ARCHIVUM MATHEMATICUM (BRNO)
Tomus 42 (2006), Supplement, 241 – 243

NEARLY KÄHLER AND NEARLY PARALLEL G_2-STRUCTURES ON SPHERES

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Abstract. In some other context, the question was raised how many nearly Kähler structures exist on the sphere $\mathbb{S}^6$ equipped with the standard Riemannian metric. In this short note, we prove that, up to isometry, there exists only one. This is a consequence of the description of the eigenspace to the eigenvalue $\lambda = 12$ of the Laplacian acting on 2-forms. A similar result concerning nearly parallel G_2-structures on the round sphere $\mathbb{S}^7$ holds, too. An alternative proof by Riemannian Killing spinors is also indicated.

Consider the 6-dimensional sphere $\mathbb{S}^6 \subset \mathbb{R}^7$ equipped with its standard metric. Denote by Δ the Hodge-Laplace operator acting an 2-forms of $\mathbb{S}^6$ and consider the space

$$E_{12} := \left\{\omega^2 \in \Gamma(\Lambda^2(\mathbb{S}^6)) : d * \omega^2 = 0\,, \quad \Delta(\omega^2) = 12 \cdot \omega^2\right\}.$$

This space is an SO(7)-representation. Moreover, it coincides with the full eigenspace of the Laplace operator acting on 2-forms with eigenvalue $\lambda = 12$.

Proposition 1. *The* SO(7)*-representation E_{12} is isomorphic to $\Lambda^3(\mathbb{R}^7)$. More precisely, for any 2-form $\omega^2 \in E_{12}$, there exists a unique algebraic 3-form $A \in \Lambda^3(\mathbb{R}^7)$ such that*

$$\omega^2_x(y, z) = A(x, y, z)$$

holds at any point $x \in \mathbb{S}^6$ for any two tangent vectors $y, z \in T_x(\mathbb{S}^6)$.

Proof. It is easy to check that any 2-form ω^2 on $\mathbb{S}^6$ defined by a 3-form $A \in \Lambda^3(\mathbb{R}^7)$ as indicated satisfies the differential equations $d * \omega^2 = 0$, $\Delta(\omega^2) = 12 \cdot \omega^2$. Consequently, we obtain an SO(7)-equivariant map

$$\Lambda^3(\mathbb{R}^7) \longrightarrow E_{12}\,.$$

Since $\Lambda^3(\mathbb{R}^7)$ is an irreducible SO(7)-representation, the map is injective. On the other hand, by Frobenius reciprocity, one computes the dimension of the eigenspace

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2000 *Mathematics Subject Classification.* Primary 53C25; Secondary 81T30.
Key words and phrases. nearly Kähler structures, nearly parallel G_2-structures.
Supported by the SFB 647 "Raum, Zeit,Materie" and the SPP 1154 "Globale Differentialgeometrie" of the DFG.
The paper is in final form and no version of it will be submitted elsewhere.

of the Laplace operator on 2-forms to the eigenvalue $\lambda = 12$. Its dimension equals 35. $\square$

We recall some basic properties of nearly Kähler manifolds in dimension six (see the paper [1]). Let (M^6, J, g) be a nearly Kähler 6-manifold. Then it is an Einstein space with positive scalar curvature $\mathrm{Scal} > 0$. The Kähler form Ω satisfies the differential equations

$$d * \Omega = 0\,, \quad \Delta(\Omega) = \frac{2}{5} \cdot \mathrm{Scal} \cdot \Omega\,.$$

In particular, the Kähler form Ω^{J} of *any* nearly Kähler structure $(\mathbb{S}^6, \mathrm{J}, g_{\mathrm{can}})$ on the standard sphere $\mathbb{S}^6$ is a 2-form on $\mathbb{S}^6$ satisfying the equations $d * \Omega^{\mathrm{J}} = 0$ and $\Delta(\Omega^{\mathrm{J}}) = 12 \cdot \Omega^{\mathrm{J}}$. This observation yields the following result.

Proposition 2. *The Kähler form Ω^{J} of any nearly Kähler structure $(\mathbb{S}^6, \mathrm{J}, g_{\mathrm{can}})$ on the standard sphere is given by an algebraic 3-form $A \in \Lambda^3(\mathbb{R}^7)$ via the formula*

$$\Omega^{\mathrm{J}}_x(y, z) = A(x, y, z)$$

where $x \in \mathbb{S}^6$ is a point in the sphere and $y, z \in T_x(\mathbb{S}^6)$ are tangent vectors.

Since the Kähler form Ω^{J} is a non-degenerate 2-form at any point of the sphere $\mathbb{S}^6$, the 3-form $A \in \Lambda^3(\mathbb{R}^7)$ is a non-degenerate vector cross product in the sense of Gray (see [2], [4], [5]). For purely algebraic reasons it follows that two forms of that type are equivalent under the action of the group $\mathrm{SO}(7)$. Finally, we obtain the following

Theorem 1. *Let $(\mathbb{S}^6, \mathrm{J}, g_{can})$ be a nearly Kähler structure on the standard 6-sphere. Then the almost complex structure J is conjugated – under the action of the isometry group $\mathrm{SO}(7)$ – to the standard nearly Kähler structure of $\mathbb{S}^6$.*

A similar argument applies in dimension seven, too.

Theorem 2. *Let $(\mathbb{S}^7, \omega, g_{\mathrm{can}})$ be a nearly parallel G_2-structure on the standard 7-sphere. Then it is conjugated – under the action of the isometry group $\mathrm{SO}(8)$ – to the standard nearly parallel G_2-structure of $\mathbb{S}^7$.*

Remark. Nearly Kähler structures in dimension six and nearly parallel structures in dimension seven correspond to Riemannian Killing spinors. It is well-known that the isometry group of the spheres $\mathbb{S}^6$ and $\mathbb{S}^7$ acts transitively on the set of Killing spinor of length one. This observation yields a second proof of the latter Theorems (see [3] and [6]). Moreover, this argument proves that on a space different from the sphere the nearly Kähler structure ($n = 6$) is uniquely determined by the metric. Indeed, the space of Killing spinors is one-dimensional.

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