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FINELY DIFFERENTIABLE MONOGENIC FUNCTIONS

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Abstract. Since 1970's B. Fuglede and others have been studying finely holomorhic functions, i.e., 'holomorphic' functions defined on the so-called fine domains which are not necessarily open in the usual sense. This note is a survey of finely monogenic functions which were introduced in [12] like a higher dimensional analogue of finely holomorphic functions.

1. Introduction

At the beginning of the 20th century É. Borel worked on the idea that certain holomorphic functions can be continued beyond their classical maximal domain of existence to a larger (not necessarily open) domain, see [2]. A significant progress in the same direction was made not earlier than in 1970's when B. Fuglede extended the notion of holomorphic functions to those defined on domains from a topology finer than the Euclidean one, namely, the fine topology of potential theory. A very deep theory of the so-called finely holomorphic functions has been developed since then, see [4], [5] or [8].

The Clifford analysis may be considered as a higher dimensional analogue of classical complex analysis. In the Clifford analysis, functions called here monogenic are counterparts of holomorphic ones. In [12], finely monogenic functions were introduced. This note presents results on finely monogenic functions obtained in [11], [12] and [13]. In [11], a special case of dimension 4 is considered.

2. Monogenic functions

The Clifford analysis studies functions taking values in Clifford modules that we are going to introduce now, see e.g. [9, Chapter 2]. Consider a real (or complex) finite dimensional Hilbert space $\mathcal{H} = \big(\mathcal{H}, (\cdot,\cdot)\big)$. Denote by $\mathcal{L}(\mathcal{H})$ the algebra of linear operators on $\mathcal{H}$ and by $\bar{a}$ the adjoint operator to $a \in \mathcal{L}(\mathcal{H})$, i.e., $(au, v) = (u, \bar{a}v)$, $u, v \in \mathcal{H}$. In what follows, we suppose that $m \geq 1$ and $\mathcal{H}$ is a (left) $C\ell_m$-module, i.e., there are skew-adjoint operators $e_1, \dots, e_m$ in $\mathcal{L}(\mathcal{H})$ (i.e., $\bar{e}_j = -e_j$) such that

$$e_j^2 = -e_0 , \quad e_j e_k = -e_k e_j$$

The paper is in final form and no version of it will be submitted elsewhere.

for $j,k = 1,\dots,m$, $j \neq k$ where e_0 is the identity operator on $\mathcal{H}$. Then the Euclidean space $\mathbb{R}^{m+1}$ can be naturally embedded into $\mathcal{L}(\mathcal{H})$ in the following way:

$$x = (x_0,\dots,x_m) \cong x_0e_0 + \dots + x_me_m\,.$$

Low dimensional examples of Clifford modules are the complex plane $\mathbb{C}$ and the skew field of real quaternions $\mathbb{H}$. Indeed, the complex plane $\mathbb{C}$ is a $C\ell_1$-module with $e_1(z) = iz$. As to the latter case, recall that the field $\mathbb{H}$ can be viewed as the Euclidean space $\mathbb{R}^4$ endowed with a non-commutative multiplication. A quaternion x can be written in the form $x = x_0 + x_1i + x_2j + x_3k$ where x_0, x_1, x_2, x_3 are real numbers and i, j, k are the imaginary units such that

$$i^2 = j^2 = k^2 = -1\,, \quad ij = -ji = k\,, \quad jk = -kj = i\,, \quad ki = -ik = j\,.$$

Then $\mathbb{H}$ is a $C\ell_3$-module with $e_1(x) = ix$, $e_2(x) = jx$ and $e_3(x) = kx$. In a general dimension, examples of Clifford modules are given by the corresponding real and complex Clifford algebras and spinor spaces, see e.g. [9, p. 60].

Now we are ready to introduce monogenic functions. Given an open set $G \subset \mathbb{R}^{m+1}$ and a $C\ell_m$-module $\mathcal{H}$, denote by $\mathcal{C}^1(G)$ the set of continuously differentiable functions $f : G \to \mathcal{H}$ and define the Cauchy-Riemann operator by

$$\mathbf{D} = \sum_{j=0}^{m} e_j \, \frac{\partial}{\partial x_j}\,.$$

Definition 1. A function $f \in \mathcal{C}^1(G)$ is called monogenic if $\mathbf{D}f = 0$ on G.

It is well known that a function f is monogenic if and only if both f and $xf(x)$ are harmonic where $xf(x) := (x_0e_0 + \dots + x_me_m)f(x)$. Let us remark that the Clifford analysis includes classical complex analysis. Indeed, in the complex case monogenic functions coincide with holomorphic ones. Furthermore, in the quaternionic case the so-called quaternionic analysis developed by R. Fueter in 1930's is obtained, see e.g. [9, Chapter 2] for details.

3. Fine topology

For an account of the fine topology, we refer to [1, Chapter 7]. The fine topology $\mathcal{F}$ in $\mathbb{R}^{m+1}$ is the weakest topology making all subharmonic functions in $\mathbb{R}^{m+1}$ continuous. It is strictly finer than the Euclidean topology in $\mathbb{R}^{m+1}$. For example, if M is a dense countable subset of an open set $G \subset \mathbb{R}^{m+1}$, then $U := G \setminus M$ is a finely open set but it has no interior points in the usual sense.

Let $U \subset \mathbb{R}^{m+1}$ be finely open and $f : U \to \mathbb{R}^k$. Then we call f finely continuous on U if f is continuous from U endowed with the fine topology to $\mathbb{R}^k$ with the Euclidean topology. Denote by $\mathcal{F}_x$ the family of fine neighbourhoods of a point $x \in \mathbb{R}^{m+1}$. The fine limit of f at a point $x \in U$ can be characterised as the usual limit along some fine neighbourhood of x, i.e., there is $V \in \mathcal{F}_x$ such that

$$\operatorname*{fine-lim}_{y\to x} f(y) = \lim_{y\to x, y\in V} f(y)\,,$$

see [1, p. 207]. Moreover, we call a linear map $L : \mathbb{R}^{m+1} \to \mathbb{R}^k$ the fine differential of f at a point $x \in U$ if

$$\text{fine-}\lim_{y\to x} \frac{f(y) - f(x) - L(y-x)}{|y-x|} = 0\,.$$

Here $|x|$ is the Euclidean norm of $x \in \mathbb{R}^{m+1}$. We write fine-$df(x)$ for L and set, for $j = 0, \dots, m$,

$$\text{fine-}\frac{\partial f}{\partial x_j}(x) := \text{fine-}df(x)(\mathbf{e}_j)$$

where the vectors $\mathbf{e}_0 \dots, \mathbf{e}_m$ form the standard basis of $\mathbb{R}^{m+1}$.

In 1970-80's a very deep theory of finely holomorhic functions has been developed, see e.g. [4], [5] or [8]. Recall that, given a finely open subset V of the complex plane $\mathbb{C}$, a function $f : V \to \mathbb{C}$ is called finely holomorphic provided that f has a finely continuous fine derivative f' on V. Here

$$f'(z) = \text{fine-}\lim_{w\to z} \frac{f(w) - f(z)}{w - z},\ z \in V\,.$$

Moreover, in [4], B. Fuglede proved the following theorem.

Theorem 1. *A function f is finely holomorphic on a finely open set $V \subset \mathbb{C}$ if and only if, for each $z \in V$, there is $K \in \mathcal{F}_z$ and $F \in \mathcal{C}^1(\mathbb{C})$ such that $F = f$ on K and $\bar\partial F = 0$ on K where $z = x + iy \in \mathbb{C}$ and*

$$\bar\partial F := \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right)\,.$$

Finely holomorphic functions are closely related to finely harmonic ones. Indeed, a function f is finely holomorphic on a finely open set $V \subset \mathbb{C}$ if and only if, the functions f and $zf(z)$ are both finely harmonic on V. For an account of finely harmonic functions, we refer to [3]. Let us recall that a function f is finely harmonic on a finely open set $U \subset \mathbb{R}^{m+1}$ if and only if, for every $x \in U$, there is $V \in \mathcal{F}_x$ such that $f|_V$, the restriction of f to V, is a uniform limit of functions f_n harmonic on open sets V_n containing V.

4. Finely monogenic functions

Now we introduce finely monogenic functions. In what follows, we suppose that $\mathcal{H}$ is a $C\ell_m$-module, $U \subset \mathbb{R}^{m+1}$ be finely open and $f : U \to \mathcal{H}$ unless otherwise stated.

Definition 2. A function f is called finely monogenic if f and $xf(x)$ are both finely harmonic on U.

When $m = 1$ we get nothing else but finely holomorphic functions introduced by B. Fuglede, see [4], [5] or [8]. A function f is monogenic on a usual open set $G \subset \mathbb{R}^{m+1}$ if and only if f is finely monogenic and locally bounded on G because the same is true even for finely harmonic functions. Moreover, when $m = 1$ we do not need to assume local boundedness of f. See [3, Theorem 10.16].

Recall that the Sobolev space $W^{1,2}(\mathbb{R}^{m+1})$ consists of (Lebesgue) measurable functions F whose second power is integrable on $\mathbb{R}^{m+1}$ together with second powers of its first weak derivatives. Denote by $W^{1,2}_{\text{f-loc}}(U)$ the set of functions f on U satisfying that, for each $x \in U$, there exist $K \in \mathcal{F}_x$ and $F \in W^{1,2}(\mathbb{R}^{m+1})$ such that $F = f$ on K. For an account of the Sobolev spaces on fine domains, we refer to [10]. Now we are ready to state other characterisations of finely monogenic functions, see [12].

Theorem 2. *The following statements are equivalent to each other:*

(a) *f is finely monogenic on U,*

(b) *f is finely continuous on U, $f \in W^{1,2}_{\text{f-loc}}(U)$ and $\mathbf{D}f = 0$ on U,*

(c) *f is finely harmonic on U and* fine-$\mathbf{D}f = 0$ *almost everywhere on U, i.e., except for a Lebesgue null set. Here*

$$\text{fine-}\mathbf{D}f = \sum_{j=0}^{m} e_j \,\text{fine-}\frac{\partial f}{\partial x_j}$$

at each point where f is finely differentiable.

To state our next result we need some notation. Let us denote by fine-$\mathcal{C}^1(U)$ the set of all functions f finely differentiable everywhere on U whose fine differential fine-df is finely continuous on U. Moreover, $\mathcal{C}^1_{\text{f-loc}}(U)$ stands for the set of all functions f on U such that, for each $x \in U$, there is $K \in \mathcal{F}_x$ and $F \in \mathcal{C}^1(\mathbb{R}^{m+1})$ with $F = f$ on K. Then the following theorem is proved in [13].

Theorem 3. *Let $m \geq 1$ and $U \subset \mathbb{R}^{m+1}$ be finely open. Then*

$$\mathcal{C}^1_{\text{f-loc}}(U) = \text{fine-}\mathcal{C}^1(U) \cap W^{1,2}_{\text{f-loc}}(U)\,.$$

In the case where $m = 1$, it is true even that $\mathcal{C}^1_{\text{f-loc}}(U) = \text{fine-}\mathcal{C}^1(U)$.

Let us remark that Theorem 3 for $m = 1$ is essentially due to B. Fuglede. If $m \geq 2$, then it seems to be open whether

$$\text{fine-}\mathcal{C}^1(U) \subset W^{1,2}_{\text{f-loc}}(U)$$

or not. Since finely holomorphic functions are infinitely fine differentiable (in particular, they belong to fine-$\mathcal{C}^1(U)$) the following result generalises Theorem 1 mentioned above.

Theorem 4. *A function f is finely monogenic and $f \in \text{fine-}\mathcal{C}^1(U)$ if and only if $f \in \mathcal{C}^1_{\text{f-loc}}(U)$ and* fine-$\mathbf{D}f = 0$ *on U.*

Proof. Let us notice that, by Theorem 2 (b), any finely monogenic function f belongs to $W^{1,2}_{\text{f-loc}}(U)$. Now it is easy to see that Theorem 4 is a direct consequence of Theorem 3 stated above. □

In comparison with finely harmonic functions, finely holomorphic functions are infinitely fine differentiable everywhere and have the unique continuation property, i.e., any finely holomorphic function f on a fine domain U in $\mathbb{C}$ is uniquely determined by its values in some fine neighbourhood of a point of U, see [5] or [4]. It would be interesting to clear up to what extent these properties remain true for finely monogenic functions in higher dimensions.

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