z,w\rangle-\omega(z,x)\langle y,w\rangle\\
&\qquad\quad+2\omega(z,x)\langle y,w\rangle+\omega(z,y)\langle x,w\rangle-\omega(x,y)\langle z,w\rangle=0\,,\\
&R_4(x,y,z,w)+R_4(y,z,x,w)+R_4(z,x,y,w)\\
&\qquad=\omega(x,w)\langle y,z\rangle-\omega(y,w)\langle x,z\rangle\\
&\qquad\quad+\omega(y,w)\langle z,x\rangle-\omega(z,w)\langle y,x\rangle\\
&\qquad\quad+\omega(z,w)\langle x,y\rangle-\omega(x,w)\langle z,y\rangle=0\,.
\end{aligned}$$

We complete the proof of Assertion (1) by computing:

$$\begin{aligned}
\rho_{14}(R_1)(y,z)&=\textstyle\sum_{ij}\Xi^{ij}\{\psi(e_i,e_j)\langle y,z\rangle-\psi(y,e_j)\langle e_i,z\rangle\}\\
&=\tau(\psi)\langle y,z\rangle-\psi(y,z)=-\psi(y,z)\,,\\
\rho_{23}(R_1)(x,w)&=\textstyle\sum_{ij}\Xi^{ij}\{\psi(x,w)\langle e_i,e_j\rangle-\psi(e_i,w)\langle x,e_j\rangle\}\\
&=(m-1)\psi(x,w)\,,\\
\rho_{14}(R_2)\psi(y,z)&=\textstyle\sum_{ij}\Xi^{ij}\{\langle e_i,e_j\rangle\psi(y,z)-\langle y,e_j\rangle\psi(e_i,z)\}\\
&=(m-1)\psi(y,z)\,,\\
\rho_{23}(R_2)(x,w)&=\textstyle\sum_{ij}\Xi^{ij}\{\langle x,w\rangle\psi(e_i,e_j)-\langle e_i,w\rangle\psi(x,e_j)\}\\
&=\tau(\psi)\langle x,w\rangle-\psi(x,w)=-\psi(x,w)\,,\\
\rho_{13}(R_3)(y,w)&=\textstyle\sum_{ij}\Xi^{ij}\{2\omega(e_i,y)\langle e_j,w\rangle+\omega(e_i,e_j)\langle y,w\rangle-\omega(y,e_j)\langle e_i,w\rangle\}\\
&=-3\omega(y,w)\,,\\
\rho_{34}(R_3)(x,y)&=\textstyle\sum_{ij}\Xi^{ij}\{2\omega(x,y)\langle e_i,e_j\rangle+\omega(x,e_i)\langle y,e_j\rangle-\omega(y,e_i)\langle x,e_i\rangle\}\\
&=2(m+1)\omega(x,y)\,,
\end{aligned}$$

$$\rho_{13}(R_4)(y,w)=\textstyle\sum_{ij}\Xi^{ij}\{\omega(e_i,w)\langle y,e_j\rangle-\omega(y,w)\langle e_i,e_j\rangle\}=(1-m)\omega(y,w)\,,$$

$$\rho_{34}(R_4)(x,y)=\textstyle\sum_{ij}\Xi^{ij}\{\omega(x,e_j)\langle y,e_i\rangle-\omega(y,e_j)\langle x,e_i\rangle\}=2\omega(x,y)\,.$$

We now prove Assertion (2). We show the first sequence splits by computing:

$$\begin{aligned}\tfrac{1}{m(m-1)}\tau(\rho_{14}(\sigma_1\langle\cdot,\cdot\rangle))&=\tfrac{1}{m(m-1)}\textstyle\sum_{ijkl}\Xi^{il}\Xi^{jk}\{\Xi_{il}\Xi_{jk}-\Xi_{ik}\Xi_{jl}\}\\&=\tfrac{1}{m(m-1)}\textstyle\sum_{ij}\{\delta_i^i\delta_j^j-\delta_i^j\delta_j^i\}=1\,.\end{aligned}$$

As the determinants of the two coefficient matrices in Assertion (1) are non-zero, the desired splitting of the second and of the third sequences follows. $\square$

Proof of Theorem 2.5 (2a). By Lemma 2.7, $\mathbb{R}$ has multiplicity 1, $S_0^2(V^*,\langle\cdot,\cdot\rangle)$ has multiplicity 2, and $\Lambda^2(V^*)$ has multiplicity 2 in the decomposition of $\mathfrak{r}(V)$ as an $O(\langle\cdot,\cdot\rangle)$ module. These modules are irreducible and

$$\dim\{\mathbb{R}\}=1,\quad\dim\{S_0^2(V^*,\langle\cdot,\cdot\rangle)\}=\tfrac{(m-1)(m+2)}{2},\quad\dim\{\Lambda^2(V^*)\}=\tfrac{m(m-1)}{2}\,.$$

Theorem 2.5 (2a) now follows from Theorem 2.5 (1). $\square$

3. Algebraic Curvature Tensors

In this section, we study the quadralinear forms with the same symmetries as those of the Levi-Civita connection of a pseudo-Riemannian manifold.

3.1. The action of the general linear group on $\mathfrak{a}(V)$.

Theorem 3.1. *$\mathfrak{a}(V)$ is an irreducible $GL(V)$ module.*

We postpone the proof of this result until Section 5 as we must first establish some additional notation.

3.2. The action of $O(V,\langle\cdot,\cdot\rangle)$ on $\mathfrak{a}(V)$.

Let

(3.a) $\quad(\mathrm{id}\otimes\pi_s)(R)(x,y,z,w):=\frac12\{R(x,y,z,w)+R(x,y,w,z)\}\quad$ for $\quad R\in\mathfrak{r}(V)\,.$

If $\phi,\psi\in S^2(V^*)$, one can define an algebraic curvature tensor $\phi\wedge\psi\in\mathfrak{a}(V)$ by:

$$\begin{aligned}\{\phi\wedge\psi\}(x,y,z,w):=\tfrac12\{&\phi(x,w)\psi(y,z)-\phi(x,z)\psi(y,w)\\&+\phi(y,z)\psi(x,w)-\phi(y,w)\psi(x,z)\}\,.\end{aligned}\tag{3.b}$$

(This has a different normalizing constant than the usual Kulkarni-Nomizu product). These tensors arise naturally. If L is the second fundamental form of a hypersurface M in $\mathbb{R}^{m+1}$, then

$$R_M=L\wedge L\,.$$

Define:

$$\begin{aligned}\mathfrak{w}(V,\langle\cdot,\cdot\rangle)&:=\ker\{\rho_{14}\}\cap\mathfrak{a}(V)\,,\\\sigma_{\mathrm{id}\otimes\pi_s}(S)_{ijkl}&:=S_{ijkl}+\tfrac12\{S_{kjil}+S_{ikjl}-S_{ljik}-S_{iljk}\}\,,\\\sigma_{\mathfrak{a},\rho_{14}}(\psi)&:=\tfrac{2}{m-2}\psi\wedge\langle\cdot,\cdot\rangle-\tfrac{\tau(\psi)}{(m-1)(m-2)}\langle\cdot,\cdot\rangle\wedge\langle\cdot,\cdot\rangle\,.\end{aligned}\tag{3.c}$$

Theorem 3.2.

(1) *Let* $R \in \otimes^4 V^*$ *satisfy Equations* (1.c) *and* (1.d). *Then Equations* (1.e) *and* (1.f) *are equivalent.*

(2) *The maps* $\mathrm{id} \otimes \pi_s$ *and* ρ_{14} *define* $GL(V)$ *and* $O(V, \langle\cdot,\cdot\rangle)$ *equivariant short exact sequences, respectively,*

$$0 \to \mathfrak{a}(V) \to \mathfrak{r}(V) \xrightarrow{\mathrm{id} \otimes \pi_s} \Lambda^2(V^*) \otimes S^2(V^*) \to 0\,,$$
$$0 \to \mathfrak{w}(V, \langle\cdot,\cdot\rangle) \to \mathfrak{a}(V) \xrightarrow{\rho_{14}} S^2(V^*) \to 0\,.$$

which are equivariantly split, respectively, by the maps $\sigma_{\mathrm{id} \otimes \pi_s}$ *and* $\sigma_{\mathfrak{a}, \rho_{14}}$.

(3) *This gives an* $O(V, \langle\cdot,\cdot\rangle)$ *equivariant decomposition of*

$$\mathfrak{a}(V) \approx \mathfrak{w}(V, \langle\cdot,\cdot\rangle) \;\oplus\; S^2_0(V^*, \langle\cdot,\cdot\rangle) \;\oplus\; \{\mathbb{R}\}$$

as the direct sum of irreducible $O(V, \langle\cdot,\cdot\rangle)$ *modules where*

$$\dim\{\mathfrak{w}(V, \langle\cdot,\cdot\rangle)\} = \tfrac{1}{12} m(m+1)(m+2)(m-3)\,, \quad \dim\{\mathbb{R}\} = 1\,,$$
$$\dim\{S^2_0(V^*, \langle\cdot,\cdot\rangle)\} = \tfrac{1}{2}(m-1)(m+2)\,, \qquad \dim\{\mathfrak{a}(V)\} = \tfrac{1}{12} m^2(m^2-1)\,.$$

Proof. It is immediate that (1.c) and (1.e) imply Equation (1.f). Conversely, suppose that Equations (1.c), (1.d), and (1.f) hold. We use the following notation:

$$\begin{aligned}
R(\xi_1,\xi_2,\xi_3,\xi_4) &= a_1\,, \quad & R(\xi_3,\xi_4,\xi_1,\xi_2) &= a_1 + \varepsilon_1\,,\\
R(\xi_1,\xi_3,\xi_2,\xi_4) &= a_2\,, \quad & R(\xi_2,\xi_4,\xi_1,\xi_3) &= a_2 + \varepsilon_2\,,\\
R(\xi_2,\xi_3,\xi_1,\xi_4) &= a_3\,, \quad & R(\xi_1,\xi_4,\xi_2,\xi_3) &= a_3 + \varepsilon_3\,.
\end{aligned}$$

We establish Assertion (1) by showing $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$. We compute:

$$\begin{aligned}
0 &= R(\xi_1,\xi_2,\xi_3,\xi_4) + R(\xi_2,\xi_3,\xi_1,\xi_4) + R(\xi_3,\xi_1,\xi_2,\xi_4)\\
&= a_1 + a_3 - a_2\,,\\
0 &= R(\xi_1,\xi_2,\xi_4,\xi_3) + R(\xi_2,\xi_4,\xi_1,\xi_3) + R(\xi_4,\xi_1,\xi_2,\xi_3)\\
&= -a_1 + a_2 - a_3 + \varepsilon_2 - \varepsilon_3 = \varepsilon_2 - \varepsilon_3\,,\\
0 &= R(\xi_1,\xi_3,\xi_4,\xi_2) + R(\xi_3,\xi_4,\xi_1,\xi_2) + R(\xi_4,\xi_1,\xi_3,\xi_2)\\
&= -a_2 + a_1 + a_3 + \varepsilon_1 + \varepsilon_3 = \varepsilon_1 + \varepsilon_3\,,\\
0 &= R(\xi_2,\xi_3,\xi_4,\xi_1) + R(\xi_3,\xi_4,\xi_2,\xi_1) + R(\xi_4,\xi_2,\xi_3,\xi_1)\\
&= -a_3 - a_1 + a_2 - \varepsilon_1 + \varepsilon_2 = -\varepsilon_1 + \varepsilon_2\,.
\end{aligned}$$

This yields the equations $0 = \varepsilon_2 - \varepsilon_3 = \varepsilon_1 + \varepsilon_3 = -\varepsilon_1 + \varepsilon_2$ from which it follows that $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$; this proves Assertion (1).

Let $S \in \Lambda^2(V^*) \otimes S^2(V^*)$. We compute:

$$\begin{aligned}
&\sigma_{\mathrm{id}\otimes\pi_s}(S)_{ijkl} + \sigma_{\mathrm{id}\otimes\pi_s}(S)_{jikl} \\
&\qquad = S_{ijkl} + \tfrac{1}{2}(S_{kjil} + S_{ikjl} - S_{ljik} - S_{iljk}) \\
&\qquad\quad + S_{jikl} + \frac{1}{2}(S_{kijl} + S_{jkil} - S_{lijk} - S_{jlik}) = 0\,, \\
&\sigma_{\mathrm{id}\otimes\pi_s}(S)_{ijkl} + \sigma_{\mathrm{id}\otimes\pi_s}(S)_{jkil} + \sigma_{\mathrm{id}\otimes\pi_s}(S)_{kijl} \\
&\qquad = S_{ijkl} + \tfrac{1}{2}(S_{kjil} + S_{ikjl} - S_{ljik} - S_{iljk}) \\
&\qquad\quad + S_{jkil} + \tfrac{1}{2}(S_{ikjl} + S_{jikl} - S_{lkji} - S_{jlki}) \\
&\qquad\quad + S_{kijl} + \tfrac{1}{2}(S_{jikl} + S_{kjil} - S_{likj} - S_{klij}) = 0\,.
\end{aligned}$$

This shows that $\sigma_{\mathrm{id}\otimes\pi_s}$ takes values in $\mathfrak{r}(V)$. Let $\alpha(S) := \sigma_{\mathrm{id}\otimes\pi_s}S - S$. Then

$$\alpha(S)_{ijkl} := \tfrac{1}{2}(S_{kjil} + S_{ikjl} - S_{ljik} - S_{iljk}) \in \Lambda^2(V^*) \otimes \Lambda^2(V^*)\,. \tag{3.d}$$

The map α will also play a role in Section 4.3. Since $\mathrm{id}\otimes\pi_s$ vanishes on the space $\Lambda^2(V^*) \otimes \Lambda^2(V^*)$, one has that

$$(\mathrm{id}\otimes\pi_s)(\sigma_{\mathrm{id}\otimes\pi_s}(S)) = (\mathrm{id}\otimes\pi_s)(S) + (\mathrm{id}\otimes\pi_s)\alpha(S) = S\,.$$

This shows that $\mathrm{id}\otimes\pi_s$ is an equivariant splitting. We refer to Singer and Thorpe [5] or to Strichartz [7] for the proof of the remaining assertions. □

Proof of Theorem 2.5 (2b). Because $\mathfrak{w}$ is the space of *Weyl conformal tensors*,

$$\dim\{\mathfrak{w}(V, \langle\cdot,\cdot\rangle)\} = \tfrac{1}{12}m(m+1)(m-3)(m+2) = \dim\{W_6\}\,.$$

Since $\mathfrak{w}\big(V, \langle\cdot,\cdot\rangle\big)$ is an irreducible $O(V, \langle\cdot,\cdot\rangle)$ module, we may use Theorem 2.5 (1) to identify $W_6 = \mathfrak{w}(V, \langle\cdot,\cdot\rangle)$. □

Theorem 2.1 generalizes immediately to this setting:

Theorem 3.3. *Let $A \in \mathfrak{a}(V)$ be given. Regard V as a smooth manifold in its own right. Let 0 be the origin of V and identify $T_0V = V$. There exists a pseudo-Riemannian metric g defined on V so that $R^g_0 = A$ where R^g_0 is the curvature tensor of the associated Levi-Civita connection.*

Proof. Let $\{e_i\}$ be an orthonormal basis for V. Let x_i be the associated coordinate system. We define the germ of a pseudo-Riemannian metric on V by setting

$$g_{ab} = g(\partial_{x_a}, \partial_{x_b}) := \langle e_a, e_b\rangle - \tfrac{1}{3}\textstyle\sum_{cd} A_{acdb}x_cx_d\,.$$

Clearly $g_{ab} = g_{ba}$. As $g|_{T_0V} = \langle\cdot,\cdot\rangle$, g is non-degenerate near 0. One may then use a partition of unity to extend g to be non-degenerate on all of V without changing it near 0. One has

$$\Gamma_{ijk} := g(\nabla_{\partial_{x_i}}\partial_{x_j}, \partial_{x_k}) = \tfrac{1}{2}(\partial_{x_i}g_{jk} + \partial_{x_j}g_{ik} - \partial_{x_k}g_{ij})\,.$$

Since $\Gamma_{ijk}(0)=0$, one has

$$\begin{aligned}R_{ijkl}(0) &:= R(\partial_{x_i},\partial_{x_j},\partial_{x_k},\partial_{x_l})(0)=\{\partial_{x_i}\Gamma_{jkl}-\partial_{x_j}\Gamma_{ikl}\}(0)\\ &=\tfrac12\{\partial_{x_i}(\partial_{x_j}g_{kl}+\partial_{x_k}g_{jl}-\partial_{x_l}g_{jk})-\partial_{x_j}(\partial_{x_i}g_{kl}+\partial_{x_k}g_{il}-\partial_{x_l}g_{ik})\}(0)\\ &=\tfrac16\{-A_{jikl}-A_{jkil}+A_{jilk}+A_{jlik}+A_{ijkl}+A_{ikjl}-A_{ijlk}-A_{iljk}\}\\ &=\tfrac16\{4A_{ijkl}-2A_{iljk}-2A_{iklj}\}=A_{ijkl}\,.\end{aligned}$$

The desired result now follows. □

The following result was first proved by Fiedler [3] using Young symmetrizers; subsequently Gilkey [4] established it using a direct construction and Díaz-Ramos and García-Río [2] derived it from the Nash embedding theorem. We adopt the notation of Equation (3.b) to define $\phi\wedge\psi\in\mathfrak{a}(V)$ for $\phi,\psi\in S^2(V^*)$.

Theorem 3.4. $\mathfrak{a}(V)=\operatorname{Span}_{\mathbb{R}}\{\phi\wedge\phi:\phi\in S^2(V^*)\}$.

We use Theorem 3.2 to establish a slightly stronger version of Theorem 3.4:

Theorem 3.5.

(1) *If $A\in\mathfrak{a}(V)$, there is a finite collection of elements $\phi_\nu\in S^2(V^*)$ such that* $\operatorname{Rank}\{\phi_\nu\}=2$ *and such that $A=\sum_\nu\phi_\nu\wedge\phi_\nu$.*

(2) *Suppose given (p,q) with $2\le p+q\le m$. Let $S^2_{(p,q)}(V^*)$ be the set of all symmetric bilinear forms on V of signature (p,q). Then*

$$\mathfrak{a}(V)=\operatorname{Span}_{\phi\in S^2_{(p,q)}(V^*)}\{\phi\wedge\phi\}\,.$$

Proof. Consider the following $GL(V)$ invariant subspace of $\mathfrak{a}(V)$:

$$\mathfrak{b}(V):=\operatorname{Span}_{\mathbb{R}}\{\phi\wedge\phi:\phi\in S^2(V^*),\operatorname{Rank}\{\phi\}=2\}\,.$$

We apply Theorem 3.1 to show $\mathfrak{b}(V)=\mathfrak{a}(V)$. This shows that we may express any $A\in\mathfrak{a}(V)$ in the form $c_1\phi_1\wedge\phi_1+\cdots+c_k\phi_k\wedge\phi_k$ where the ϕ_ν are symmetric bilinear forms of rank 2 and where the $c_\nu\in\mathbb{R}$. By rescaling the ϕ_ν, we may assume that the $c_\nu=\pm1$. Set $\alpha_1:=e^1\otimes e^1+e^2\otimes e^2$ and $\alpha_2:=e^1\otimes e^2+e^2\otimes e^1$. We have

$$(\alpha_1\wedge\alpha_1)(e_1,e_2,e_2,e_1)=+1\quad\text{and}\quad(\alpha_2\wedge\alpha_2)(e_1,e_2,e_2,e_1)=-1\,.$$

Thus $\alpha_1\wedge\alpha_1=-\alpha_2\wedge\alpha_2$. Consequently, by replacing a definite form by an indefinite form or an indefinite form by a definite form if necessary, we can change the sign and assume that all the constants c_ν are equal to 1. Assertion (1) now follows.

To prove Assertion (2), we set

$$\mathfrak{b}(V):=\operatorname{Span}_{\phi\in S^2_{(p,q)}(V^*)}\{\phi\wedge\phi\}\,.$$

As this is a non-empty $GL(V)$ invariant subspace of $\mathfrak{a}(V)$, Theorem 3.1 shows $\mathfrak{a}(V)=\mathfrak{b}(V)$ as desired. □

4. Affine Curvature Tensors in the Algebraic Setting

4.1. **The action of the general linear group on $\mathfrak{F}(V)$.** We adopt the notion of Equation (2.a) to define $\mathfrak{U}(V)$; the geometrical significance of this subspace is given in Lemma 2.4.

Use Equations (1.g) and (2.b) to define ρ_{14}, π_a, and π_s. Let $\sigma_{\pi_a\circ\rho_{14}}$ and $\sigma_{\pi_s\circ\rho_{14}}$ be as in Theorem 2.3. The following is an immediate consequence of Theorem 2.3:

Theorem 4.1. *We have the following $GL(V)$ equivariant short exact sequences*

$$0 \to \mathfrak{F}(V) \to \mathfrak{R}(V) \xrightarrow{\pi_a\circ\rho_{14}} \Lambda^2(V^*) \to 0\,,$$
$$0 \to \mathfrak{U}(V) \to \mathfrak{F}(V) \xrightarrow{\pi_s\circ\rho_{14}} S^2(V^*) \to 0$$

which are equivariantly split by the maps $\sigma_{\pi_a\circ\rho_{14}}$ and $\sigma_{\pi_s\circ\rho_{14}}$, respectively. This gives a $GL(V)$ equivariant decomposition of

$$\mathfrak{F}(V) = \mathfrak{U}(V) \oplus S^2(V^*)$$

as the direct sum of irreducible $GL(V)$ modules where

$$\begin{aligned}
&\dim\{\mathfrak{U}(V)\} = \tfrac{m^2(m^2-4)}{3}\,,\\
&\dim\{\dim\{S^2(V^*)\} = \tfrac{1}{2}m(m+1),\\
&\dim\{\mathfrak{F}(V)\} = \tfrac{m(m-1)(2m^2+2m-3)}{6}\,.
\end{aligned}$$

We use this result to generalize Theorem 3.4 to the setting at hand. We exploit in an essential way that the space $\mathfrak{A}(V,\langle\cdot,\cdot\rangle)$ depends non-trivially on the particular bilinear form which is chosen. Let $\mathcal{G}_{(p,q)}(V)$ be the set of non-degenerate bilinear forms on V of signature (p,q). Let $\mathcal{G}_{(p,q)}(M)$ be the set of all pseudo-Riemannian metrics on a smooth m-dimensional manifold M of signature (p,q). If $g \in \mathcal{G}_{(0,m)}(M)$ and if $P \in M$, let $\mathcal{R}(g,P)$ be the curvature operator of the Levi-Civita connection defined by g.

Theorem 4.2.

1. *If $p+q=m$, then $\mathfrak{F}(V) = \operatorname{Span}_{\langle\cdot,\cdot\rangle\in\mathcal{G}_{(p,q)}}\{\mathfrak{A}(V,\langle\cdot,\cdot\rangle)\}$.*
2. *We have that $\mathfrak{F}(T_PM) = \operatorname{Span}_{g\in\mathcal{G}_{(0,m)}(M)}\{\mathcal{R}(g,P)\}$.*

Proof. Let

$$\mathfrak{B}(V) := \operatorname{Span}_{\langle\cdot,\cdot\rangle\in\mathcal{G}_{(p,q)}}\{\mathfrak{A}(V,\langle\cdot,\cdot\rangle)\}\,.$$

Let $\Psi \in GL(V)$. If $\mathcal{A} \in \mathfrak{A}(V,\langle\cdot,\cdot\rangle)$, then

$$\Psi^*\mathcal{A} \in \mathfrak{A}(V,\Psi^*\langle\cdot,\cdot\rangle)\,.$$

Thus $\mathfrak{B}(V)$ is invariant under the action of $GL(V)$. Since $\mathfrak{B}(V) \neq \{0\}$, Theorem 4.1 shows exactly one of the following alternatives holds:

1. $\mathfrak{B}(V) = \ker\{\pi_s\circ\rho_{14}\}$.
2. $\mathfrak{B}(V) \approx S^2(V^*)$.
3. $\mathfrak{B}(V) = \mathfrak{F}(V)$.

If $\langle\cdot,\cdot\rangle \in \mathcal{G}_{(p,q)}(V)$, let $\mathcal{A}_{\langle\cdot,\cdot\rangle} \in \mathfrak{A}(V,\langle\cdot,\cdot\rangle)$ be the associated algebraic curvature operator of constant sectional curvature:

$$\mathcal{A}_{\langle\cdot,\cdot\rangle}(x,y)z := \langle y,z\rangle x - \langle x,z\rangle y\,.$$

Since $\rho_{14}(\mathcal{A}_{\langle\cdot,\cdot\rangle}) = (m-1)\langle\cdot,\cdot\rangle$, $\mathfrak{B}(V) \neq \ker\{\rho_{14}\}$. This eliminates the first possibility. Since $m \geq 4$, $m(m+1) > 6$. Consequently,

$$\dim\{\mathfrak{B}(V)\} \geq \dim\{\mathfrak{A}(V,\langle\cdot,\cdot\rangle)\} = \tfrac{m^2(m^2-1)}{12} > \tfrac{m(m+1)}{2} = \dim\{S^2(V^*)\}\,.$$

This eliminates the second possibility. Thus the third possibility holds; this proves Assertion (1).

Let $V = T_PM$. Let $g_0 \in \mathcal{G}_{(0,m)}(T_PM)$. By Theorem 3.3,

$$\mathfrak{A}(V,g_0) = \cup_{g\in\mathcal{G}_{(0,m)},g|_{T_PM}=g_0}\{\mathcal{R}(g,P)\}\,.$$

Assertion (2) now follows from Assertion (1). □

4.2. **Centro affine geometry.** Let $h \in S^2(V^*)$ and let $\mathcal{C} \in S^2(V^*)\otimes V$. Define

$$\begin{aligned}\mathcal{R}_h(x,y)z &:= h(y,z)x - h(x,z)y\,,\\ \mathcal{R}_\mathcal{C}(w,v)u &:= \mathcal{C}(v,\mathcal{C}(w,u)) - \mathcal{C}(w,\mathcal{C}(v,u))\,.\end{aligned}$$

The decomposition of Theorem 4.1 has geometric significance. Let h be the centroaffine metric, let ∇ be the induced connection, and let ∇^* be the conormal connection. Then $\mathcal{R}_h$ is the curvature operator of both ∇ and of ∇^* while the Riemannian curvature tensor of the associated Levi-Civita connection is given by $\mathcal{R}_C + \mathcal{R}_h$.

Theorem 4.3.

(1) $\mathcal{R}_h \in \sigma_{\pi_s\circ\rho_{14}}S^2(V^*)$ *and* $\sigma_{\pi_s\circ\rho_{14}}S^2(V^*) = \operatorname{Span}_{h\in S^2(V^*)}\{\mathcal{R}_h\}$.

(2) $\mathcal{R}_\mathcal{C} \in \mathfrak{F}(V)$ *and* $\mathfrak{F}(V) = \operatorname{Span}_{\mathcal{C}\in S^2(V^*)\otimes V}\{\mathcal{R}_\mathcal{C}\}$.

Proof. Assertion (1) follows from the discussion given to establish Theorem 4.1. We begin the proof of Assertion (2) by computing:

$$\begin{aligned}&\mathcal{R}_\mathcal{C}(v,w)u = \mathcal{C}(w,\mathcal{C}(v,u)) - \mathcal{C}(v,\mathcal{C}(w,u)) = -\mathcal{R}_\mathcal{C}(w,v)u\,,\\ &\quad\mathcal{R}_\mathcal{C}(w,v)u + \mathcal{R}_\mathcal{C}(v,u)w + \mathcal{R}_\mathcal{C}(u,w)v = \mathcal{C}(v,\mathcal{C}(w,u)) - \mathcal{C}(w,\mathcal{C}(v,u))\\ &\qquad\qquad + \mathcal{C}(w,\mathcal{C}(u,v)) - \mathcal{C}(u,\mathcal{C}(w,v)) + \mathcal{C}(u,\mathcal{C}(v,w)) - \mathcal{C}(v,\mathcal{C}(u,w))\\ &\qquad\qquad = 0\,.\end{aligned}$$

Let $C(e_i,e_j) = \sum_k C_{ij}{}^k e_k$ where $\{e_i\}$ is a basis for V. We show that $\mathcal{R}_\mathcal{C} \in \mathfrak{F}(V)$ by checking:

$$\begin{aligned}\mathcal{R}_\mathcal{C}(e_i,e_j)e_k &= \textstyle\sum_{l,n}\{C_{jl}{}^nC_{ik}{}^l - C_{il}{}^nC_{jk}{}^l\}e_n,\\ \rho_{34}(\mathcal{R}_\mathcal{C})(e_i,e_j) &= \textstyle\sum_{k,l}\{C_{jl}{}^kC_{ik}{}^l - C_{il}{}^kC_{jk}{}^l\} = 0\,.\end{aligned}$$

Let $\mathfrak{B}(V) := \operatorname{Span}_{\mathcal{C}\in S^2(V^*)\otimes V}\{\mathcal{R}_\mathcal{C}\}$. For $\varepsilon \neq 0$, let the non-zero components of $\mathcal{C}$ be given by:

$$C_{21}{}^1 = C_{12}{}^1 = C_{11}{}^2 = C_{31}{}^1 = C_{13}{}^1 = C_{11}{}^3 = \varepsilon\,.$$

We have

$$\rho_{14}(\mathcal{R}_\mathcal{C})(e_2,e_2)=\textstyle\sum_{l,i}\{C_{2l}{}^iC_{i2}{}^l-C_{il}{}^iC_{22}{}^l\}=\varepsilon^2\neq 0\,.$$

This shows that $\rho_{14}(\mathcal{R}_\mathcal{C})(e_2,e_2)\neq 0$. Consequently

$$\sigma_{\pi_s\circ\rho_{14}}S^2(V^*)\subset\mathfrak{B}(V)\,.$$

We also compute

$$\begin{aligned}\mathcal{R}_\mathcal{C}(e_1,e_2)e_1&=\textstyle\sum_{l,n}\{C_{2l}{}^nC_{11}{}^l-C_{1l}{}^nC_{21}{}^l\}e_n\\&=C_{21}{}^1C_{11}{}^1e_1-C_{11}{}^2C_{21}{}^1e_2-C_{11}{}^3C_{21}{}^1e_3\\&=-\varepsilon^2(e_2+e_3)\,.\end{aligned}$$

If $\mathcal{R}_\mathcal{C}\in\sigma_{\pi_s\circ\rho_{14}}S^2(V^*)$, then $\mathcal{R}_\mathcal{C}(e_1,e_2)e_1\in\operatorname{Span}\{e_1,e_2\}$ which is false. Thus

$$\sigma_{\pi_s\circ\rho_{14}}S^2(V^*)\not\subset\mathfrak{B}(V)\,.$$

The desired result now follows. □

4.3. **The action of $O(V,\langle\cdot,\cdot\rangle)$ on $\mathfrak{F}(V)$.** We can use Theorems 2.5 and 4.1 to see that there is an $O(V,\langle\cdot,\cdot\rangle)$ equivariant orthogonal decomposition of

$$\begin{aligned}\mathfrak{f}(V,\langle\cdot,\cdot\rangle)\approx{}&\mathfrak{w}(V,\langle\cdot,\cdot\rangle)\oplus\mathbb{R}\oplus S_0^2(V^*,\langle\cdot,\cdot\rangle)\\&\oplus S_0^2(V^*,\langle\cdot,\cdot\rangle)\oplus\Lambda^2(V^*)\oplus W_7\oplus W_8\end{aligned}$$

is a direct sum of 7 irreducible $O(V,\langle\cdot,\cdot\rangle)$ modules. Since $S_0^2(V^*,\langle\cdot,\cdot\rangle)$ is repeated with multiplicity 2, the decomposition is not unique.

We now make this decomposition a bit more explicit to identify the factors W_7 and W_8. We adopt the notation of Equation (3.a) and let $\mathrm{id}\otimes\pi_s$ symmetrize the last two components of $T\in\otimes^4V^*$. Let $\sigma_{\mathrm{id}\otimes\pi_s}$ be the splitting of Equation (3.c). Finally, let α be the map of Equation (3.d).

Lemma 4.4. *We have an $O(V,\langle\cdot,\cdot\rangle)$ equivariant short exact sequence*

$$0\to\mathfrak{a}(V)\to\mathfrak{f}(V,\langle\cdot,\cdot\rangle)\xrightarrow{\mathrm{id}\otimes\pi_s}\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle)\to 0$$

which is equivariantly split by the map $\sigma_{\mathrm{id}\otimes\pi_s}$.

Proof. Let $F\in\mathfrak{f}(V,\langle\cdot,\cdot\rangle)$. We have

$$\begin{aligned}&(\mathrm{id}\otimes\pi_s)(F)(x,y,z,w)=\tfrac12\{F(x,y,z,w)+F(x,y,w,z)\},\\&(\mathrm{id}\otimes\pi_s)(F)=0\quad\Leftrightarrow\quad F(x,y,z,w)=-F(x,y,w,z)\ \forall\ x,y,z,w\in V\,.\end{aligned}$$

This implies $F\in\mathfrak{a}(V)$. Conversely, if $F\in\mathfrak{a}(V)$, then $\rho_{34}(F)=0$ and $(\mathrm{id}\otimes\pi_s)F=0$ and hence $F\in\mathfrak{f}(V,\langle\cdot,\cdot\rangle)$. Thus

$$\ker\{\mathrm{id}\otimes\pi_s\}\cap\mathfrak{f}(V,\langle\cdot,\cdot\rangle)=\mathfrak{a}(V)\,.$$

Furthermore

$$\rho_{34}(F)=(\mathrm{id}\otimes\mathrm{Tr})((\mathrm{id}\otimes\pi_s)F)$$

and consequently $(\mathrm{id}\otimes\pi_s)$ takes values in $\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle)$.

In the proof of Theorem 3.2, we showed that $\sigma_{\mathrm{id}\otimes\pi_s}$ takes values in $\mathfrak{r}(V)$ and that $(\mathrm{id}\otimes\pi_s)\sigma_{\mathrm{id}\otimes\pi_s}$ is the identity on $\Lambda^2(V)\otimes S^2(V^*)$. Thus $\sigma_{\mathrm{id}\otimes\pi_s}S\in\mathfrak{f}(V,\langle\cdot,\cdot\rangle)$ if and only if $S\in\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle)$. □

This shows that

$$\mathfrak{f}(V,\langle\cdot,\cdot\rangle)\approx\mathfrak{a}(V)\oplus\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle)\,,$$

so

$$\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle)\approx S_0^2(V^*,\langle\cdot,\cdot\rangle)\oplus\Lambda^2(V^*)\oplus W_7\oplus W_8\,.$$

We therefore study $\Lambda^2(V^*)\otimes S^2(V^*)$ as an $O(V,\langle\cdot,\cdot\rangle)$ module and identify the copies of $\Lambda^2(V^*)$ and $S_0^2(V^*,\langle\cdot,\cdot\rangle)$ in $\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle)$. Let

$$\Theta\in\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle),\quad \psi\in S_0^2(V^*,\langle\cdot,\cdot\rangle),\quad \omega\in\Lambda^2(V^*)\,.$$

Let Ξ be as in Equation (1.r). Define:

$$\begin{aligned}
\pi_{1,s}(\Theta)_{jk}&:=(\pi_s(\rho_{14}\theta))_{jk}=\tfrac12\textstyle\sum_{il}\Xi^{il}\{\Theta_{ijkl}+\Theta_{ikjl}\}\,,\\
\pi_{1,a}(\Theta)_{jk}&:=(\pi_a(\rho_{14}\theta))_{jk}=\tfrac12\textstyle\sum_{il}\Xi^{il}\{\Theta_{ijkl}-\Theta_{ikjl}\}\,,\\
\pi_{\Lambda}(\Theta)_{ijkl}&:=\tfrac12(\Theta_{kjil}+\Theta_{ikjl}-\Theta_{ljik}-\Theta_{iljk})\,,\\
\sigma_{\pi_{1,s}}(\psi)_{ijkl}&:=\tfrac1m\{\Xi_{il}\psi_{jk}-\Xi_{jl}\psi_{ik}+\Xi_{ik}\psi_{jl}-\Xi_{jk}\psi_{il}\}\,,\\
\sigma_{\pi_{1,a}}(\omega)_{ijkl}&:=\tfrac{m}{m^2-4}\{\Xi_{il}\omega_{jk}+\Xi_{ik}\omega_{jl}-\Xi_{jl}\omega_{ik}-\Xi_{jk}\omega_{il}+\tfrac4m\omega_{ij}\Xi_{kl}\}\,,\\
\sigma_{\pi_\Lambda}(\Theta)_{ijkl}&:=\tfrac12(\Theta_{kjil}-\Theta_{kijl})\,,\\
\Lambda_0^2(\Lambda^2(V^*))&:=\left\{\Theta:\Theta_{ijkl}=-\Theta_{jikl}=-\Theta_{klij}\,,\ \textstyle\sum_{il}\Xi^{il}\Theta_{ijkl}=0\right\}\,,\\
\mathfrak{S}(V,\langle\cdot,\cdot\rangle)&:=\ker\{\pi_{1,s}\}\cap\ker\{\pi_{1,a}\}\cap\ker\{\pi_\Lambda\}\cap\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle)\,.
\end{aligned}$$

Lemma 4.5. *We have $O(V,\langle\cdot,\cdot\rangle)$ equivariant short exact sequences*

$$\begin{aligned}
&0\to\ker\{\pi_{1,s}\}\to\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle)\xrightarrow{\pi_{1,s}}S_0^2(V^*,\langle\cdot,\cdot\rangle)\to0\,,\\
&0\to\ker\{\pi_{1,a}\}\to\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle)\xrightarrow{\pi_{1,a}}\Lambda^2(V^*)\to0\,,\\
&0\to\ker\{\pi_{1,a}\}\cap\ker\{\pi_\Lambda\}\to\ker\{\pi_{1,a}\}\xrightarrow{\pi_\Lambda}\Lambda_0^2(\Lambda^2(V^*))\to0\,.
\end{aligned}$$

These sequences are equivariantly split, respectively, by $\sigma_{\pi_{1,s}}$, $\sigma_{\pi_{1,a}}$, and σ_{π_Λ}. This gives an $O(V,\langle\cdot,\cdot\rangle)$ equivariant decomposition of

$$\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle)\approx S_0^2(V^*,\langle\cdot,\cdot\rangle)\oplus\Lambda^2(V^*)\oplus\Lambda_0^2(\Lambda^2(V^*))\oplus\mathfrak{S}(V,\langle\cdot,\cdot\rangle)$$

as the direct sum of irreducible $O(V,\langle\cdot,\cdot\rangle)$ modules where

$$\begin{aligned}
&\dim\{S_0^2(V^*,\langle\cdot,\cdot\rangle)\}=\tfrac{(m-1)(m+2)}{2},\\
&\dim\{\Lambda^2(V^*)\}=\tfrac{m(m-1)}{2},\\
&\dim\{\Lambda_0^2(\Lambda^2(V^*))\}=\tfrac{m(m-1)(m-3)(m+2)}{8},\\
&\dim\{\mathfrak{S}(V,\langle\cdot,\cdot\rangle)\}=\tfrac{(m-1)(m-2)(m+1)(m+4)}{8},\\
&\dim\{\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle)\}=\tfrac{m(m-1)^2(m+2)}{8}\,.
\end{aligned}$$

We have $W_8\approx\Lambda_0^2(\Lambda^2(V^*))$ and $W_7\approx\mathfrak{S}(V,\langle\cdot,\cdot\rangle)$.

Proof. It is clear that $\pi_{1,s}$ takes values in $S^2(V^*)$. Let Ξ be as in Equation (1.r). We show that $\pi_{1,s}$ takes values in $S_0^2(V^*,\langle\cdot,\cdot\rangle)$ by checking:

$$\begin{aligned}\mathrm{Tr}\{\pi_{1,s}(\Theta)\} &= \tfrac12\textstyle\sum_{ijkl}\Xi^{il}\Xi^{jk}\{\Theta_{ijkl}+\Theta_{ikjl}\}\\ &= \textstyle\sum_{ijkl}\Xi^{il}\Xi^{jk}\Theta_{ijkl} = \sum_{ijkl}\Xi^{jk}\Xi^{il}\Theta_{jilk}\\ &= -\textstyle\sum_{ijkl}\Xi^{jk}\Xi^{il}\Theta_{ijkl} = -\mathrm{Tr}\{\pi_{1,s}(\Theta)\}\,.\end{aligned}$$

It is clear that $\sigma_{\pi_{1,s}}$ takes values in $\Lambda^2(V^*)\otimes S^2(V^*)$. We verify that $\sigma_{\pi_{1,s}}$ takes values in $\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle)$ by checking the trace condition:

$$\begin{aligned}\textstyle\sum_{kl}\Xi^{kl}\sigma_{\pi_{1,s}}(\psi)_{ijkl} &= \tfrac1m\textstyle\sum_{kl}\Xi^{kl}\{\Xi_{il}\psi_{jk}-\Xi_{jl}\psi_{ik}+\Xi_{ik}\psi_{jl}-\Xi_{jk}\psi_{il}\}\\ &= \tfrac1m\{\psi_{ji}-\psi_{ij}+\psi_{ji}-\psi_{ij}\}=0\,.\end{aligned}$$

We check that $\sigma_{\pi_{1,s}}$ is a splitting by verifying:

$$\begin{aligned}\pi_{1,s}(\sigma_{\pi_{1,s}}(\psi))_{jk} &= \tfrac1{2m}\textstyle\sum_{il}\Xi^{il}\{\Xi_{il}\psi_{jk}-\Xi_{jl}\psi_{ik}+\Xi_{ik}\psi_{jl}-\Xi_{jk}\psi_{il}\\ &\quad+\Xi_{il}\psi_{kj}-\Xi_{kl}\psi_{ij}+\Xi_{ij}\psi_{kl}-\Xi_{kj}\psi_{il}\}\\ &= \tfrac1{2m}\{m\psi_{jk}-\psi_{jk}+\psi_{jk}-\Xi_{jk}\,\mathrm{Tr}\{\psi\}\\ &\quad+m\psi_{kj}-\psi_{kj}+\psi_{kj}-\Xi_{kj}\,\mathrm{Tr}\{\psi\}\}\\ &= \psi_{jk}\,.\end{aligned}$$

Clearly $\pi_{1,a}$ takes values in $\Lambda^2(V^*)$ and $\sigma_{\pi_{1,a}}$ takes values in $\Lambda^2(V^*)\otimes S^2(V^*)$. We check the trace condition by computing:

$$\begin{aligned}\{(\mathrm{id}\otimes\mathrm{Tr})(\sigma_{\pi_{1,a}}(\omega))\}_{ij} &= \tfrac{m}{m^2-4}\textstyle\sum_{kl}\Xi^{kl}\{\Xi_{il}\omega_{jk}+\Xi_{ik}\omega_{jl}-\Xi_{jl}\omega_{ik}-\Xi_{jk}\omega_{il}+\frac4m\omega_{ij}\Xi_{kl}\}\\ &= \tfrac{m}{m^2-4}\{\omega_{ji}+\omega_{ji}-\omega_{ij}-\omega_{ij}+\tfrac4m m\omega_{ij}\}\\ &= \tfrac{m}{m^2-4}(-4+\tfrac4m m)\omega_{ij}=0\,.\end{aligned}$$

To check $\sigma_{\pi_{1,a}}$ is a splitting, we compute:

$$\begin{aligned}\pi_{1,a}(\sigma_{\pi_{1,a}}(\omega))_{jk} &= \tfrac12\tfrac{m}{m^2-4}\textstyle\sum_{il}\Xi^{il}\{\Xi_{il}\omega_{jk}+\Xi_{ik}\omega_{jl}-\Xi_{jl}\omega_{ik}-\Xi_{jk}\omega_{il}+\frac4m\omega_{ij}\Xi_{kl}\\ &\quad-\Xi_{il}\omega_{kj}-\Xi_{ij}\omega_{kl}+\Xi_{kl}\omega_{ij}+\Xi_{kj}\omega_{il}-\tfrac4m\omega_{ik}\Xi_{jl}\}\\ &= \tfrac12\tfrac{m}{m^2-4}\{m\omega_{jk}+\omega_{jk}-\omega_{jk}+\tfrac4m\omega_{kj}-m\omega_{kj}-\omega_{kj}+\omega_{kj}-\tfrac4m\omega_{jk}\}\\ &= \tfrac{m}{m^2-4}\{m-\tfrac4m\}\omega_{jk}=\omega_{jk}\,.\end{aligned}$$

Let $S\in\ker\{\pi_{1,a}\}\cap\{\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle)\}$. To check that π_Λ takes values in $\Lambda_0^2(\Lambda^2(V^*))$, we compute:

$$\begin{aligned}\pi_\Lambda(S)_{ijkl} &= \tfrac12(S_{kjil}+S_{ikjl}-S_{ljik}-S_{iljk})\,,\\ \pi_\Lambda(S)_{jikl} &= \tfrac12(S_{kijl}+S_{jkil}-S_{lijk}-S_{jlik})=-\pi_\Lambda(S)_{ijkl}\,,\\ \pi_\Lambda(S)_{klij} &= \tfrac12(S_{ilkj}+S_{kilj}-S_{jlki}-S_{kjli})=-\pi_\Lambda(S)_{ijkl}\,,\\ \rho_{14}(\pi_\Lambda(S))_{jk} &= \tfrac12\textstyle\sum_{il}\Xi^{il}\{S_{kjil}+S_{ikjl}-S_{ljik}-S_{iljk}\}\\ &= \{\tfrac12\rho_{34}(S)+\pi_{1,a}(S)\}_{jk}=0\,.\end{aligned}$$

Let $T \in \Lambda_0^2(\Lambda^2(V^*))$. To check σ_{π_Λ} takes values in $\Lambda^2(V^*) \otimes S_0^2(V^*, \langle\cdot,\cdot\rangle)$, we compute:

$$\begin{aligned}
\sigma_{\pi_\Lambda}(T)_{ijkl} &= \tfrac{1}{2}(T_{kjil} - T_{kijl})\,, \\
\sigma_{\pi_\Lambda}(T)_{jikl} &= \tfrac{1}{2}(T_{kijl} - T_{kjil}) = -\sigma_{\pi_\Lambda}(T)_{ijkl}\,, \\
\sigma_{\pi_\Lambda}(T)_{ijlk} &= \tfrac{1}{2}(T_{ljik} - T_{lijk}) = \tfrac{1}{2}(T_{jkli} - T_{iklj}) \\
&= \tfrac{1}{2}(T_{kjil} - T_{kijl}) = \sigma_{\pi_\Lambda}(T)_{ijlk}\,, \\
\textstyle\sum_{kl} \Xi^{kl}\sigma_{\pi_\Lambda}(T)_{ijkl} &= \frac{1}{2}\sum_{kl} \Xi^{kl}(T_{kjil} - T_{kijl}) = 0\,.
\end{aligned}$$

Finally, we verify that σ_{π_Λ} is a splitting by computing

$$\begin{aligned}
\{\pi_\Lambda(\sigma_{\pi_\Lambda}(T))\}_{ijkl} &= \tfrac{1}{2}(\sigma_{\pi_\Lambda}(T)_{kjil} + \sigma_{\pi_\Lambda}(T)_{ikjl} - \sigma_{\pi_\Lambda}(T)_{ljki} - \sigma_{\pi_\Lambda}(T)_{ilkj}) \\
&= \tfrac{1}{4}(T_{ijkl} - T_{ikjl} + T_{jkil} - T_{jikl} - T_{kjli} + T_{klji} - T_{klij} + T_{kilj}) \\
&= T_{ijkl}\,.
\end{aligned}$$

We compute dimensions:

$$\begin{aligned}
\dim\{\Lambda^2(V^*)\} &= \tfrac{1}{2}m(m-1)\,, \\
\dim\{\Lambda^2(\Lambda^2(V^*))\} &= \tfrac{1}{2}\{\tfrac{1}{2}m(m-1)\}\{\tfrac{1}{2}m(m-1)-1\}\,, \\
\dim\{\Lambda_0^2(\Lambda^2(V^*))\} &= \dim\{\Lambda^2(\Lambda^2(V^*))\} - \dim\{\Lambda^2(V^*)\} \\
&= \tfrac{1}{2}\{\tfrac{1}{2}m(m-1)\}\{\tfrac{1}{2}m(m-1)-1\} - \tfrac{1}{2}m(m-1) \\
&= \{\tfrac{1}{2}m(m-1)\}\{\tfrac{1}{4}m(m-1) - \tfrac{1}{2} - 1\} \\
&= \tfrac{1}{8}\{m(m-1)\}\{m(m-1)-6\} = \tfrac{1}{8}m(m-1)(m-3)(m+2) \\
&= \dim\{W_8\}
\end{aligned}$$

and

$$\begin{aligned}
\dim\{\mathfrak{S}(V,\langle\cdot,\cdot\rangle)\} =&\ \dim\{\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle)\} - \dim\{\Lambda_0^2(\Lambda^2(V^*))\} \\
&- \dim\{S_0^2(V^*,\langle\cdot,\cdot\rangle)\} - \dim\{\Lambda^2(V^*)\} \\
=&\ \dim\{\Lambda^2(V^*)\otimes S_0^2(V^*,\langle\cdot,\cdot\rangle)\} \\
&- \dim\{\Lambda^2(\Lambda^2(V^*))\} - \dim\{S_0^2(V^*,\langle\cdot,\cdot\rangle\} \\
=&\ \tfrac{m(m-1)(m-1)(m+2)}{4} - \tfrac{m(m-1)(m(m-1)-2)}{8} - \tfrac{(m-1)(m+2)}{2} \\
=&\ \tfrac{m-1}{8}\{2m(m-1)(m+2) - m(m-2)(m+1) - 4(m+2)\} \\
=&\ \tfrac{(m-1)(m-2)(m+1)(m+4)}{8} = \dim\{W_7\}\,.
\end{aligned}$$

The remaining assertions now follow from Theorem 2.5 (1); this also establishes Theorem 2.5 (2c). $\square$

As an immediate consequence, we have

Theorem 4.6.

(1) *There is an $O(V,\langle\cdot,\cdot\rangle)$ equivariant orthogonal decomposition of*

$$\mathfrak{F}(V)\approx\mathfrak{f}(V)=W_1\oplus W_2\oplus W_4\oplus W_5\oplus W_6\oplus W_7\oplus W_8$$

as the direct sum of irreducible $O(V,\langle\cdot,\cdot\rangle)$ modules where:

$$\begin{array}{ll}\dim\{W_1\}=1\,, & \dim\{W_2\}=\dim\{W_5\}=\frac{(m-1)(m+2)}{2}\,,\\ \dim\{W_4\}=\frac{m(m-1)}{2}\,, & \dim\{W_6\}=\frac{m(m+1)(m-3)(m+2)}{12}\,,\\ \dim\{W_7\}=\frac{(m-1)(m-2)(m+1)(m+4)}{8}\,, & \dim\{W_8\}=\frac{m(m-1)(m-3)(m+2)}{8}\,.\end{array}$$

(2) *There are the following isomorphisms as $O(\langle\cdot,\cdot\rangle)$ modules:*
   (a) $W_1\approx\mathbb{R}$, $W_2\approx W_5\approx S_0^2(V^*,\langle\cdot,\cdot\rangle)$, *and* $W_4\approx\Lambda^2(V^*)$.
   (b) $W_6\approx\mathfrak{w}(V,\langle\cdot,\cdot\rangle)$ *is the space of Weyl conformal curvature tensors.*
   (c) $W_7\approx\mathfrak{S}(V,\langle\cdot,\cdot\rangle)$ *and* $W_8\approx\Lambda_0^2(\Lambda^2(V^*))$.

5. The proof of Theorem 3.1

Let $\mathfrak{b}$ be a non-empty subspace of $\mathfrak{a}(V)$ which is invariant under the action of $GL(V)$. We must show that $\mathfrak{b}=\mathfrak{a}(V)$. Choose a positive definite inner product $\langle\cdot,\cdot\rangle$ on V. Then $\mathfrak{b}$ is invariant under the action of $O(V,\langle\cdot,\cdot\rangle)$ as well. Let $\pi_{\mathfrak{w}}$, π_0, and $\pi_{\mathbb{R}}$ be the projections on the appropriate module summands in the decomposition of Theorem 3.2 (3);

$$\begin{aligned}&\pi_{\mathbb{R}}(R):=\tau(\rho_{14}(R))\,,\qquad \pi_0(R):=\rho_{14}(R)-\tfrac{1}{m}\tau(\rho_{14}(R))\langle\cdot,\cdot\rangle\,,\\ &\pi_{\mathfrak{w}}(R):=R-\sigma_{\mathfrak{a},\rho_{14}}(\rho_{14}(R))\quad\text{where}\\ &\sigma_{\mathfrak{a},\rho_{14}}(\psi):=\tfrac{2}{m-2}\psi\wedge\langle\cdot,\cdot\rangle-\tfrac{\tau(\psi)}{(m-1)(m-2)}\langle\cdot,\cdot\rangle\wedge\langle\cdot,\cdot\rangle\,.\end{aligned}$$

Since $O(V,\langle\cdot,\cdot\rangle)$ is a compact Lie group acting orthogonally, the projections are orthogonal projections. Furthermore:

$$\begin{aligned}\pi_{\mathfrak{w}}(\mathfrak{b})\neq\{0\}&\Rightarrow\mathfrak{w}(V,\langle\cdot,\cdot\rangle)\subset\mathfrak{b}\,,\\ \pi_0(\mathfrak{b})\neq\{0\}&\Rightarrow\sigma_{\mathfrak{a},\rho_{14}}(S_0^2(V,\langle\cdot,\cdot\rangle))\subset\mathfrak{b}\,,\\ \pi_{\mathbb{R}}(\mathfrak{b})\neq\{0\}&\Rightarrow\sigma_{\mathfrak{a},\rho_{14}}(\langle\cdot,\cdot\rangle)\subset\mathfrak{b}\,.\end{aligned}$$

Let $\{e_i\}$ be an orthonormal basis for V. Let $\{\lambda_i\}$ be distinct positive constants. Define $\Theta\in GL(V)$ by setting:

$$\Theta(e_i)=\lambda_i e_i\,.$$

Suppose $\pi_{\mathbb{R}}(\mathfrak{b})\neq\{0\}$. The component corresponding to $\mathbb{R}$ in $\mathfrak{a}(V)$ is generated by $A:=\langle\cdot,\cdot\rangle\wedge\langle\cdot,\cdot\rangle$. Consequently $A\in\mathfrak{b}$; the non-zero components of $\Theta^*(A)$ and $\rho_{14}(\Theta^*(A))$ are, up to the usual $\mathbb{Z}_2$ symmetries and modulo a suitable normalizing constant which plays no role, given by

$$\Theta^*(A)(e_i,e_j,e_j,e_i)=\lambda_i^2\lambda_j^2\quad\text{and}\quad\rho_{14}(\Theta^*(A))(e_i,e_i)=\lambda_i^2\textstyle\sum_{j\neq i}\lambda_j^2\,.$$

This shows the projection of $\Theta^*(A)$, and hence of $\mathfrak{b}$, on $S_0(V^*,\langle\cdot,\cdot\rangle)$ is non-zero. Let A_1 be the algebraic curvature tensor whose only non-zero component, up to the usual $\mathbb{Z}_2$ symmetries, is $A_1(e_1, e_2, e_2, e_1)$. As $\mathfrak{b}$ is closed, we show that $A_1 \in \mathfrak{b}$ by taking the limit

$$\lambda_1 \to 1, \quad \lambda_2 \to 1, \quad \lambda_j \to 0 \text{ for } j \geq 3\,.$$

As $\{A_1 - \sigma_{\mathfrak{a},\rho_{14}}(\rho_{14}(A_1))\}(e_1, e_3, e_3, e_1) \neq 0$, one has $\pi_{\mathfrak{w}}(\mathfrak{b}) \neq 0$. We summarize:

$$\pi_{\mathbb{R}}(\mathfrak{b}) \neq 0 \quad \Rightarrow \quad \mathfrak{b} = \mathfrak{a}(V)\,.$$

Suppose $\pi_0(\mathfrak{b}) \neq 0$. Then $\sigma_{\mathfrak{a},\rho_{14}}(S_0^2(V^*, \langle\cdot,\cdot\rangle)) \subset \mathfrak{b}$. Define $\psi \in S_0^2(V^*, \langle\cdot,\cdot\rangle)$ with non-zero components

$$\psi(e_1, e_1) = 1, \quad \psi(e_2, e_2) = 1, \quad \text{and} \quad \psi(e_3, e_3) = -2\,.$$

Let $A = \sigma_{\mathfrak{a},\rho_{14}}(\psi) = \frac{2}{m-2}\psi \wedge \langle\cdot,\cdot\rangle \in \mathfrak{b}$. We compute:

$$\begin{aligned}
\Theta^*(A)(e_i, e_j, e_k, e_l) &= \lambda_i\lambda_j\lambda_k\lambda_l \tfrac{2}{m-2}\{\psi(e_i, e_l)\delta_{jk} + \psi(e_j, e_k)\delta_{il} \\
&\quad - \psi(e_i, e_k)\delta_{jl} - \psi(e_j, e_l)\delta_{ik}\}\,, \\
\tau(\rho_{14}(\Theta^*(A))) &= \tfrac{2}{m-2}\textstyle\sum_{i,j} \Theta^* A(e_i, e_j, e_j, e_i) \\
&= \tfrac{2}{m-2}\{(\lambda_1^2 + \lambda_2^2 - 2\lambda_3^2)(\textstyle\sum_j \lambda_j^2)\} - \tfrac{2}{m-2}\textstyle\sum_i\{\lambda_1^4 + \lambda_2^4 - 2\lambda_3^4\}\,.
\end{aligned}$$

This is non-zero for generic values of $\vec{\lambda}$. This shows $\pi_{\mathbb{R}}(\mathfrak{b}) \neq \{0\}$. Combining this result with the result of the previous paragraph yields:

$$\rho_{14}(\mathfrak{b}) \neq \{0\} \quad \Rightarrow \quad \mathfrak{b} = \mathfrak{a}(V)\,.$$

Finally, suppose $\pi_{\mathfrak{w}}(\mathfrak{b}) \neq 0$. Then $\mathfrak{w}(V, \langle\cdot,\cdot\rangle) \subset \mathfrak{b}$. Let $A \in \mathfrak{a}$ be defined with non-zero components, up to the usual $\mathbb{Z}_2$ symmetries, by

$$A(e_1, e_3, e_4, e_1) = +1 \quad \text{and} \quad A(e_2, e_3, e_4, e_2) = -1\,.$$

Then $\rho_{14}(A) = 0$ so $A \in \mathfrak{w}(V, \langle\cdot,\cdot\rangle)$. We have

$$\Theta^*(A)(e_1, e_3, e_4, e_1) = \lambda_1^2\lambda_3\lambda_4 \quad \text{and} \quad \Theta^*(A)(e_2, e_3, e_4, e_2) = \lambda_2^2\lambda_3\lambda_4\,.$$

Thus $\rho_{14}(\Theta^*(A))(e_3, e_4) = \lambda_3\lambda_4(\lambda_1^2 - \lambda_2^2) \neq 0$. Since $\rho_{14}(\Theta^*(A)) \neq 0$ we may conclude that $\mathfrak{b} = \mathfrak{a}(V)$. □

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