ARCHIVUM MATHEMATICUM (BRNO)
Tomus 42 (2006), Supplement, 197 – 211

ALGEBRAIC ANALYSIS OF THE RARITA–SCHWINGER SYSTEM IN REAL DIMENSION THREE

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Abstract. In this paper we use the explicit description of the Spin–$\frac{3}{2}$ Dirac operator in real dimension 3 appeared in [17] to perform the algebraic analysis of the space of nullsolution of the system of equations given by several Rarita–Schwinger operators. We make use of the general theory provided by [10] and some standard Gröbner Bases techniques. Our aim is to show that such operator shares many of the algebraic properties of the Dirac operator in real dimension four. In particular, we prove the exactness of the associated algebraic complex, a duality result and we explicitly describe the space of polynomial solutions.

1. Introduction

A large class of invariant differential operators in parabolic geometries can be constructed by taking suitable projections of the covariant derivative onto sections of irreducible representation bundles defined on the corresponding homogeneous space. See for example [24] for a general construction of such operators. The first and maybe simplest example is given by the Dirac operator, which is an invariant first order operator with respect to the action of the Spin group on spaces of functions with values in the basic spinor representation. Its analytic and algebraic properties have been widely investigated in the literature (see for instance [15] for an analytic approach and [10, 23] for the algebraic treatment). The Rarita–Schwinger operator constitutes a generalization of the Dirac operator to the case of higher dimensional representations of the Spin group, and so its study is the first natural step in the research on the so called higher spin differential operators. Many of the relevant analytical properties of the space of nullsolutions of such operators on manifolds, as well as eigenvalue problems on the sphere and characterization of the polynomial solutions have been already illustrated in [3] and later by the authors of [5].

A thorough and self-contained treatment of the theory underlying an algebraic and computational approach can be found in [10] where numerous examples of application are presented, including the case of the Dirac operator and some of its

The author is part of the Eduard Čech center and is sponsored by the relative grants.
The paper is in final form and no version of it will be submitted elsewhere.

variations. The aim of this paper is to use such algebraic methods for the study of linear constant coefficient partial differential operators and apply them to the case of the system of differential equations associated to the Rarita–Schwinger operator in one or more vector variables, in real dimension 3, namely the system

$$\left\{ \begin{array}{rcl} \mathcal{R}_1(D)f & = & g_1 \\ \mathcal{R}_2(D)f & = & g_2 \\ & \vdots & \\ \mathcal{R}_n(D)f & = & g_n \end{array} \right. \tag{1}$$

where $\mathcal{R}_i$ is the Rarita–Schwinger operator with respect to the i-th vector variable, and sections $f, g_1, \ldots, g_n$ are in a sheaf of generalized function on $(\mathbb{R}^3)^n$ such as polynomial functions $\mathcal{P}$, analytical functions $\mathcal{A}$, smooth functions $\mathcal{E}$, distributions $\mathcal{D}'$ or any of the sheaves described in [10] for which, essentially, there exists a notion of Fourier transform that is also an isometry. Thanks to the work of Y. Homma [17] we are able to start with a local coordinate expression of the Rarita–Schwinger operator and from this we can carry on the explicit computational algebraic analysis of the operator. Among the problems that we are able to consider for this case, in Proposition 3.2 we describe some of the compatibility conditions of the system (1). We also present, as a consequence of Theorem 3.1, a proof of the exactness of the complex associated to (1) in the general case, and some cohomological properties and duality theorems for the space of solutions of the homogenous version of (1) in the case $n = 2$ (see Corollary 3.3). An alternative construction of a Dolbealut-like sequence of invariant operators is in principle possible using techniques of representation theory and the orbits of the action of the Weyl group associated to the Lie group with respect to which the initial operator is invariant. For the case of the Dirac operator in real dimension four, also known as the Cauchy-Fueter operator, this apparently different approach coming from representation theory has been compared to the algebraic analysis of the operator using computational tools [4, 8]. It is clearly shown in these papers that such approaches are in fact equivalent. We then believe that our results from this paper can be easily translated into the language of invariant differential operators for parabolic geometries. We may return on this subject in a future work.

Acknowledgments. I want to thank Jarolím Bureš for the many useful discussion on this topic and for providing me with the necessary references. I would also like to acknowledge Irene Sabadini for reading an earlier version of this paper and for precious remarks.

2. The flat version of the Rarita–Schwinger operator

Let (ρ, V_ρ) be an irreducible (complex) unitary representation of the group *Spin*(3). A 3-dimensional spin manifold M is a manifold endowed with a principal Spin bundle $S_\rho(M)$, whose sections are smooth maps that to each point of the manifold associate a vector of the spin representation V_ρ. Let us recall that the group *Spin*(3) can be thought of as $SU(2)$ and hence its irreducible representations are the $m+1$ dimensional spaces V_m of polynomials in z with degree less than or equal to m. We will denote the spin bundle induced by V_1 on M by S_1 and similarly by S_m the spin bundle induced by the irreducible representation V_m. For each choice of a Spin bundle on the manifold, the classical covariant derivative ∇ on M gives rise to the so called lifted covariant derivative

$$\nabla^S : \Gamma(S_m) \to \Gamma(S_m \otimes T^*(M)).$$

The tensor product of the space S_m with the cotangent bundle splits into several irreducible *Spin*(3)-modules S_ν, following the well known Clebsch-Gordan formula, so it is possible to define projections $\pi_{m,\nu}$ from $\Gamma(S_m \otimes T^*(M))$ to $\Gamma(S_\nu)$. For instance, the classical Dirac operator is defined as

$$\mathcal{D} := \pi_{1,1} \circ \nabla^S : \Gamma(S_1) \to \Gamma(S_1) \tag{2}$$

and the Rarita–Schwinger operator as

$$\mathcal{R} := \pi_{3,3} \circ \nabla^S : \Gamma(S_3) \to \Gamma(S_3)\,. \tag{3}$$

If $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$, a local formula for the Dirac operator is given by

$$\mathcal{D} = \sum_{i=1}^{3} e_i \cdot \nabla_{e_i}\,, \tag{4}$$

where the multiplication by e_i is given by the Clifford multiplication on the half-spinor space which comes from the identification of the real clifford algebra Cl(3) as $SU(2) \oplus SU(2)$ as follows. An element of the basis e_i is identified with $(\sigma_i, -\sigma_i)$ where the Pauli matrices σ_i are

$$\sigma_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The action of an element e_i on a vector v of the half-spin representation is then defined by $e_i \cdot v := \sigma_i v$. For an element $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SU(2)$, with $a, b, c, d \in \mathbb{C}$, and a vector z^k in V_m we define the representation ρ_m

$$\rho_m(h) z^k = (bz + d)^{m-k}(az + c)^k.$$

According to [17], given that we have the isomorphism of representations

$$(\rho_2, V_2) \simeq (ad, \mathbb{R}^3 \otimes \mathbb{C})\,,$$

the space $V_m \otimes \mathbb{R}^3$ (actually, its complexification) decomposes, as representation space, as follows

$$(\rho_m, V_m) \otimes (\rho_2, V_2) \simeq (\rho_{m+2}, V_{m+2}) \oplus (\rho_m, V_m) \oplus (\rho_{m-2}, V_{m-2})$$

and each vector $v \otimes X \in V_m \otimes \mathbb{R}^3$ splits consequently into three pieces

$$v \otimes X = (v \otimes X)^+ + (v \otimes X)^0 + (v \otimes X)^-\,.$$

For the sake of the representation of the Rarita–Schwinger operator, we will only need to describe the projection onto the middle component. The operators associated to the other two invariant projections are respectivley overdetermined and underdetermined. We will not analyze the properties of such operators here, see the webpage [12] in which some related calculations are presented. For each vector X in $\mathbb{R}^3$, we then have a morphism $\rho_m^0(X) : V_m \to V_m$, defined by $\rho_m^0(X)(v) = (v \otimes X)^0$, which is called *Clifford homomorphism*. The following follows directly from [17], Prop. 3.1, and can be directly checked with Mathematica using the "Clebsch-Gordan coefficients formula" package.

Proposition 2.1. *The Clifford homomorphism ρ^0_m in terms of the basis $\{e_i\}_{1\le i\le 3}$ of $\mathbb{R}^3$ and the basis $\{z^k\}_{0\le k\le m}$ of V_m is given by*

$$\begin{cases} \rho^0_m(\frac{e_1}{2})z^k &= i(k-\frac{m}{2})z^k \\ \rho^0_m(\frac{e_2}{2}+i\frac{e_3}{2})z^k &= -kz^{k-1} \\ \rho^0_m(\frac{e_2}{2}-i\frac{e_3}{2})z^k &= (m-k)z^{k+1}\,. \end{cases}$$

If $m=3$, the matrices A_i, $i=1\dots 3$, associated to the linear maps $\rho^0_3(e_i)$ with respect to such basis are

$$A_1 := \begin{pmatrix} -3i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 3i \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 3 & 0 & -2 & 0 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A_3 := \begin{pmatrix} 0 & i & 0 & 0 \\ 3i & 0 & 2i & 0 \\ 0 & 2i & 0 & 3i \\ 0 & 0 & i & 0 \end{pmatrix}.$$

Definition 2.2. The general higher-spin Dirac operator acting on sections $\Gamma(S_m)$ is, in coordinates,

$$\mathcal{D}^m = \sum_{i=1}^{3} \rho^0_m(e_i)\cdot\nabla_{e_i}\,. \tag{5}$$

Remark 2.3. Note that in the case $m=1$ we simply reobtain the Clifford homomorphism given by the standard Clifford action $e_i\cdot$ as discussed above, so $\mathcal{D}^1$ is the classical Dirac operator (2).

Definition 2.4. The Rarita–Schwinger operator is the higher spin Dirac operator with $m=3$, hence its explicit form is given by

$$\mathcal{R} = \sum_{i=1}^{3} \rho^0_3(e_i)\cdot\nabla_{e_i}\,. \tag{6}$$

Let us now endow the space $\mathbb{R}^3$ with coordinates x_1,x_2,x_3 and let $R=\mathbb{C}[x_1,x_2,x_3]$ be a ring of polynomials. The directional derivative ∇_{e_i} with respect to this system of coordinates will be denoted by ∂_i. Let $f\in\Gamma(S_3)$ written as $f=f_0+f_1z+f_2z^2+f_3z^3$ and $f_j=f_j(x_1,x_2,x_3)$. Then the matrix form of the Rarita–Schwinger operator is

$$P(D) = \begin{pmatrix} -3i\partial_1 & -\partial_2+i\partial_3 & 0 & 0 \\ 3\partial_2+3i\partial_3 & -i\partial_1 & -2\partial_2+2i\partial_3 & 0 \\ 0 & 2\partial_2+2i\partial_3 & i\partial_1 & -3\partial_2+3i\partial_3 \\ 0 & 0 & \partial_2+i\partial_3 & 3i\partial_1 \end{pmatrix}$$

and its symbol obtained via Fourier transform up to a scalar, is the matrix:

$$P = \begin{pmatrix} -3ix_1 & -x_2+ix_3 & 0 & 0 \\ 3x_2+3ix_3 & -ix_1 & -2x_2+2ix_3 & 0 \\ 0 & 2x_2+2ix_3 & ix_1 & -3x_2+3ix_3 \\ 0 & 0 & x_2+ix_3 & 3ix_1 \end{pmatrix} \tag{7}$$

In order to perform the algebraic analysis of the Rarita–Schwinger operator using Gröbner Basis techniques, let us consider the module $\langle P^t \rangle$ generated by the rows of P. The use of a computer algebra software to analyze such a module will be very useful for our scope. We have utilized CoCoA [6], whose current version does not support complex coefficients. The only coefficient field of characteristic zero available is $\mathbb{Q}$. Although in this case it would suffice to handle polynomials with coefficients in $\mathbb{Z}[i]$ (and this can be implemented "by hand" by just adding a new variable i and requiring that $i^2+1=0$), we are still able to avoid imaginary numbers and find an expression for the matrix which contains only integers, thanks to a suitable change of coordinates. Here we present two possible choices.

First method. The multiplication by a complex number $z = a+ib$ with $a, b \in \mathbb{R}$ can be thought of as a $\mathbb{R}$–linear operation whose associated matrix is $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. Therefore if we consider each scalar function $f_j \in \mathbb{C}$ as $u_j + iv_j$ with $u_j, v_j \in \mathbb{R}$, we can rewrite the matrix $P(D)$, as follows:

$$P(D)_{\mathbb{R}} = \begin{pmatrix} 0 & 3\partial_1 & -\partial_2 & -\partial_3 & 0 & 0 & 0 & 0 \\ -3\partial_1 & 0 & \partial_3 & -\partial_2 & 0 & 0 & 0 & 0 \\ 3\partial_2 & -3\partial_3 & 0 & \partial_1 & -2\partial_2 & -2\partial_3 & 0 & 0 \\ 3\partial_3 & 3\partial_2 & -\partial_1 & 0 & 2\partial_3 & -2\partial_2 & 0 & 0 \\ 0 & 0 & 2\partial_2 & -2\partial_3 & 0 & -\partial_1 & -3\partial_2 & -3\partial_3 \\ 0 & 0 & 2\partial_3 & 2\partial_2 & \partial_1 & 0 & 3\partial_3 & -3\partial_2 \\ 0 & 0 & 0 & 0 & \partial_2 & -\partial_3 & 0 & -3\partial_1 \\ 0 & 0 & 0 & 0 & \partial_3 & \partial_2 & 3\partial_1 & 0 \end{pmatrix}$$

which, apart from replacing f_0 and f_3 with $3f_0$ and $3f_3$ and after Fourier transform, can be seen as the skew-symmetric matrix

$$P_{\mathbb{R}} = \begin{pmatrix} 0 & x_1 & -x_2 & -x_3 & 0 & 0 & 0 & 0 \\ -x_1 & 0 & x_3 & -x_2 & 0 & 0 & 0 & 0 \\ x_2 & -x_3 & 0 & x_1 & -2x_2 & -2x_3 & 0 & 0 \\ x_3 & x_2 & -x_1 & 0 & 2x_3 & -2x_2 & 0 & 0 \\ 0 & 0 & 2x_2 & -2x_3 & 0 & -x_1 & -x_2 & -x_3 \\ 0 & 0 & 2x_3 & 2x_2 & x_1 & 0 & x_3 & -x_2 \\ 0 & 0 & 0 & 0 & x_2 & -x_3 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & x_3 & x_2 & x_1 & 0 \end{pmatrix} \tag{8}$$

This representation of the Rarita–Schwinger operator is sure more symmetric than (7). However it has the disadvantage of being computationally more expensive since (8) has twice the rank of (7).

Second method. We apply the following change of coordinates within $\mathbb{C}[x_1, x_2, x_3]$:

$$X_1 = -ix_1, \qquad X_2 = x_2 + ix_3, \qquad X_3 = -x_2 + ix_3\,,$$

together with the change of basis such that $F = 3f_0 + f_1 z + f_2 z^2 + 3f_3 z^3$. The matrix P becomes

$$P = \begin{pmatrix} X_1 & X_3 & 0 & 0 \\ X_2 & X_1 & 2X_3 & 0 \\ 0 & 2X_2 & -X_1 & X_3 \\ 0 & 0 & X_2 & -X_1 \end{pmatrix}. \tag{9}$$

From now on, given that this representation is much more computationally suitable for our scope, we will use (9) as the symbol matrix for the Rarita–Schwinger operator. The algebraic analysis of the system of equations given by $P(D)f = 0$ amounts then to the study of the minimal free resolution of the module $M_1 := R^4/\langle P^t\rangle$. Since $\mathrm{Det}(P) = X_1^4 + 2X_1^2X_2X_3 + X_2^2X_3^2 = (X_1^2 + X_2X_3)^2 = -(x_1^2 + x_2^2 + x_3^2) \neq 0$, using Proposition 2.2 from [7] we obtain that the module has no syzygies and therefore its free resolution is

$$0 \longrightarrow R^4 \xrightarrow{P_1^t} R^4 \longrightarrow M_1 \longrightarrow 0\,.$$

From the vanishing of the first module of syzygies it also follows that $\mathrm{Hom}_R(M_1, R) = 0$ but $0 \neq \mathrm{Ext}^1_R(M_1, R) = R^4/\langle P_1\rangle$ so according to the terminology of [10] the system of equation is not overdetermined. This also shows, in particular, that the nullsolutions of the Rarita–Schwinger operator can have compact singularities, by using Theorem 2.1.14 of [10].

Before we proceed to the analysis of the system given by several Rarita–Schwinger operators in more variables, let us make the following remark. Henceforth we will denote by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the imaginary unit quaternions. Consider the Moisil-Theodorescu operator

$$\mathcal{D} := \mathbf{i}\partial_1 + \mathbf{j}\partial_2 + \mathbf{k}\partial_3$$

which is the Dirac operator in real dimension 3 action on (smooth) functions $f : \mathbb{R}^3 \longrightarrow \mathbb{H}$. Its symbol matrix has the following form [22]:

$$U := \begin{pmatrix} 0 & X_1 & X_2 & X_3 \\ X_1 & 0 & -X_3 & X_2 \\ X_2 & X_3 & 0 & -X_1 \\ X_3 & -X_2 & X_1 & 0 \end{pmatrix}$$

where X_1, X_2, X_3 are real coordinates and $\partial_1, \partial_2, \partial_3$ are the respective directional derivatives. The authors of [22] have shown that the module associated to one (resp. several) Moisil-Theodorescu operators has the same minimal free resolution of the one associated to the Cauchy-Fueter in one (several) quaternionic variables:

$$\partial_{\bar{q}} := \partial_0 + \mathbf{i}\partial_1 + \mathbf{j}\partial_2 + \mathbf{k}\partial_3$$

at least in terms of Betti numbers and degrees of the maps. Obviously the two modules differ (as they have, for instance, different characteristic varieties), but it is easy to see that the descriptions of the two resolutions are tightly related.

Definition 2.5. Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring and let $P(D)$ and $Q(D)$ be two differential operators with symbol matrices P and Q respectively. We say that P has *the same analytical structure* of Q if their free resolution have the same Betti numbers and the same degrees. Two operators are called *similar* if there exists a map $\phi \in \mathrm{End}(R)$ such that $P = \widetilde{\phi}(Q)$ where $\widetilde{\phi}$ is the natural extension of ϕ to the ring of matrices with entries in R.

In this language, the Moisil-Theodorescu operator not only has the same analytical structure of the Cauchy-Fueter operator, but they are also similar, ϕ being the map such that $\phi(X_0) = 0$ and all other variables are fixed. More in general, it follows directly from Lemma 3.15 of [16] that whenever the module associated to Q is obtained from the module associated to P via the quotient with an element which is not a zerodivisor in $\mathrm{coker}(P^t)$ (such as X_0 in the case of the Cauchy-Fueter operator), then the two operators automatically have the same analytical structure. This is a case in which the similarity of the two operators is in fact a stronger condition than having the same analytical structure. To prove that the two properties are actually independent, we will show in the next paragraphs that the Rarita–Schwinger operator has the same analytical structure of the Moisil-Theodorescu operator, even in several variables.

3. Rarita-Schwinger operators in several vector variables

In complete analogy with the holomorphic (respectively monogenic) case, the case of nullsolutions to the system of equations defined by one complex (resp. vector) variable is somehow special. In several variables, one observes that the system of differential equations describing the kernel of the operator becomes overdetermined and the length of the resolution is actually greater than one, leading to non trivial compatibility conditions for the case of the non-homogeneous system. Let us investigate this matter for the Rarita–Schwinger operator. Let f be a smooth function defined on $(\mathbb{R}^3)^n$, $n > 1$, with values in $V_3 \simeq \mathbb{H}$ (this last isomorphism being realized by the correspondence $1 \mapsto 1/3$, $z \mapsto \mathbf{i}$, $z^2 \mapsto \mathbf{j}$ and $z^3 \mapsto 1/3\mathbf{k}$. Let $\{X_{ji} \,|\, i = 1 \ldots 3,\ j = 1 \ldots n\}$ be real coordinates on $\mathbb{R}^{3n}$ and let $P_j(D)$ be the Rarita–Schwinger operator with respect to the real variables (X_{j1}, X_{j2}, X_{j3}), whose symbol is

$$P_j = \begin{pmatrix} X_{j1} & X_{j3} & 0 & 0 \\ X_{j2} & X_{j1} & 2X_{j3} & 0 \\ 0 & 2X_{j2} & -X_{j1} & X_{j3} \\ 0 & 0 & X_{j2} & -X_{j1} \end{pmatrix}.$$

Let $g = (g_1, \ldots, g_n)^t$ be some data of the same type of f and consider the inhomogeneous system of equations

$$\begin{cases} P_1(D)f = g_1 \\ P_2(D)f = g_2 \\ \quad\vdots \\ P_n(D)f = g_n \end{cases} \tag{10}$$

which can be written in compact form as $P(D)f = g$ where $P = [P_1^t \dots P_n^t]^t$ is the block matrix containing the P_j's in a column.

Theorem 3.1. *Let $R = \mathbb{C}[X_{10}, \dots, X_{13}, \dots, X_{n0}, \dots, X_{n3}]$ be a polynomial ring and let P be as above. Let Q be the $4n$ by 4 matrix associated to the Cauchy–Fueter operator in n quaternionic variables $q_j = X_{j0} + \mathbf{i}X_{j1} + \mathbf{j}X_{j2} + \mathbf{k}X_{j3}$ as in* [1]. *Then P has the same analytical structure of Q.*

Before we actually prove the result, let us focus on the case $n = 2$. For the Cauchy-Fueter operator in 2 quaternionic variables, an interesting property of the compatibility conditions of the system is that they only arise from the centrality of the Laplace operator $\Delta_i = \partial\bar{q}_i\partial q_i$. This is also the case for the Dirac operator in any number of vector variables and in high dimension. For the Dirac operator, in the terminology of [21] such relations are called *radial*. See also [11], Chapter 6, for further details on radial syzygies and an alternative way to compute them. Let us study the case of the Rarita–Schwinger operator in two variables and see whether it is possible to associate to such operator an opportune "laplacian" which commutes with it. Computations with CoCoA show that the free resolution for the module $M_2 := R^4/\langle P^t \rangle$ in this case is

$$0 \longrightarrow R(-4)^4 \longrightarrow R(-3)^8 \longrightarrow R(-1)^8 \longrightarrow R^4 \longrightarrow M_2 \longrightarrow 0 \tag{11}$$

so that we have 2 "quaternionic" quadratic syzygies at the first step and one "quaternionic" linear syzygy that closes the complex. By the term "quaternionic" syzygy we mean that we have divided the dimensions in (11) by a factor of 4 in order to express the compatibility conditions of the system (10) only in terms of the vector data g_1, g_2. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are the Rarita–Schwinger operators with respect to the first and the second set of variables respectively, then we have that

Proposition 3.2. *The compatibility conditions of the system* (10) *are given by*

$$(\mathcal{C}_{12} + \mathcal{R}_1\widetilde{\mathcal{R}}_2)g_1 - \mathcal{R}_1\widetilde{\mathcal{R}}_1 g_2 = 0$$

and

$$\mathcal{R}_2\widetilde{\mathcal{R}}_2 g_1 + (\mathcal{C}_{12} - \mathcal{R}_2\widetilde{\mathcal{R}}_1)g_2 = 0\,.$$

Furthermore, naming h_1 and h_2 the left hand sides of the two previous equations respectively, we have that the second compatibility conditions of the system are given by the single equation

$$\mathcal{R}'_1 h_2 - \mathcal{R}'_2 h_1 = 0\,.$$

Here the operators $\mathcal{R}'_j$ and $\widetilde{\mathcal{R}}_j$ are given, in real form, by

$$\widetilde{\mathcal{R}}_j := \begin{pmatrix} 5\partial_{j1} & -5\partial_{j3} & 0 & 0 \\ \partial_{j2} & -\partial_{j1} & -2\partial_{j3} & 0 \\ 0 & 2\partial_{j2} & -\partial_{j1} & -\partial_{j3} \\ 0 & 0 & 5\partial_{j2} & 5\partial_{j1} \end{pmatrix}, \mathcal{R}'_j := \begin{pmatrix} \partial_{j1} & -\partial_{j3} & 0 & 0 \\ -\partial_{j2} & \partial_{j1} & 2\partial_{j3} & 0 \\ 0 & 2\partial_{j2} & -\partial_{j1} & -\partial_{j3} \\ 0 & 0 & -\partial_{j2} & -\partial_{j1} \end{pmatrix},$$

$j = 1, 2$ and $\mathcal{C}_{ij} := \widetilde{\mathcal{R}}_i\mathcal{R}_j - \widetilde{\mathcal{R}}_j\mathcal{R}_i = 5(\partial_{i2}\partial_{j3} - \partial_{i3}\partial_{j2})I_4$, $i \neq j$, where I_4 is the identity matrix of rank 4.

Proof. The proof can be obtained by comparing the syzygies with the ones obtained with CoCoA. They only differ by some scalar coefficients hence the module they generate is exactly the same. As an alternative proof, one could just calculate the graded Betti numbers (i.e. the ranks of the free modules in the resolution) with the use of the computer and discover that the eight first (independent) syzygies are quadratic and that the four second syzygy are linear. Therefore, since the relations above are obviously syzygies and they are independent[1], one concludes that those are all the possible syzygies at each step. □

The complex associated to two Rarita–Schwinger operators then shares some of the properties of the Cauchy-Fueter complex for two operators. It is immediate for example to show, using CoCoA, that $\mathrm{Ext}^1(M_2, R) = 0$ and that $\mathrm{Ext}^2(M_2, R) \simeq R^4/\langle (P')^t \rangle$ where $(P')^t$ is the block matrix given by $[P_1'\ P_2']$ and P_i' is the symbol of $\mathcal{R}_i'$, $i = 1, 2$. This fact has two noteworthy consequences.

Corollary 3.3. *Let $\mathcal{E}$ be the sheaf of smooth functions on $\mathbb{R}^6$ and let $\mathcal{D}'$ be the sheaf of distributions. If $\mathcal{S}$ is a sheaf, denote by $\mathcal{S}^{\mathcal{R}}$ the sheaf of nullsolutions to the Rarita-Schwinger system for $n = 2$. Then for each open set in $\mathbb{R}^6$ of the type $U \setminus K$ where U is an open convex and K a compact set in U, and for each element $f \in \Gamma(U \setminus K, \mathcal{S}^{\mathcal{R}})$ there exist a unique extension $f \in \Gamma(U, S^{\mathcal{R}})$. Let us consider the matrix $(P')^t$ as previously defined and let $\mathcal{S}^{(P')^t}$ denote the space of solutions to the system $(P')^t(D)f = 0$ on the sheaf $\mathcal{S}$. Let K be a compact convex set in $\mathbb{R}^6$. Then*

$$H^3_K(\mathbb{R}^6, \mathcal{E}^{(P')^t}) \cong [\mathcal{D}'^{\mathcal{R}}(K)]' \tag{12}$$

and

$$H^3_K(\mathbb{R}^6, D'^{\mathcal{R}}) \cong [\mathcal{E}^{(P')^t}(K)]' \,.$$

Proof. These facts are immediate consequences of Theorems 2.1.11 and 2.1.12 of [10]. □

Remark 3.4. We refer the reader to the paper [9] for more details on what can be said about quaternionic hyperfunctions. We believe that a similar theory could be carried out for the Rarita–Schwinger operator in one and two vector variables, but we defer this matter to a future paper.

We now present some technical lemmas that will lead to the proof of Theorem 3.1. The first one gives a Gröbner Basis for the module associated to several *Rarita-Swinger* operators. For the terminology and the main results on Gröbner Bases we refer the reader to [18].

Lemma 3.5. *Let $R = \mathbb{C}[X_{11}, \ldots, X_{13}, \ldots, X_{n1}, \ldots, X_{n3}]$ be a ring of polynomials and let $P_1, \ldots, P_n$ be the symbols associated to n Rarita-Schwinger operators. Let M_n be the module generated by the rows of such matrices. The reduced degree*

[1] independence of syzygies can be proved by just saying that the matrices corresponding to the maps in the free resolution do not contain nonzero constants

reverse lexicographic Gröbner Basis for M_n is given by the rows of the matrices P_t, $t = 1, \dots, n$ together with the rows of the matrices

$$C_{rs} = \widetilde{P}_r P_s - \widetilde{P}_s P_r\,, \qquad 1 \le r < s \le n\,,$$

where the matrices $\widetilde{P}_j$, $j = 1, \dots, n$ are the symbols of the operators $\widetilde{\mathcal{R}}_j$.

Proof. The statement can be verified directly with CoCoA for $n \le 4$. For the general case, we can see that the S-polynomials generated by any two rows of the matrices P_i give rise to the rows of C_{rs}. Therefore, considering Buchberger's algorithms for the computation of a reduced Gröbner Basis, we have so far generated elements of the Gröbner Basis of M_n by adding the rows of C_{rs}. To prove that they are the only elements of the reduced Gröbner Basis, we need to show that all their S-polynomials reduce to zero. An S-polynomial generated by a row of P_i and a row of C_{rs} is computed and reduced to zero as in the case $n = 2$ or $n = 3$, depending on the number of different indices in the triple (i, r, s). An S-polynomial generated by two rows of C_{rs} is computed and reduced to zero as in the case $n = 2$, $n = 3$ or $n = 4$ depending on the number of different indices. This completes the proof. □

Remark 3.6. The proof of the previous lemma relies on an argument that has been employed many times in the literature for the algebraic analysis of differential operators, [2, 13, 22]. We believe that it can be generalized and applied to all such cases in which there exists, together with the original operators, say $\mathcal{D}_i$, another operator $\widetilde{\mathcal{D}}_i$ such that the "generalized commutator" $\mathcal{C}_{ij} := \widetilde{\mathcal{D}}_i \mathcal{D}_j - \widetilde{\mathcal{D}}_j \mathcal{D}_i$ is diagonal. This and some related considerations will not be presented here for the sake of brevity, but will be investigated in our future work.

From the calculation of the Gröbner Basis for the module M_n it follows the expression of the Hilbert series of R^4/M_n. Remember that the *Hilbert function* $h_N : \mathbb{Z} \longrightarrow \mathbb{N}$ associated to a finitely generated $\mathbb{Z}$-graded R-module $N = \bigoplus_{d \in \mathbb{Z}} N_d$ is simply $h_N(d) := \dim_{\mathbb{C}}(N_d)$ and the associate *Hilbert series* is $H_M(z) := \sum\limits_{d \in \mathbb{Z}} h_M(d) z^d$.

Lemma 3.7. *The Hilbert series of the module R^4/M_n is given by*

$$\mathcal{H}_{R^4/M_n}(z) = 4\frac{1 + (n-1)z}{(1-z)^{n+1}}.$$

Moreover, the module is Cohen-Macaulay with Castelnuovo-Mumford regularity equal to two.

Proof. Let us first calculate the monomial module $\mathrm{LT}(M_n)$, i.e. the leading term module of M_n with respect to the degree reverse lexicographic term order on R. It follows from Lemma 3.5 that such a module is generated by the set of terms $\{X_{i1}e_t,\, X_{h3}X_{k2}e_t \,|\, i = 1 \dots n, t = 1 \dots 4, 1 \le h < k \le n\}$, where e_t is the t-th element of the canonical basis of R^4. Let I_n be the ideal $\{X_{i1},\, X_{h3}X_{k2} \,|\, i = 1 \dots n, t = 1 \dots 4, 1 \le h < k \le n\}$. Then from a well known result of Macaulay ([18], Theorem 1.5.7) and from the additivity of the Hilbert function it follows

$$\mathcal{H}_{R^4/M_n}(z) = 4\mathcal{H}_{R/I_n}(z).$$

The ideal I_n is the same initial ideal that one obtains for the module associated to the Moisil-Theodorescu operator in several quaternionic variables (see [22]). The method we present here to calculate the Hilbert series, though, is slightly different and more direct than the one presented in the aforementioned paper. First one shows that the set of polynomials

$$\{X_{12},\, X_{n3},\, X_{13}+X_{22}, \ldots,\, X_{n-13}+X_{n2}\}$$

is a maximal regular sequence in R^4/M_n. So the depth of the module is $n+1$. This is equal to the cardinality of a maximal subset of independent indeterminates for I_n, namely $\mathcal{Y} = \{X_{12}\} \cup \{X_{i3} \mid 1 \leq i \leq n\}$, and this number equals the dimension of the affine algebra R/I_n (see [19]) and hence equals the dimension of R^4/M_n. This shows that the module is Cohen-Macaulay and that the denominator of its Hilbert series must be $(1-z)^{n+1}$. Let J be the ideal generated by the above maximal sequence. If we look at the so called *Artinian reduction* of the module, namely $A_n := R^4/JM_n$, then A_n is zerodimensional by Cohen-Macaulyness and therefore $\mathcal{H}_{A_n}$ is just a polynomial in z. Such a polynomial is precisely the numerator of $\mathcal{H}_{R^4/M_n}$. The non constant elements of A_n are the classes of the elements $X_{k3}e_t$, $k = 1 \ldots n-1$, $t = 1 \ldots 4$, which are $4n-4$ elements of degree one, and it is immediate to check that A_n does not contain quadratic elements. Therefore, $\mathcal{H}_{A_n} = 4 + 4(n-1)z$.
The fact that the module is Cohen-Macaulay follows because the length of a maximal regular sequence for $R^4/\operatorname{LT}(M_n)$ coincides with the denominator of the Hilbert series. Finally, using Proposition 4.14 from [16] and the fact that the elements of the regular sequence are linear, we obtain that the Castelnuovo-Mumford regularity of M_n is two, but this is obviously the same as the regularity of $R^4/\operatorname{LT}(M_n)$. □

Now that we have the Hilbert series of the module associated to the Rarita–Schwinger in several variables, we can finally prove the main result of this paper.

Proof of Theorem 3.1**.** To show that the Cauchy-Fueter operator and the Rarita–Schwinger operator have the same analytical structure, we just need to prove that the Betti numbers β_j of the module M_n coincides with those found in [2]. For the case $n = 1$, the proof follows from the computation in Section 2. Let us then deal with the case $n > 1$. First note that by the Auslander-Buchsbaum formula, the length of the free resolution is $3n-(n+1)$ because the proof of Lemma 3.7 shows that the depth of the module is $n+1$. The regularity of the module being two, we have that the maps of its free resolution are all linear except for the first syzygies that are quadratic. This allows then to calculate the explicit form of the Betti numbers, using Corollary 1.10 from [16] which gives the Betti numbers of a module if one knows its Hilbert function and the degrees of the maps at each step of the resolution. We can then apply such result to this case and get that $\beta_0 = 4$, $\beta_1 = 4n$ and

$$\beta_j = 4n\binom{2n-1}{j}\frac{j-1}{j+1}, \quad j > 1\,.$$

□

Another important fact that can be deduced from the expression of the Hilbert series of R^4/M_n is that the cohomology modules $\mathrm{Ext}^i(R^4/M_n, R)$ are all zero except for the one corresponding to the last spot of the resolution. In other words

Corollary 3.8. *The complex associated to $n > 1$ Rarita-Schwinger operators*

$$0 \longrightarrow R^4 \longrightarrow R^{4n}(-1) \longrightarrow R^{\beta_1}(-3) \longrightarrow \ldots \longrightarrow R^{\beta_n}(-2n) \longrightarrow 0$$

is exact, except at the last spot.

Proof. The dimension of the associated module is $n+1$ as calculated in Lemma 3.7, and this is the same as the dimension of the characteristic variety of the module M_n. Applying [20], Corollary 1, p. 377, we obtain that the first nontrivial cohomology corresponds to the index $3n - (n+1) = 2n - 1$. □

In particular, the Hartogs' phenomenon and all the cohomology vanishing results described in [10] hold for nullsolutions of the Rarita–Schwinger operator in several variables.

4. Polynomial solutions to the Rarita–Schwinger system

As a last application of the calculation of the Hilbert series, we show that it is possible to find the dimension of the spaces of homogeneous polynomial solutions to the Rarita–Schwinger operator in any number of variables. Let us introduce the Weyl algebra

$$A := \mathbb{C}\big[X_{i1}, \ldots, X_{i3}, \partial_{i1}, \ldots, \partial_{i3} \mid i = 1 \ldots n\big]$$

and denote by R the ring of polynomials as before. R can be seen as an A-module in a canonical way. Let $P(D)$ be a linear constant coefficient partial differential operator with symbol $P = [P_1^t \ldots P_n^t]^t \in \mathrm{Mat}_{4n\times 4}(R)$. Then let us denote by $\mathcal{N}$ the (left) A–module

$$\mathcal{N} = A^4 / \sum_{j=1}^{n} A^4 \cdot P_j(D) = A^4/\mathrm{im}\,(P(D)^t)\,,$$

and consider the vector space

$$\mathcal{P} := \mathrm{Hom}_A(\mathcal{N}, R)$$

which, adapting the proof of Proposition 2.1.1 from [10], is isomorphic to the space of polynomial solutions to the system of equations $P(D)f = 0$.

Proposition 4.1. *The $\mathbb{C}$–spaces $\mathcal{P}$ and $\mathcal{M} = R^4/im\,(P^t)$ are isomorphic.*

Proof. It suffices to construct the map Φ from $\mathrm{Hom}_A(\mathcal{N}, R)$ to $\mathcal{M}$ which sends the morphism $\varphi \in \mathrm{Hom}_A(\mathcal{N}, R)$ defined by $\varphi(\bar{e}_i) = f_i$ to the element $\Phi(\varphi) = (\bar{f}_1, \ldots, \bar{f}_4) \in \mathcal{M}$. Here e_i is the i-th canonical basis element of A^4. Then it is immediate to see that this is the isomorphism required. □

Since the isomorphism Φ is obviously graded (the degree of a homogeneous map φ in $\mathcal{N}$ just being the degree of $f_i = \varphi(\bar{e}_i)$), we obtain a method to describe the homogeneous components of the space of polynomial solution by just studying the homogenous components of $\mathcal{M}$. These are finite dimensional vector spaces and their dimensions are given, by definition, by the values of the Hilbert function. We can then prove the following result.

Theorem 4.2. *Let P be the symbol matrix associated to the Rarita–Schwinger operator in n variables in real dimension 3. Let $\mathcal{P}$ be the space of polynomial solutions to the system $P(D)f = 0$, let $d \in \mathbb{N}$ be a natural number and let $[\mathcal{P}]_d$ be the homogeneous component of degree d. Then*

$$\dim_{\mathbb{C}}([\mathcal{P}]_d) = 4(d+1)\binom{d+n-1}{n-1}.$$

Proof. Given the definition of Hilbert function and Proposition 4.1 we have that $h_{\mathcal{P}} = h_{\mathcal{M}}$ so we just need to find the values of the Hilbert function $h_{\mathcal{M}}(d)$ for each $d \in \mathbb{N}$. Let use distinguish two cases for n.
$n = 1$. The Hilbert series associated to $\mathcal{M}$ is $\mathcal{H}(z) = 4\frac{1}{(1-z)^2}$ hence the values $h(d)$ can be reconstructed by the formula

$$h(d) = \frac{1}{d!}\frac{\partial^d \mathcal{H}}{\partial z^d}(0) = 4\frac{1}{d!}\frac{(d+1)!}{(1-z)^{d+2}}\Big|_{z=0} = 4(d+1)\,.$$

$n > 1$. We proceed in a similar way to calculate the d-th derivative of the Hilbert series. By induction on d, with some simple calculations, we get

$$\frac{\partial^d}{\partial z^d}\mathcal{H}(z) = 4n(n+1)\cdots(n+d-1)\frac{(d+1)+(n-1)z}{(1-z)^{n+d+1}}$$

from which it follows immediately

$$h(d) = \frac{1}{d!}\frac{\partial^d \mathcal{H}}{\partial z^d}(0) = 4\frac{1}{d!}\frac{(n+d-1)!}{(n-1)!}(d+1) = 4(d+1)\binom{d+n-1}{n-1}.$$

□

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and let $P \in \mathrm{Mat}_{r,s}(R)$ be the symbol matrix of a linear differential operator. Many computer algebra softwares allow the computation of polynomial solutions, up to a given degree, to the system of PDEs given by $P(D)f = 0$. It is interesting to note that the computation of the so called Noetherian operators [14] associated to $P(D)$ can be related to this problem. If we want to find the space of all polynomial solutions $\mathcal{P}_{<d}$ of degree (strictly) less than $d > 0$, then it suffices to consider the operator

$$P_{<d} := \begin{bmatrix} P \\ T \end{bmatrix}$$

where T is the matrix whose rows are of the type $te_i = (0, \ldots, 0, t, 0, \ldots, 0)$, $i = 1..s$, and t varies among all power products in R whose degree is d. It follows that (smooth) solutions to the new system

$$P_{<d}(D)f = 0$$

are polynomial functions of degree less than d which also satisfy $P(D)f = 0$. The module $R^s/im(T^t)$ is a finite dimensional $\mathbb{C}$-space (it is generated by all the polynomial vectors of R^s whose degree is less than d), and hence the space $R^s/im(P^t_{<d})$ is finite dimensional as well, or *zerodimensional* in the language of [14]. Its associated characteristic variety, i.e. the set in $\mathbb{C}^n$ where the matrix $P_{<d}$ is not of maximal rank, is the origin. One can then apply the methods described in the aforementioned paper (see section 3.4, l.c.) to find an explicit expression of the solutions to the system $P_{<d}(D)f = 0$ via the calculation of the Noetherian operators associated to the module. It is interesting to notice that this is consistent with the description of Theorem 4.2. In fact, the dimension of the space $\mathcal{P}_{<d+1}$ is equal to the multiplicity of $\mathcal{M}_{<d+1} := R^s/im(P^t_{<d+1})$, which equals its dimension as a vector space. But this is given by the sum of the values of the Hilbert function

$$\sum_{i=1}^{\infty} h_{\mathcal{M}_{<d+1}}(i) = \sum_{i=1}^{d} h_{\mathcal{M}}(i)\,.$$

The same holds for the space $\mathcal{P}_{<d}$ and by subtracting the two multiplicities we obtain another proof of the fact that the space of polynomial solutions of degree *exactly* d is $h_{\mathcal{M}}(d) = \dim_{\mathbb{C}}(\mathcal{M}_d)$.

References

[1] Adams, W. W., Berenstein, C. A., Loustaunau, P., Sabadini, I., Struppa, D. C., *Regular functions of several quaternionic variables and the Cauchy–Fueter complex*, J. Geom. Anal. **9** (1999), 1–16.

[2] Adams, W. W., Loustaunau, P., *Analysis of the module determining the properties of regular functions of several quaternionic variables*, Pacific J. Math. **196** (2000), no. 1, 1–15.

[3] Bureš, J., *The Rarita-Schwinger operator and spherical monogenic forms*, Complex Variables Theory Appl. **43** (2000), no. 1, 77–108.

[4] Bureš, J., Damiano, A., Sabadini, I., *Explicit resolutions for the complex several Fueter operators*, J. Geom. Phys. **57** 3 (2007), 765–775.

[5] Bureš, J., Sommen, F., Souček, V., Van Lancker, P., *Rarita-Schwinger type operators in Clifford analysis*, J. Funct. Anal. **185** (2001), no. 2, 425–455.

[6] CoCoATeam, CoCoA, A software package for COmputations in COmmutative Algebra, freely available at `http://cocoa.dima.unige.it`

[7] Colombo, F., Damiano, A., Sabadini, I., Struppa, D. C., *A surjectivity theorem for differential operators on spaces of regular functions*, Complex Variables Theory Appl. **50** (2005), no. 6, 389–400.

[8] Colombo, F., Souček, V., Struppa, D. C., *Invariant resolutions for several Fueter operators*, J. Geom. Phys. **56** 7 (2006), 1175–1191.

[9] Colombo, F., Damiano, A., Sabadini, I., Struppa, D. C., *A new Dolbeault complex in quaternionic and Clifford analysis*, to appear in Proceedings Fifth ISAAC Congress, Catania, 2005.

[10] Colombo, F., Sabadini, I., Sommen, F., Struppa, D. C., *Analysis of Dirac systems and computational algebra*, Progress in Mathematical Physics, Vol. 39, *Birkhäuser*, Boston, 2004.

[11] Damiano, A., *Computational Approach to some Problems in Algebraic Analysis*, Ph.D. Dissertation, George Mason University, 2005.

[12] Damiano, A., Mannino, S., CoAlA , A web page for COmputational ALgebraic Analysis available at `http://www.tlc185.com/coala`.

[13] Damiano, A., Sabadini, I., Struppa, D. C., *New algebraic properties of biregular functions in* $2n$ *quaternionic variables*, Compl. Var. Ell. Eq. **51** (2006), No. 5–6, 497–510. 2006.

[14] Damiano, A., Sabadini, I., Struppa, D. C., *Computational methods for the construction of a class of noetherian operators*, to appear in Exp. Math.

[15] Delanghe, R., Sommen, F., Soucek, V., *Clifford Algebra and Spinor-valued Functions*, Math. Appl. **53**, Kluwer Academic Publishers, 1992.

[16] Eisenbud, D., *The Geometry of Syzygies*, Graduate Texts in Mathematics, Vol. 229, *Springer-Verlag*, New York, 2005.

[17] Homma, Y., *The Higher Spin Dirac Operators on* 3*–Dimensional Manifolds.* Tokyo J. Math. **24** (2001), no. 2, 579–596.

[18] Kreuzer, M., Robbiano, L., *Computational Commutative Algebra 1*, Springer, 2000.

[19] Kreuzer, M., Robbiano, L., *Computational Commutative Algebra 2*, Springer, 2005.

[20] Palamodov, V. P., *Linear Differential Operators with Constant Coefficients*, Springer Verlag, New York 1970.

[21] Sabadini, I., Sommen, F., Struppa, D. C., *The Dirac complex on abstract vector variables: megaforms*, Exp. Math., **12** (2003), 351–364.

[22] Sabadini, I., Shapiro, M., Struppa, D. C., *Algebraic analysis of the Moisil-Theodorescu system*, Complex Variables Theory Appl. **40** (2000), 333–357.

[23] Sabadini, I., Sommen, F., Struppa, D. C., Van Lancker, P., *Complexes of Dirac operators in Clifford algebras*, Math. Z., **239** (2002), 293–320.

[24] Souček, V., *Invariant operators and Clifford analysis*, Adv. Appl. Clifford Algebras **11** (2001), no. S1, 37–52.

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