ARCHIVUM MATHEMATICUM (BRNO)
Tomus 42 (2006), Supplement, 5 – 84

THE SRNÍ LECTURES ON NON-INTEGRABLE GEOMETRIES WITH TORSION

ILKA AGRICOLA

Abstract. This review article intends to introduce the reader to non-integrable geometric structures on Riemannian manifolds and invariant metric connections with torsion, and to discuss recent aspects of mathematical physics—in particular superstring theory—where these naturally appear.

Connections with skew-symmetric torsion are exhibited as one of the main tools to understand non-integrable geometries. To this aim a a series of key examples is presented and successively dealt with using the notions of intrinsic torsion and characteristic connection of a G-structure as unifying principles. The General Holonomy Principle bridges over to parallel objects, thus motivating the discussion of geometric stabilizers, with emphasis on spinors and differential forms. Several Weitzenböck formulas for Dirac operators associated with torsion connections enable us to discuss spinorial field equations, such as those governing the common sector of type II superstring theory. They also provide the link to Kostant's cubic Dirac operator.

Contents

2000 *Mathematics Subject Classification.* Primary 53-02 (C, D); Secondary 53C25-30, 53D15, 81T30.

Key words and phrases. metric connection with torsion; intrinsic torsion; G-structure; characteristic connection; superstring theory; Strominger model; parallel spinor; non-integrable geometry; integrable geometry; Berger's holonomy theorem; naturally reductive space; hyper-Kähler manifold with torsion; almost metric contact structure; G_2-manifold; Spin(7)-manifold; SO(3)-structure; 3-Sasakian manifold.

Supported by the SFB 647 "Space—Time—Matter" of the DFG and the Junior Research Group "Special Geometries in Mathematical Physics" of the Volkswagen Foundation.

The paper is in final form and no version of it will be submitted elsewhere.

1. Background and motivation

1.1. **Introduction.** Since Paul Dirac's formulation in 1928 of the field equation for a quantized electron in flat Minkowski space, Dirac operators on Riemannian manifolds have become a powerful tool for the treatment of various problems in geometry, analysis and theoretical physics. Meanwhile, starting from the fifties the French school founded by M. Berger had developed the idea that manifolds should be subdivided into different classes according to their holonomy group. The name *special (integrable) geometries* has become customary for those which are not of general type. Already at that early stage there were hints that parallel spinor fields would induce special geometries, but this idea was not further investigated. At the beginning of the seventies, A. Gray generalized the classical holonomy concept by introducing a classification principle for *non-integrable special Riemannian geometries* and studied the defining differential equations of each class. The connection between these two lines of research in mathematical physics became clear in the eighties in the context of twistor theory and the study of small eigenvalues of the Dirac operator, mainly developed by the Berlin school around Th. Friedrich. In the case of homogeneous manifolds, integrable geometries correspond to symmetric spaces, whose classification by E. Cartan is a milestone in 20th century differential geometry. The much richer class of homogeneous reductive spaces—which is inaccessible to any kind of classification—has been studied intensively since the mid-sixties, and is a main source of examples for non-integrable geometries.

The interest in non-integrable geometries was revived in the past years through developments of superstring theory. Firstly, integrable geometries (Calabi-Yau manifolds, Joyce manifolds etc.) are exact solutions of the Strominger model (1986), though with vanishing B-field. If one deforms these vacuum equations and looks for models with non-trivial B-field, a new mathematical approach going back a decade implies that solutions can be constructed geometrically from non integrable geometries with torsion. In this way, manifolds *not* belonging to the field of algebraic geometry (integrable geometries) become candidates for interesting models in theoretical physics.

Before discussing the deep mathematical and physical backgrounds, let us give a—very intuitive—explanation of why the traditional Yang-Mills approach needs modification in string theory and how torsion enters the scene. Point particles move along world-lines, and physical quantities are typically computed as line integrals of some potential that is, mathematically speaking, just a 1-form. The associated field strength is then its differential—a 2-form—and interpreted as the curvature of some connection. In contrast, excitations of extended 1-dimensional objects (the 'strings') are 'world-surfaces', and physical quantities have to be surface integrals of certain potential 2-forms. Their field strengths are thus 3-forms and cannot be interpreted as curvatures anymore. The key idea is to supply the (pseudo)-Riemannian manifold underlying the physical model with a non-integrable G-structure admitting a 'good' metric G-connection ∇ with torsion, which in turn will play the role of a B-field strength; and the art is to choose the G-structure so that the connection ∇ admits the desired parallel objects, in particular spinors, interpreted as supersymmetry transformations.

My warmest thanks go to all colleagues whose countless remarks and corrections helped improving the quality of this text, in particular to Simon Chiossi, Richard Cleyton, Thomas Friedrich, Mario Kassuba, Nils Schoemann (Humboldt University Berlin) as well as Paweł Nurowski and Andrzej Trautman (Warsaw University).

I am also grateful to my Czech friends for the unabated perseverance in organizing the excellent Srní Winter Schools for over two decades and for inviting me to give a series of lectures on this topic in January 2006. My notes for these lectures constituted the basis of the present article.

1.2. **Mathematical motivation.** From classical mechanics, it is a well-known fact that symmetry considerations can simplify the study of geometric problems—for example, Noether's theorem tells us how to construct first integrals, like momentum, from invariance properties of the underlying mechanical system. In fact, beginning from 1870, it became clear that the principle organizing geometry ought to be the study of its symmetry groups. In his inaugural lecture at the University of Erlangen in 1872, which later became known as the "Erlanger Programm", Felix Klein said [Kl72, p. 34]:

> *Let a manifold and on it a transformation group be given; the objects belonging to the manifold ought to be studied with respect to those properties which are not changed by the transformations of the group*[1].

Hence the classical symmetry approach in differential geometry was based on the *isometry group* of a manifold, that is, the group of all transformations acting on the given manifold.

By the mid-fifties, a second intrinsic group associated to a Riemannian manifold turned out to be deeply related to fundamental features like curvature and parallel objects. The so-called *holonomy group* determines how a vector can change under parallel transport along a closed loop inside the manifold (only in the flat case the transported vector will coincide with the original one). Berger's theorem (1955) classifies all possible restricted holonomy groups of a simply connected, irreducible and non-symmetric Riemannian manifold (M, g) (see [Ber55], [Sim62] for corrections and simplifications in the proof and [Br96] for a status report). The holonomy group can be either $\mathrm{SO}(n)$ in the generic case or one of the groups listed in Table 1 (here and in the sequel, ∇^g denotes the Levi-Civita connection). Manifolds having one of these holonomy groups are called manifolds with special (integrable) holonomy, or *special (integrable) geometries* for short. We put the case $n = 16$ and $\mathrm{Hol}(M) = \mathrm{Spin}(9)$ into brackets, because Alekseevski and Brown/Gray showed independently that such a manifold is necessarily symmetric ([Ale68], [BG72]). The point is that Berger proved that the groups on this list were the only possibilities, but he was not able to show whether they actually occurred as holonomy groups of compact manifolds. It took another thirty years to find out that—with the exception of Spin(9)—this is indeed the case: The existence of metrics with holonomy $\mathrm{SU}(m)$ or $\mathrm{Sp}(m)$ on compact manifolds followed from Yau's solution of the Calabi Conjecture posed in 1954 [Yau78]. Explicit non-compact metrics with holonomy G_2 or Spin(7) are due to R. Bryant [Br87] and R. Bryant and S. Salamon [BrS89], while compact manifolds with holonomy G_2 or Spin(7) were constructed by D. Joyce only in 1996 (see [Joy96a], [Joy96b] [Joy96c] and the book [Joy00], which also contains a proof of the Calabi Conjecture). Later, compact exceptional holonomy manifolds have also been constructed by other methods by Kovalev ([Kov03]).

[1] „Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben; man soll die der Mannigfaltigkeit angehörigen Gebilde hinsichtlich solcher Eigenschaften untersuchen, die durch die Transformationen der Gruppe nicht geändert werden.“

$\dim M$	$\mathrm{Hol}(M)$	name	parallel object	curvature
$4n$	$\mathrm{Sp}(n)\mathrm{Sp}(1)$	quaternionic-Kähler manifold	$-$	$\mathrm{Ric} = \lambda g$
$2n$	$\mathrm{U}(n)$	Kähler manifold	$\nabla^g J = 0$	$-$
$2n$	$\mathrm{SU}(n)$	Calabi-Yau manifold	$\nabla^g J = 0$	$\mathrm{Ric} = 0$
$4n$	$\mathrm{Sp}(n)$	hyper-Kähler manifold	$\nabla^g J = 0$	$\mathrm{Ric} = 0$
7	G_2	parallel G_2-manifold	$\nabla^g \omega^3 = 0$	$\mathrm{Ric} = 0$
8	Spin(7)	parallel Spin(7)-manifold	$\nabla^g \beta^4 = 0$	$\mathrm{Ric} = 0$
[16]	[Spin(9)]	[parallel Spin(9)-manifold]	$-$	$-$

TABLE 1. Possible Riemannian holonomy groups ('Berger's list').

As we will explain later, the General Holonomy Principle relates manifolds with $\mathrm{Hol}(M) = \mathrm{SU}(n), \mathrm{Sp}(n), G_2$ or Spin(7) with ∇^g-parallel spinors (see Section 3). Already in the sixties it had been observed that the existence of such a spinor implies in turn the vanishing of the Ricci curvature ([Bon66] and Proposition 2.2) and restricts the holonomy group of the manifold ([Hit74], [McKW89]), but the difficulties in constructing explicit compact manifolds with special integrable Ricci-flat metrics inhibited further research on the deeper meaning of this result.

There was progress in this direction only in the homogeneous case. Symmetric spaces are the "integrable" geometries inside the much larger class of homogeneous reductive spaces. Given a non-compact semisimple Lie group G and a maximal compact subgroup K such that $\mathrm{rank}\, G = \mathrm{rank}\, K$, consider the associated symmetric space G/K. The Dirac operator can be twisted by a finite-dimensional irreducible unitary representation τ of K, and it was shown by Parthasarathy, Wolf, Atiyah and Schmid that for suitable τ most of the discrete series representations of G can be realized on the L^2-kernel of this twisted Dirac operator ([Par72], [Wol74], [AS77]). The crucial step is to relate the square of the Dirac operator with the Casimir operator Ω_G of G; for trivial τ, the corresponding formula reads

$$D^2 \;=\; \Omega_G + \frac{1}{8}\,\mathrm{scal}\,. \tag{1}$$

Meanwhile many people began looking for suitable generalizations of the classical holonomy concept. One motivation for this was that the notion of Riemannian holonomy is too restrictive for vast classes of interesting Riemannian manifolds; for example, contact or almost Hermitian manifolds cannot be distinguished merely by their holonomy properties (they have generic holonomy $\mathrm{SO}(n)$), and the Levi-Civita connection is not adapted to the underlying geometric structure (meaning that the defining objects are not parallel).

In 1971 A. Gray introduced the notion of *weak holonomy* ([Gra71]), "one of his most original concepts" and "an idea much ahead of its time" in the words of N. Hitchin [Hit01]. This concept turned out to yield interesting non-integrable geometries in dimensions $n \leq 8$ and $n = 16$. In particular, manifolds with weak holonomy $\mathrm{U}(n)$ and G_2 became known as *nearly Kähler* and *nearly parallel* G_2-manifolds, respectively. But whereas metrics of compact Ricci-flat integrable geometries have not been realized explicitly (so far), there are many well-known homogeneous reductive examples of non-integrable geometries ([Gra70], [Fer87], [BFGK91], [FKMS97], [BG99], [Fin05] and many others). The relation to Dirac operators emerged shortly after Th. Friedrich proved in 1980 a seminal inequality for the first eigenvalue λ_1 of the Dirac operator on a compact Riemannian manifold M^n of non-negative curvature [Fri80]

$$(\lambda_1)^2 \ \geq \ \frac{n}{4(n-1)} \min_{M^n}(\mathrm{scal}) \,, \tag{2}$$

In this estimate, equality occurs precisely if the corresponding eigenspinor ψ satisfies the Killing equation

$$\nabla^g_X \psi \ = \ \pm\frac{1}{2}\sqrt{\frac{\min(\mathrm{scal})}{n(n-1)}} X \cdot \psi \ =: \ \mu X \cdot \psi \,.$$

The first non-trivial compact examples of Riemannian manifolds with Killing spinors were found in dimensions 5 and 6 in 1980 and 1985, respectively ([Fri80], [FG85]). The link to non-integrable geometry was established shortly after; for instance, a compact, connected and simply connected 6-dimensional Hermitian manifold is nearly Kähler if and only if it admits a Killing spinor with real Killing number μ [Gru90]. Similar results hold for Einstein-Sasaki structures in dimension 5 and nearly parallel G_2-manifolds in dimension 7 ([FK89], [FK90]). Remarkably, the proof of inequality (2) relies on introducing a suitable spin connection—an idea much in line with recent developments. A. Lichnerowicz established the link to twistor theory by showing that on a compact manifold the space of twistor spinors coincides—up to a conformal change of the metric—with the space of Killing spinors [Lich88].

1.3. **Physical motivation – torsion in gravity.** The first attempts to introduce torsion as an additional 'datum' for describing physics in general relativity goes back to Cartan himself [Car24a]. Viewing torsion as some intrinsic angular momentum, he derived a set of gravitational field equations from a variational principle, but postulated that the energy-momentum tensor should still be divergence-free, a condition too restrictive for making this approach useful. The idea was taken up again in broader context in the late fifties. The variation of the scalar curvature (and an additional Lagrangian generating the energy-momentum tensor) on a space-time endowed with a metric connection with torsion yielded the two fundamental equations of *Einstein-Cartan theory*, first formulated by Kibble [Kib61] and Sciama (see his article in [In62]). The first equation is (formally) Einstein's classical field equation of general relativity with an effective energy momentum tensor T_{eff} depending on torsion, the second one can be written in index-free notation as

$$Q(X,Y) + \sum_{i=1}^{n} \big(Q(Y,e_i) \lrcorner\, e_i\big) \cdot X - \big(Q(X,e_i) \lrcorner\, e_i\big) \cdot Y \;=\; 8\pi S(X,Y)\,.$$

Here, Q denotes the torsion of the new connection ∇, S the spin density and $e_1, \ldots, e_n$ any orthonormal frame. A. Trautman provided an elegant formulation of Einstein-Cartan theory in the language of principal fibre bundles [Tra73a]. The most striking predictions of Einstein-Cartan theory are in cosmology. In the presence of very dense spinning matter, nonsingular cosmological models may be constructed because the effective energy momentum tensor T_{eff} does not fulfill the conditions of the Penrose-Hawking singularity theorems anymore [Tra73b]. The first example of such a model was provided by W. Kopczyński [Kop73], while J. Tafel found a large class of such models with homogeneous spacial sections [Taf75]. For a general review of gravity with spin and torsion including extensive references, we refer to the article [HHKN76].

In the absence of spin, the torsion vanishes and the whole theory reduces to Einstein's original formulation of general relativity. In practice, torsion turned out to be hard to detect experimentally, since all tests of general relativity are based on experiments in empty space. Einstein-Cartan theory is pursued no longer, although some concepts that it inspired are still of relevance (see [HMMN95] for a generalization with additional currents and shear, [Tra99] for optical aspects, [RT03] for the link to the classical theory of defects in elastic media). Yet, it may be possible that Einstein-Cartan theory will prove to be a better classical limit of a future quantum theory of gravitation than the theory without torsion.

1.4. **Physical motivation – torsion in superstring theory.** Superstring theory (see for example [GSW87], [LT89]) is a physical theory aiming at describing nature at small distances ($\simeq\ 10^{-25}$ m). The concept of point-like elementary particles is replaced by one-dimensional objects as building blocks of matter—the so-called strings. Particles are then understood as resonance states of strings and can be described together with their interactions up to very high energies (small distances) without internal contradictions. Besides gravitation, string theory incorporates many other gauge interactions and hence is an excellent candidate for a more profound description of matter than the standard model of elementary particles. Quantization of superstrings is only possible in the critical dimension 10, while M-theory is a non-perturbative description of superstrings with "geometrized" coupling, and lives in dimension 11. Mathematically speaking, a 10-

# of inv. spinors	stabilizer groups
1	$\mathrm{Spin}(7) \ltimes \mathbb{R}^8$
2	G_2, $\mathrm{SU}(4) \ltimes \mathbb{R}^8$
3	$\mathrm{Sp}(2) \ltimes \mathbb{R}^8$
4	$\mathrm{SU}(3)$, $(\mathrm{SU}(2) \times \mathrm{SU}(2)) \ltimes \mathbb{R}^8$
8	$\mathrm{SU}(2)$, $\mathbb{R}^8$
16	$\{e\}$

Table 2. Possible stabilizers of invariant spinors inside $\mathrm{Spin}(9,1)$.

or 11-dimensional configuration space Y (a priori not necessarily smooth) is assumed to be the product

$$Y^{10,11} \;=\; V^{3\text{-}5} \times M^{5\text{-}8}$$

of a low-dimensional spacetime V describing the 'external' part of the theory (typically, Minkowski space or a space motivated from general relativity like anti-de-Sitter space), and a higher-dimensional 'internal space' M with some special geometric structure. The metric is typically a direct or a warped product. On M, internal symmetries of particles are described by parallel spinor fields, the most important of which being the existing supersymmetries: a spinor field has spin $1/2$, so tensoring with it swaps bosons and fermions. By the General Holonomy Principle (see Theorem 2.7), the holonomy group has to be a subgroup of the stabilizer of the set of parallel spinors inside $\mathrm{Spin}(9,1)$. These are well known and summarized in Table 2. We shall explain how to derive this result and how to understand the occurring semidirect products in Section 3.4.

Since its early days, string theory has been intricately related with some branches of algebraic geometry. This is due to the fact that the integrable, Ricci-flat geometries with a parallel spinor field with respect to the Levi-Civita connection are exact solutions of the Strominger model for a string vacuum with vanishing B-field and constant dilaton. This rich and active area of mathematical research lead to interesting developments such as the discovery of mirror symmetry.

1.5. **First developments since 1980.** In the early eighties, several physicists independently tried to incorporate torsion into superstring and supergravity theories in order to get a more physically flexible model, possibly inspired by the developments in classical gravity ([VanN81], [GHR84], [HP87], [dWSHD87], [Roc92]). In fact, simple supergravity is equivalent to Einstein-Cartan theory with a massless, anticommuting Rarita-Schwinger field as source. But contrary to general relativity, one difficulty stems from the fact that there are several models in superstring theory (type I, II, heterotic…) that vary in the excitation spectrum and the possible interactions.

In his article "Superstrings with torsion" [Str86], A. Strominger describes the basic model in the common sector of type II superstring theory as a 6-tuple $(M^n, g, \nabla, T, \Phi, \Psi)$ consisting of a Riemannian spin manifold (M^n, g), a 3-form T, a dilaton function Φ and a spinor field Ψ. The field equations can be written in the following form (recall that ∇^g denotes the Levi-Civita connection):

$$\mathrm{Ric}_{ij} - \frac{1}{4} T_{imn} T_{jmn} + 2 \nabla^g_i \partial_j \Phi = 0\,, \quad \delta(e^{-2\Phi} T) = 0\,,$$

$$(\nabla^g_X + \frac{1}{4} X \lrcorner T)\psi = 0, \quad (2d\Phi - T) \cdot \psi = 0\,.$$

If one introduces a new metric connection ∇ whose torsion is given by the 3-form T,

$$\nabla_X Y := \nabla^g_X Y + \frac{1}{2} T(X, Y, -)\,,$$

one sees that the third equation is equivalent to $\nabla\Psi = 0$. The remaining equations can similarly be rewritten in terms of ∇. For constant dilaton Φ, they take the particularly simple form [IP01]

$$\text{(3)} \qquad \mathrm{Ric}^{\nabla} = 0\,, \quad \delta^g(T) = 0\,, \quad \nabla\Psi = 0\,, \quad T \cdot \Psi = 0\,,$$

and the second equation ($\delta^g(T) = 0$) now follows from the first equation ($\mathrm{Ric}^{\nabla} = 0$). For M compact, it was shown in [Agr03, Theorem 4.1] that a solution of all equations necessarily forces $T = 0$, i. e. an integrable Ricci-flat geometry with classical holonomy given by Berger's list. By a careful analysis of the integrability conditions, this result could later be extended to the non-compact case ([AFNP05], see also Section 5.5). Together with the well-understood Calabi-Yau manifolds, Joyce manifolds with Riemannian holonomy G_2 or Spin(7) thus became of interest in recent times (see [AW01], [CKL01]). From a mathematical point of view, this result stresses the importance of tackling easier problems first, for example partial solutions. As first step in the investigation of metric connections with totally skew-symmetric torsion, Dirac operators, parallel spinors etc., Th. Friedrich and S. Ivanov proved that many non-integrable geometric structures (almost contact metric structures, nearly Kähler and weak G_2-structures) admit a unique invariant connection ∇ with totally skew-symmetric torsion [FI02], thus being a natural replacement for the Levi-Civita connection. Non-integrable geometries could then be studied by their holonomy properties.

In fact, in mathematics the times had ben ripe for a new look at the intricate relationship between holonomy, special geometries, spinors and differential forms: in 1987, R. Bryant found the first explicit local examples of metrics with exceptional Riemannian holonomy (see [Br87] and [BrS89]), Chr. Bär described their relation to Killing spinors via the cone construction [Bär93]. Building on the insightful vision of Gray, S. Salamon realized the centrality of the concept of *intrinsic torsion* ([Sal89] and, for recent results, [Fin98], [CS02], [CS04]). Swann successfully tried *weakening holonomy* [Sw00], and N. Hitchin characterized non-integrable geometries as critical points of some linear functionals on differential forms [Hit01]. In particular, he motivated a generalization of Calabi-Yau-manifolds [Hit00] and of

G_2-manifolds [Wi04], and discovered a new, previously unknown special geometry in dimension 8 ("weak PSU(3)-structures", see also [Wi06]). Friedrich reformulated the concepts of non-integrable geometries in terms of principal fiber bundles [Fri03b] and discussed the exceptional dimension 16 suggested by A. Gray years before ([Fr01], [Fr03a]). Analytic problems—in particular, the investigation of the Dirac operator—on non-integrable Riemannian manifolds contributed to a further understanding of the underlying geometry ([Bis89], [AI00], [Gau97]). Finally, the Italian school and collaborators devoted over the past years a lot of effort to the explicit construction of homogeneous examples of non-integrable geometries with special properties in small dimensions (see for example [AGS00], [FG03], [FPS04], [Sal01] and the literature cited therein), making it possible to test the different concepts on explicit examples.

The first non-integrable geometry that raised the interest of string theorists was the squashed 7-sphere with its weak G_2-structure, although the first steps in this direction were still marked by confusion about the different holonomy concepts. A good overview about G_2 in string theory is the survey article by M. Duff ([Duf02]). It includes speculations about possible applications of weak Spin(9)-structures in dimension 16, which a priori are of too high dimension to be considered in physics. In dimension three, it is well known (see for example [SSTP88]) that the Strominger equation $\nabla\Psi = 0$ can be solved only on a compact Lie group with biinvariant metric, and that the torsion of the invariant connection ∇ coincides with the Lie bracket. In dimension four, the Strominger model leads to a HKT structure (see Section 2.4 for more references), i.e. a hyper-Hermitian structure that is parallel with respect to ∇, and—in the compact case—the manifold is either a Calabi-Yau manifold or a Hopf surface [IP01]. Hence, the first interesting dimension for further mathematical investigations is five.

Obviously, besides the basic correspondence outlined here, there is still much more going on between special geometries and detailed properties of physical models constructed from them. Some weak geometries have been rederived by physicists looking for partial solutions by numerical analysis of ODE's and heavy special function machinery [GKMW01].

As an example of the many interesting mathematical problems appearing in the context of string theory, the physicists Ramond and Pengpan observed that there is an infinite set of irreducible representations of Spin(9) partitioned into triplets $\mathcal{S} = \cup_i\{\mu^i, \sigma^i, \tau^i\}$, whose representations are related in a remarkable way. For example, the infinitesimal character value of the Casimir operator is constant on triplets, and $\dim\mu^i + \dim\sigma^i = \dim\tau^i$ if numbered appropriately. These triplets are used to describe massless supermultiplets, for example $N = 2$ hypermultiplets in $(3+1)$ dimensions with helicity U(1) or $N = 1$ supermultiplets in eleven dimensions, where SO(9) is the light-cone little group [BRX02]. To explain this fact, B. Kostant introduced an element in the tensor product of the Clifford algebra and the universal enveloping algebra of a Lie group called "Kostant's cubic Dirac operator", and derived a striking formula for its square ([GKRS98], [Kos99]). The triplet structure of the representations observed for Spin(9) is due to the fact that the Euler characteristic of $F_4/\mathrm{Spin}(9)$ is three, hence the name "Euler multiplets" has become common for describing this effect. In Section 5.3, we will show that Kostant's operator may be interpreted as the symbol of a usual Dirac operator which is induced by a non-standard connection on a homogeneous naturally reductive space ([Agr02],[Agr03]). In particular, this Dirac operator satisfies a remarkably simple formula which is a direct generalization of Parthasarathy's formula on symmetric spaces [Par72]. This established the link between Kostant's algebraic considerations and recent models in string theory; in particular, it made homogeneous naturally reductive spaces to key examples for string theory and allowed the derivation of strong vanishing theorems on them. In representation theory this opened the possibility to realize infinite-dimensional representations in kernels of twisted Dirac operators on homogeneous spaces ([HP02], [MZ04]), as it had been carried out on symmetric spaces in the seventies ([Par72], [Wol74], [AS77]).

2. Metric connections with torsion

2.1. **Types of connections and their lift into the spinor bundle.** Let us begin with some general remarks on torsion. The notion of torsion of a connection was invented by Elie Cartan, and appeared for the first time in a short note at the Académie des Sciences de Paris in 1922 [Car22]. Although the article contains no formulas, Cartan observed that such a connection may or may not preserve geodesics, and initially turns his attention to those who do so. In this sense, Cartan was the first to investigate this class of connections. At that time, it was not customary — as it became in the second half of the 20th century — to assign to a Riemannian manifold only its Levi-Civita connection. Rather, Cartan demands (see [Car24b]):

> *Given a manifold embedded in affine (or projective or conformal etc.) space, attribute to this manifold the affine (or projective or conformal etc.) connection that reflects in the simplest possible way the relations of this manifold with the ambient space*[2].

He then goes on to explain in very general terms how the connection should be *adapted* to the geometry under consideration. We believe that this point of view should be taken into account in Riemannian geometry, too.

We now give a short review of the 8 classes of geometric torsion tensors. Consider a Riemannian manifold (M^n, g). The difference between its Levi-Civita connection ∇^g and any linear connection ∇ is a $(2,1)$-tensor field A,

$$\nabla_X Y = \nabla^g_X Y + A(X,Y)\,, \quad X, Y \in TM^n\,.$$

The vanishing of the symmetric or the antisymmetric part of A has immediate geometric interpretations. The connection ∇ is torsion-free if and only if A is symmetric. The connection ∇ has the same geodesics as the Levi-Civita connection

[2] ≪Étant donné une variété plongée dans l'espace affine (ou projectif, ou conforme etc.), attribuer à cette variété la connexion affine (ou projective, ou conforme etc.) qui rende le plus simplement compte des relations de cette variété avec l'espace ambiant.≫

∇^g if and only if A is skew-symmetric. Following Cartan, we study the algebraic types of the torsion tensor for a metric connection. Denote by the same symbol the $(3,0)$-tensor derived from a $(2,1)$-tensor by contraction with the metric. We identify TM^n with $(TM^n)^*$ using g from now on. Let $\mathcal{T}$ be the $n^2(n-1)/2$-dimensional space of all possible torsion tensors,

$$\mathcal{T} = \left\{T \in \otimes^3 TM^n \mid T(X,Y,Z) = -T(Y,X,Z)\right\} \;\cong\; \Lambda^2 TM^n \otimes TM^n\,.$$

A connection ∇ is metric if and only if A belongs to the space

$$\mathcal{A}^g := TM^n \otimes (\Lambda^2 TM^n) = \left\{A \in \otimes^3 TM^n \mid A(X,V,W) + A(X,W,V) = 0\right\}.$$

In particular, $\dim \mathcal{A}^g = \dim \mathcal{T}$, reflecting the fact that metric connections can be uniquely characterized by their torsion.

Proposition 2.1 ([Car25, p.51], [TV83], [Sal89])**.** *The spaces $\mathcal{T}$ and $\mathcal{A}^g$ are isomorphic as $\mathrm{O}(n)$ representations, an equivariant bijection being*

$$\begin{aligned} T(X,Y,Z) &= A(X,Y,Z) - A(Y,X,Z)\,,\\ 2\,A(X,Y,Z) &= T(X,Y,Z) - T(Y,Z,X) + T(Z,X,Y)\,. \end{aligned}$$

For $n \geq 3$, they split under the action of $\mathrm{O}(n)$ into the sum of three irreducible representations,

$$\mathcal{T} \cong TM^n \oplus \Lambda^3(M^n) \oplus \mathcal{T}'.$$

The last module will also be denoted $\mathcal{A}'$ if viewed as a subspace of $\mathcal{A}^g$ and is equivalent to the Cartan product of representations $TM^n \otimes \Lambda^2 TM^n$,

$$\mathcal{T}' = \Big\{T \in \mathcal{T} \mid \overset{X,Y,Z}{\mathfrak{S}}\ T(X,Y,Z) = 0,\ \sum_{i=1}^{n} T(e_i, e_i, X) = 0\ \forall X,Y,Z\Big\}$$

for any orthonormal frame $e_1, \ldots, e_n$. For $n = 2$, $\mathcal{T} \cong \mathcal{A}^g \cong \mathbb{R}^2$ is $\mathrm{O}(2)$-irreducible.

The eight classes of linear connections are now defined by the possible components of their torsions T in these spaces. The nice lecture notes by Tricerri and Vanhecke [TV83] use a similar approach in order to classify homogeneous spaces by the algebraic properties of the torsion of the canonical connection. They construct homogeneous examples of all classes, study their "richness" and give explicit formulas for the projections on every irreducible component of $\mathcal{T}$ in terms of $\mathrm{O}(n)$-invariants.

Definition 2.1 (Connection with vectorial torsion)**.** The connection ∇ is said to have *vectorial torsion* if its torsion tensor lies in the first space of the decomposition in Proposition 2.1, i.e. if it is essentially defined by some vector field V on M. The tensors A and T can then be directly expressed through V as

$$A(X,Y) \;=\; g(X,Y)V - g(V,Y)X, \quad T(X,Y,Z) \;=\; g\big(g(V,X)Y - g(V,Y)X, Z\big).$$

These connections are particularly interesting on surfaces, in as much that *every* metric connection on a surface is of this type.

In [TV83], F. Tricerri and L. Vanhecke showed that if M is connected, complete, simply-connected and V is ∇-parallel, then (M,g) has to be isometric to the hyperbolic space. V. Miquel studied in [Miq82] and [Miq01] the growth of geodesic balls of such connections, but did not investigate the detailed shape of geodesics. The study of the latter was outlined in [AT04] (see Example 2.7), whereas [AF05] and [IPP05] are devoted to holonomy aspects and a possible role in superstring theory.

Notice that there is some similarity to Weyl geometry. In both cases, we consider a Riemannian manifold with a fixed vector field V on it ([CP99], [Gau95]). A Weyl structure is a pair consisting of a conformal class of metrics and a *torsion-free non-metric connection* preserving the conformal structure. This connection is constructed by choosing a metric g in the conformal class and is then defined by the formula

$$\nabla^{\mathrm{w}}_X Y := \nabla^g_X Y + g(X,V)\cdot Y + g(Y,V)X - g(X,Y)V\,.$$

Weyl geometry deals with the geometric properties of these connections, but in spite of the resemblance, it turns out to be a rather different topic. Yet in special geometric situations it may happen that ideas from Weyl geometry can be useful.

Definition 2.2 (Connection with skew-symmetric torsion)**.** The connection ∇ is said to have *(totally) skew-symmetric torsion* if its torsion tensor lies in the second component of the decomposition in Proposition 2.1, i.e. it is given by a 3-form. They are by now — for reasons to be detailed later — a well-established tool in superstring theory and weak holonomy theories (see for example [Str86], [LT89], [GKMW01], [CKL01], [FP02], [Duf02], [AF04a] etc.). In Examples 2.2 to 2.5, we describe large classes of interesting manifolds that carry natural connections with skew-symmetric torsion. Observe that we can characterize these connections geometrically as follows:

Corollary 2.1. *A connection ∇ on (M^n,g) is metric and geodesic-preserving if and only if its torsion T lies in $\Lambda^3(TM^n)$. In this case, $2\,A = T$ holds,*

$$\nabla_X Y = \nabla^g_X Y + \frac{1}{2}\,T(X,Y,-)\,,$$

and the ∇-Killing vector fields coincide with the Riemannian Killing vector fields.

In contrast to the case of vectorial torsion, manifolds admitting invariant metric connections ∇ with ∇-parallel skew-symmetric torsion form a vast class that is worth a separate investigation ([Ale03], [CS04], [Sch06]).

Suppose now that we are given a metric connection ∇ with torsion on a Riemannian spin manifold (M^n,g) with spin bundle ΣM^n. We slightly modify our notation and write ∇ as

$$\nabla_X Y := \nabla^g_X Y + A_X Y\,,$$

where A_X defines an endomorphism $TM^n \to TM^n$ for every X. The condition for ∇ to be metric

$$g(A_X Y, Z) + g(Y, A_X Z) = 0$$

means that A_X preserves the scalar product g, which can be expressed as $A_X \in \mathfrak{so}(n)$. After identifying $\mathfrak{so}(n)$ with $\Lambda^2(\mathbb{R}^n)$, A_X can be written relative to some orthonormal frame

$$A_X = \sum_{i<j} \alpha_{ij} e_i \wedge e_j \,.$$

Since the lift into $\mathfrak{spin}(n)$ of $e_i \wedge e_j$ is $E_i \cdot E_j/2$, A_X defines an element in $\mathfrak{spin}(n)$ and hence an endomorphism of the spinor bundle. In fact, we need not introduce a different notation for the lift of A_X. Rather, observe that if A_X is written as a 2-form,

(1) its action on a vector Y as an element of $\mathfrak{so}(n)$ is just $A_X Y = Y \lrcorner A_X$, so our connection takes on vectors the form

$$\nabla_X Y = \nabla^g_X Y + Y \lrcorner A_X \,,$$

(2) the action of A_X on a spinor ψ as an element of $\mathfrak{spin}(n)$ is just $A_X \psi = (1/2)\, A_X \cdot \psi$, where $\cdot$ denotes the Clifford product of a k-form by a spinor. The lift of the connection ∇ to the spinor bundle ΣM^n (again denoted by ∇) is thus given by

$$\nabla_X \psi = \nabla^g_X \psi + \frac{1}{2}\, A_X \cdot \psi \,.$$

We denote by $(-,-)$ the Hermitian product on the spinor bundle ΣM^n induced by g. When lifted to the spinor bundle, ∇ satisfies the following properties that are well known for the lift of the Levi-Civita connection. In fact, the proof easily follows from the corresponding properties for the Levi-Civita connection [Fri00, p. 59] and the Hermitian product [Fri00, p. 24].

Lemma 2.1. *The lift of any metric connection ∇ on TM^n into the spinor bundle ΣM^n satisfies*

$$\nabla_X(Y \cdot \psi) = (\nabla_X Y) \cdot \psi + Y \cdot (\nabla_X \psi)\,, \quad X(\psi_1, \psi_2) = (\nabla_X \psi_1, \psi_2) + (\psi_1, \nabla_X \psi_2)\,.$$

Any spinorial connection with the second property is again called *metric*. The first property (chain rule for Clifford products) makes only sense for spinorial connections that are lifts from the tangent bundle, not for arbitrary spin connections.

Example 2.1 (Connection with vectorial torsion)**.** For a metric connection with vectorial torsion given by $V \in TM$, $A_X = 2\, X \wedge V$, since

$$Y \lrcorner (2\, X \wedge V) = 2(X \wedge V)(Y,-) = (X \otimes V)(Y,-) - (V \otimes X)(Y,-) = g(X,Y)V - g(Y,V)X \,.$$

Example 2.2 (Connection with skew-symmetric torsion)**.** For a metric connection with skew-symmetric torsion defined by some $T \in \Lambda^3(M)$, $A_X = X \lrcorner T$. Examples of manifolds with a geometrically defined torsion 3-form are given in the next section.

Example 2.3 (Connection defined by higher order differential forms)**.** As example of a metric spinorial connection not induced from the tangent bundle, consider

$$\nabla_X \psi := \nabla^g_X \psi + (X \lrcorner \omega^k) \cdot \psi + (X \wedge \eta^l) \cdot \psi$$

for some forms $\omega^k \in \Lambda^k(M)$, $\eta^l \in \Lambda^l(M)$ $(k, l \geq 4)$. These are of particular interest in string theory as they are used for the description of higher dimensional membranes ([AF03], [Pu06]).

Example 2.4 (General case)**.** The class $\mathcal{A}'$ of Proposition 2.1 cannot be directly interpreted as vectors or forms of a given degree, but it is not complicated to construct elements in $\mathcal{A}'$ either. For simplicity, assume $n = 3$, and let ∇ be the metric connection with vectorial torsion $V = e_1$. Then

$$A_X = X \wedge e_1 = \Big(\sum_{i=1}^{3} g(X, e_i)e_i\Big) \wedge e_1 = g(X, e_2)\, e_2 \wedge e_1 + g(X, e_3)\, e_3 \wedge e_1 \,.$$

Thus, the new form $\tilde{A}_X := g(X, e_2)e_2 \wedge e_1 - g(X, e_3)e_3 \wedge e_1$ defines a metric linear connection as well. One easily checks that, as a tensor in $\mathcal{A}^g$, it is orthogonal to $\Lambda^3(M) \oplus \mathfrak{X}(M)$, hence it lies in $\mathcal{A}'$. Connections of this type have not yet been investigated as a class of their own, but they are used as an interesting tool in several contexts — for example, in closed G_2-geometry ([Br03], [CI03]). The canonical connection of an almost Kähler manifold is also of this type.

What makes metric connections with torsion so interesting is the huge variety of geometric situations that they unify in a mathematically useful way. Let us illustrate this fact by some examples.

2.2. **Naturally reductive spaces.** Naturally reductive spaces are a key example of manifolds with a metric connection with skew-symmetric torsion.

Consider a Riemannian homogeneous space $M = G/H$. We suppose that M is *reductive*, i.e. the Lie algebra $\mathfrak{g}$ of G splits as vector space direct sum of the Lie algebra $\mathfrak{h}$ of H and an $\mathrm{Ad}(H)$-invariant subspace $\mathfrak{m}$: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$, where $\mathrm{Ad} : H \to \mathrm{SO}(\mathfrak{m})$ is the isotropy representation of M. We identify $\mathfrak{m}$ with T_0M and we pull back the Riemannian metric $\langle\ ,\ \rangle_0$ on T_0M to an inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$. By a theorem of Wang ([KN69, Ch. X, Thm 2.1]), there is a one-to-one correspondence between the set of G-invariant metric affine connections and the set of linear mappings $\Lambda_{\mathfrak{m}} :\ \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ such that

$$\Lambda_{\mathfrak{m}}(hXh^{-1}) = \mathrm{Ad}(h)\Lambda_{\mathfrak{m}}(X)\mathrm{Ad}(h)^{-1} \quad \text{for} \quad X \in \mathfrak{m} \quad \text{and} \quad h \in H \,.$$

A homogeneous Riemannian metric g on M is said to be *naturally reductive* (with respect to G) if the map $[X, -]_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}$ is skew-symmetric,

$$g\big([X, Y]_{\mathfrak{m}}, Z\big) + g\big(Y, [X, Z]_{\mathfrak{m}}\big) = 0 \quad \text{for all} \quad X, Y, Z \in \mathfrak{m} \,.$$

The family of metric connections ∇^t defined by $\Lambda^t_{\mathfrak{m}}(X)Y := (1 - t)/2\, [X, Y]_{\mathfrak{m}}$ has then skew-symmetric torsion $T^t(X, Y) = -t[X, Y]_{\mathfrak{m}}$. The connection ∇^1 is of particular interest and is called the *canonical connection.* Naturally reductive homogeneous spaces equipped with their canonical connection are a well studied (see for example [AZ79]) generalization of symmetric spaces since they satisfy $\nabla^1 T^1 = \nabla^1 \mathcal{R}^1 = 0$, where $\mathcal{R}^1$ denotes the curvature tensor of ∇^1 (Ambrose-Singer, [AS58]). In fact, a converse holds:

Theorem 2.1 ([TV84b, Thm 2.3])**.** *A connected, simply connected and complete Riemannian manifold (M, g) is a naturally reductive homogeneous space if and only if there exists a skew-symmetric tensor field T of type $(1, 2)$ such that $\nabla := \nabla^g - T$ is a metric connection with $\nabla T = \nabla \mathcal{R} = 0$.*

The characterization of naturally reductive homogeneous spaces given in [AS58] through the property that their geodesics are orbits of one-parameter subgroups of isometries is actually wrong; Kaplan's 6-dimensional Heisenberg group is the most prominent counterexample (see [Kap83] and [TV84b]). Naturally reductive spaces have been classified in small dimensions by Kowalski, Tricerri and Vanhecke, partially in the larger context of commutative spaces (in the sense of Gel'fand): the 3-dimensional naturally reductive homogeneous spaces are $\mathrm{SU}(2)$, the universal covering group of $\mathrm{SL}(2,\mathbb{R})$ and the Heisenberg group H_3, all with special families of left-invariant metrics ([TV83]). A simply connected four-dimensional naturally reductive space is either symmetric or decomposable as direct product ([KVh83]). In dimension 5, it is either symmetric, decomposable or locally isometric to $\mathrm{SO}(3)\times \mathrm{SO}(3)/\mathrm{SO}(2)$, $\mathrm{SO}(3)\times H_3/\mathrm{SO}(2)$ (or any of these with $\mathrm{SO}(3)$ replaced by $\mathrm{SL}(2,\mathbb{R})$), to the five-dimensional Heisenberg group H_5 or to the Berger sphere $\mathrm{SU}(3)/\mathrm{SU}(2)$ (or $\mathrm{SU}(2,1)/\mathrm{SU}(2)$), all endowed with special families of metrics ([KVh85]).

Other standard examples of naturally reductive spaces are

- Geodesic spheres in two-point homogeneous spaces,with the exception of the complex and quaternionic Cayley planes [Zil82], [TV84a]
- Geodesic hyperspheres, horospheres and tubes around totally geodesic non-flat complex space forms, described and classified in detail by S. Nagai [Na95], [Na96], [Na97]
- Simply connected φ-symmetric spaces [BlVh87]. They are Sasaki manifolds with complete characteristic field for which reflections with respect to the integral curves of that field are global isometries.
- All known left-invariant Einstein metrics on compact Lie groups [AZ79]. In fact, every simple Lie group apart from $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$ carries at least one naturally reductive Einstein metric other than the biinvariant metric. Similarly, large families of naturally reductive Einstein metrics on compact homogeneous spaces were constructed in [WZ85]. In contrast, non-compact naturally reductive Einstein manifolds are necessarily symmetric [GZ84].

2.3. **Almost Hermitian manifolds.** An *almost Hermitian manifold* (M^{2n}, g, J) is a manifold with a Riemannian metric g and a g-compatible almost complex structure $J : TM^{2n} \to TM^{2n}$. We denote by $\Omega(X,Y) := g(JX,Y)$ its Kähler form and by N the Nijenhuis tensor of J, defined by

$$\begin{aligned} N(X,Y) &:= [JX,JY] - J[X,JY] - J[JX,Y] - [X,Y] \\ &= (\nabla^g_X J)JY - (\nabla^g_Y J)JX + (\nabla^g_{JX} J)Y - (\nabla^g_{JY} J)X\,, \end{aligned}$$

where the second expression follows directly from the vanishing of the torsion of ∇^g and the identity

$$(\nabla^g_X J)(Y) = \nabla^g_X(JY) - J(\nabla^g_X Y)\,. \tag{4}$$

The reader is probably acquainted with the *first canonical Hermitian connection*[3] (see the nice article [Gau97] by Paul Gauduchon, which we strongly recommend for further reading on Hermitian connections)

$$\overline{\nabla}_X Y := \nabla^g_X Y + \frac{1}{2}(\nabla^g_X J)JY\,.$$

Indeed, the condition $\overline{\nabla} J = 0$ is equivalent to the identity (4), and the antisymmetry of the difference tensor $g((\nabla^g_X J)JY, Z)$ in Y and Z can for example be seen from the standard identity[4]

$$2\,g\big((\nabla^g_X J)Y, Z\big) = d\Omega(X, Y, Z) - d\Omega(X, JY, JZ) + g\big(N(Y, Z), JX\big)\,. \tag{5}$$

Sometimes, one finds the alternative formula $-1/2J(\nabla^g_X J)Y$ for the difference tensor of $\overline{\nabla}$; but this is the same, since $J^2 = -1$ implies $\nabla^g_X J^2 = 0 = (\nabla^g_X J)J + J(\nabla_X J)$, i.e. $\nabla^g J \in \mathfrak{u}(n)^\perp \subset \mathfrak{so}(2n)$. Let us now express the difference tensor of the connection $\overline{\nabla}$ using the Nijenhuis tensor and the Kähler form. Since $\nabla^g_X \Omega(Y, Z) = g\big((\nabla^g_X J)Y, Z\big)$, the differential $d\Omega$ is just

$$d\Omega(X, Y, Z) \;=\; g\big((\nabla^g_X J)Y, Z\big) - g\big((\nabla^g_Y J)X, Z\big) + g\big((\nabla^g_Z J)X, Y\big)\,. \tag{6}$$

Together with the expression for N in terms of covariant derivatives of J, this yields

$$\begin{aligned} g\big((\nabla^g_X J)JY, Z\big) = g\big(N(X, Y), Z\big) + d\Omega(JX, JY, JZ) - g\big((\nabla^g_Y J)JZ, X\big) \\ - g\big((\nabla^g_{JZ} J)Y, X\big)\,. \end{aligned}$$

A priori, $(\nabla^g_X J)Y$ has no particular symmetry properties in X and Y, hence the last two terms cannot be simplified any further (in general, they are a mixture of the two other Cartan types). An exceptional situation occurs if M is nearly Kähler ($(\nabla^g_X J)X = 0$), for then $(\nabla^g_X J)Y = -(\nabla^g_Y J)X$ and the last two terms cancel each other. Furthermore, this antisymmetry property implies that the difference tensor is totally skew-symmetric, hence we can conclude:

Lemma 2.2. *On a nearly Kähler manifold (M^{2n}, g, J), the formula*

$$\overline{\nabla}_X Y := \nabla^g_X Y + \frac{1}{2}(\nabla^g_X J)JY = \nabla^g_X Y + \frac{1}{2}\,[N(X, Y) + d\Omega(JX, JY, J-)]$$

defines a Hermitian connection with totally skew-symmetric torsion.

This connection was first defined and studied by Alfred Gray (see [Gra70, p. 304] and [Gra76, p. 237]). It is a non-trivial result of Kirichenko that it has $\overline{\nabla}$-parallel torsion ([Kir77], see also [AFS05] for a modern index-free proof). Furthermore, it

[3] By definition, a connection ∇ is called *Hermitian* if it is metric and has ∇-parallel almost complex structure J.

[4] For a proof, see [KN69, Prop. 4.2.]. Be aware of the different conventions in this book: Ω is defined with J in the second argument, the Nijenhuis tensor is twice our N and derivatives of k-forms differ by a multiple of $1/k$, see [KN63, Prop. 3.11.].

is shown in [Gra76] that any 6-dimensional nearly Kähler manifold is Einstein and of constant type, i.e. it satisfies

$$\|(\nabla^g_X J)(Y)\|^2 = \frac{\mathrm{scal}^g}{30}\left[\|X\|^2\cdot\|Y\|^2 - g(X,Y)^2 - g(X,JY)^2\right] .$$

Together with Lemma 2.2, this identity yields by a direct calculation that *any 6-dimensional nearly Kähler manifold is also $\overline{\nabla}$-Einstein with* $\mathrm{Ric}^{\overline{\nabla}} = 2(\mathrm{scal}^g/15)\,g$ (see Theorem A.1 for the relation between Ricci tensors).

Now let us look for a Hermitian connection with totally skew-symmetric torsion on a larger class of Hermitian manifolds generalizing nearly Kähler manifolds.

Lemma 2.3. *Let (M^{2n}, g, J) be an almost Hermitian manifold with skew-symmetric Nijenhuis tensor $N(X,Y,Z) := g(N(X,Y),Z)$. Then the formula*

$$g(\nabla_X Y, Z) := g(\nabla^g_X Y, Z) + \frac{1}{2}\left[N(X,Y,Z) + d\Omega(JX,JY,JZ)\right]$$

defines a Hermitian connection with skew-symmetric torsion.

Proof. Obviously, only $\nabla J = 0$ requires a calculation. By (4) and the definition of ∇, we have

$$\begin{aligned} 2\nabla_X J(Y) &= 2g\big(\nabla_X(JY) - J(\nabla_X Y), Z\big) = 2g\big(\nabla_X(JY), Z\big) + 2g\big(\nabla_X Y, JZ\big) \\ &= 2\nabla^g_X J(Y) + N(X,JY,Z) - d\Omega(JX,Y,JZ) + N(X,Y,JZ) \\ &\quad - d\Omega(JX,JY,Z)\,. \end{aligned}$$

But from the symmetry properties of the Nijenhuis tensor and the metric, one sees

$$\begin{aligned} N(X,Y,JZ) &= g\big(N(X,Y),JZ\big) = -g\big(JN(X,Y),Z\big) \\ &= g\big(N(JX,Y),Z\big) = N(JX,Y,Z) \end{aligned}$$

and repeated application of the identity (6) for $d\Omega$ yields

$$3N(JX,Y,Z) = d\Omega(X,JY,JZ) - d\Omega(X,Y,Z) + d\Omega(JX,JY,Z) + d\Omega(JX,Y,JZ)\,.$$

Together, these two equations show that the previous expression for $2\nabla_X J(Y)$ vanishes by equation (5). □

Besides nearly Kähler manifolds, Hermitian manifolds ($N = 0$) trivially fulfill the condition of the preceding lemma and ∇ coincides then with the Bismut connection; however, in the non-Hermitian situation, ∇ is *not* in the standard family of canonical Hermitian connections that is usually considered (see [Gau97, 2.5.4]). Proposition 2 in this same reference gives the decomposition of the torsion of any Hermitian connection in its (p,q)-components and gives another justification for this precise form for the torsion. Later, we shall see that ∇ is the *only possible* Hermitian connection with skew-symmetric torsion and that the class of almost Hermitian manifolds with skew-symmetric Nijenhuis tensor is the largest possible where it is defined.

We will put major emphasis on almost Hermitian manifolds of dimension 6, although one will find some general results formulated independently of the dimension. Two reasons for this choice are that nearly Kähler manifolds are of interest only in dimension 6, and that 6 is also the relevant dimension in superstring theory.

2.4. **Hyper-Kähler manifolds with torsion (HKT-manifolds).** We recall that a manifold M is called *hypercomplex* if it is endowed with three (integrable) complex structures I, J, K satisfying the quaternionic identities $IJ = -JI = K$. A metric g compatible with these three complex structures (a so-called *hyper-Hermitian* metric) is said to be *hyper-Kähler with torsion* or just a *HKT-metric* if the Kähler forms satisfy the identity

$$I\, d\Omega_I = J\, d\Omega_J = K\, d\Omega_K\,. \tag{7}$$

Despite the misleading name, these manifolds are not Kähler (and hence even less hyper-Kähler). HKT-metrics were introduced by Howe and Papadopoulos as target spaces of some two-dimensional sigma models with $(4, 0)$ supersymmetry with Wess-Zumino term [HP96]. Their mathematical description was given by Grantcharov and Poon in [GP00] and further investigated by several authors since then (see for example [Ve02], [DF02], [PS03], [FG03], [FPS04], [IM04]). From the previous example, we can conclude immediately that

$$g(\nabla_X Y, Z) := g(\nabla^g_X Y, Z) + \frac{1}{2}\, d\Omega_I(IX, IY, IZ)$$

defines a metric connection with skew-symmetric torsion such that $\nabla I = \nabla J = \nabla K = 0$; one easily checks that ∇ is again the *only* connection fulfilling these conditions. Equation (7) implies that we could equally well have chosen J or K in the last term. In general, a hyper-Hermitian manifold will not carry an HKT-structure, except in dimension 4 where this is proved in [GT98]. Examples of homogeneous HKT-metrics can be constructed using a family of homogeneous hypercomplex structures associated with compact semisimple Lie groups constructed by Joyce [Joy92]. Inhomogeneous HKT-structures exist for example on $S^1 \times S^{4n-3}$ [GP00]. The question of existence of suitable potential functions for HKT-manifolds was first raised and discussed in the context of super-conformal quantum mechanics by the physicists Michelson and Strominger [MS00] (a maximum principle argument shows that compact HKT-manifolds do not admit global potentials); Poon and Swann discussed potentials for some symmetric HKT-manifolds [PS01], while Banos and Swann were able to show local existence [BS04].

2.5. **Almost contact metric structures.** An odd-dimensional manifold M^{2n+1} is said to carry an *almost contact structure* if it admits a $(1, 1)$-tensor field φ and a vector field ξ (sometimes called the *characteristic* or *Reeb vector field*) with dual 1-form η ($\eta(\xi) = 1$) such that $\varphi^2 = -\mathrm{Id} + \eta \otimes \xi$. Geometrically, this means that M has a preferred direction (defined by ξ) on which φ^2 vanishes, while φ behaves like an almost complex structure on any linear complement of ξ. An easy argument shows that $\varphi(\xi) = 0$ [Bla02, Thm 4.1]. If there exists in addition a φ-compatible Riemannian metric g on M^{2n+1}, i.e. satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)\,,$$

then we say that $(M^{2n+1}, g, \xi, \eta, \varphi)$ carries an *almost contact metric structure* or that it is an *almost contact metric manifold.* The condition says that ξ is a vector field of unit length with respect to g and that g is φ-compatible in the sense of Hermitian geometry on the orthogonal complement $\xi^\perp$. Unfortunately, this relatively intuitive structural concept splits into a myriad of subtypes and leads to complicated equations in the defining data (ξ, η, φ), making the investigation of almost contact metric structures look rather unattractive at first sight (see [AG86], [ChG90], [ChM92], and [Fin95] for a classification). Yet, they constitute a rich and particularly interesting class of non-integrable geometries, as they have no integrable analogue on Berger's list. An excellent general source on contact manifolds with extensive references are the books by David Blair, [Bla76] and [Bla02] (however, the classification is not treated in these).

Example 2.5. For every almost Hermitian manifold (M^{2n}, g, J), there exists an almost metric contact structure $(\tilde{g}, \xi, \eta, \varphi)$ on the cone $M^{2n} \times \mathbb{R}$ with the product metric. For any function $f \in C^\infty(M^{2n} \times \mathbb{R})$ and vector field $X \in \mathfrak{X}(M^{2n})$, it is defined by

$$\varphi\,(X, f\partial_t) = (JX, 0)\,, \quad \xi = (0, \partial_t)\;, \quad \eta\,(X, f\partial_t) = f\,.$$

Conversely, an almost metric contact structure (g, ξ, η, φ) on M^{2n+1} induces an almost Hermitian structure $(\tilde{g}, J)$ on its cone $M^{2n+1} \times \mathbb{R}$ with product metric by setting for $f \in C^\infty(M^{2n+1} \times \mathbb{R})$ and $X \in \mathfrak{X}(M^{2n+1})$

$$J(X, f\partial_t) = (\varphi X - f\xi, \eta(X)\partial_t)\,.$$

In fact, one shows that every smooth orientable hypersurface M^{2n-1} in an almost Hermitian manifold (M^{2n}, g, J) carries a canonical almost contact metric structure [Bla02, 4.5.2]. In this way, one easily constructs almost contact metric structures on compact manifolds, for example on all odd dimensional spheres.

The fundamental form F of an almost contact metric structure is defined by $F(X, Y)$
$= g\big(X, \varphi(Y)\big)$, its Nijenhuis tensor is given by a similar, but slightly more complicated formula as in the almost Hermitian case and can also be written in terms of covariant derivatives of φ,

$$\begin{aligned} N(X, Y) &:= [\varphi(X), \varphi(Y)] - \varphi[X, \varphi(Y)] - \varphi[\varphi(X), Y] + \varphi^2[X, Y] + d\eta(X, Y)\xi \\ &= (\nabla^g_X \varphi)\varphi(Y) - (\nabla^g_Y \varphi)\varphi(X) + (\nabla^g_{\varphi(X)}\varphi)Y - (\nabla^g_{\varphi(Y)}\varphi)X \\ &\quad + \eta(X)\nabla^g_Y \xi - \eta(Y)\nabla^g_X \xi\,. \end{aligned}$$

Let us emphasize some particularly interesting cases. A manifold with an almost contact metric structure $(M^{2n+1}, g, \xi, \eta, \varphi)$ is called

(1) a *normal almost contact metric manifold* if $N = 0$,
(2) a *contact metric manifold* if $2F = d\eta$.

Furthermore, a contact metric structure is said to be a *K-contact metric structure* if ξ is in addition a Killing vector field, and a *Sasaki structure* if it is normal (it is then automatically K-contact, see [Bla02, Cor. 6.3.]). *Einstein-Sasaki manifolds* are just Sasaki manifolds whose φ-compatible Riemannian metric g is Einstein.

Without doubt, the forthcoming monograph by Ch. Boyer and Kr. Galicki on Sasakian geometry [BG07] is set to become *the* standard reference for this area of contact geometry in the future; in the meantime, the reader will have to be contented with the shorter reviews [BG99] and [BG01].

Much less is known about metric connections on almost contact metric manifolds than on almost Hermitian manifolds. In fact, only the so-called *generalized Tanaka connection* (introduced by S. Tanno) has been investigated. It is a metric connection defined on the class of contact metric manifolds by the formula

$$\nabla^*_X Y := \nabla^g_X Y + \eta(X)\varphi(Y) - \eta(Y)\nabla^g_X \xi + (\nabla_X \eta)(Y)\xi$$

satisfying the additional conditions $\nabla^*\eta = 0$ (which is of course equivalent to $\nabla^*\xi = 0$), see [Ta89] and [Bla02, 10.4.]. One easily checks that its torsion is not skew-symmetric, not even in the Sasaki case. In fact, from the point of view of non-integrable structures, it seems appropriate to require in addition $\nabla^*\varphi = 0$ (compare with the almost Hermitian situation).

Following a similar but more complicated line of arguments as in the almost Hermitian case, Th. Friedrich and S. Ivanov showed:

Theorem 2.2 ([FI02, Thm 8.2.])**.** *Let $(M^{2n+1}, g, \xi, \eta, \varphi)$ be an almost contact metric manifold. It admits a metric connection ∇ with totally skew-symmetric torsion T and $\nabla\eta = \nabla\varphi = 0$ if and only if the Nijenhuis tensor N is skew-symmetric and if ξ is a Killing vector field. Furthermore, $\nabla = \nabla^g + (1/2)T$ is uniquely determined by*

$$T = \eta \wedge d\eta + d^\varphi F + N - \eta \wedge (\xi \lrcorner N)\,,$$

where $d^\varphi F$ stands for the φ-twisted derivative,

$$d^\varphi F(X, Y, Z) := -dF\big(\varphi(X), \varphi(Y), \varphi(Z)\big)\,.$$

For a Sasaki structure, $N = 0$ and $2F = d\eta$ implies $d^\varphi F = 0$, hence T is given by the much simpler formula $T = \eta \wedge d\eta$. This connection had been noticed before, for example in [KW87]. In fact, one sees that $\nabla T = 0$ holds, hence Sasaki manifolds endowed with this connection are examples of non-integrable geometries with parallel torsion. A. Fino studied naturally reductive almost contact metric structures such that φ is parallel with respect to the canonical connection in [Fin94]. In general, potentials are hardly studied in contact geometry (compare with the situation for HKT-manifolds), but a suitable analogue of the Kähler potential was constructed on Sasaki manifolds by M. Godlinski, W. Kopczynski and P. Nurowski [GKN00].

2.6. **3-Sasaki manifolds.** Similarly to HKT-manifolds and quaternionic-Kähler manifolds, it makes sense to investigate configurations with three 'compatible' almost metric contact structures $(\varphi_i, \xi_i, \eta_i)$, $i = 1, 2, 3$ on (M^{2n+1}, g) for some fixed metric g. The compatibility condition may be formulated as

$$\varphi_k = \varphi_i\varphi_j - \eta_j \otimes \xi_i = -\varphi_j\varphi_i + \eta_i \otimes \eta_j\,, \quad \xi_k = \varphi_i\xi_j = -\varphi_j\xi_i$$

for any even permutation (i, j, k) of $(1, 2, 3)$, and such a structure is called an *almost contact metric 3-structure*. By defining on the cone $M^{2n+1} \times \mathbb{R}$ three almost complex structures J_1, J_2, J_3 as outlined in Example 2.5, one sees that the cone carries an almost quaternionic structure and hence has dimension divisible by 4. Consequently, almost contact metric 3-structures exist only in dimensions $4n+3, n \in \mathbb{N}$, and it is no surprise that the structure group of its tangent bundle turns out to be contained in $\mathrm{Sp}(n) \times \{1\}$. What is surprising is the recent result by T. Kashiwada that if all three structures $(\varphi_i, \xi_i, \eta_i)$ are contact metric structures, they automatically have to be Sasakian [Ka01]. A manifold with such a structure will be called a 3-*Sasaki*(*an*) *manifold*. An earlier result by T. Kashiwada claims that any 3-Sasaki manifold is Einstein [Ka71]. The canonical example of a 3-Sasaki manifold is the sphere S^{4n+3} realized as a hypersurface in $\mathbb{H}^{n+1}$: each of the three almost complex structures forming the quaternionic structure of $\mathbb{H}^{n+1}$ applied to the exterior normal vector field of the sphere yields a vector field ξ_i $(i = 1, 2, 3)$ on S^{4n+3}, leading thus to three orthonormal vector fields on S^{4n+3}. Th. Friedrich and I. Kath showed that every compact simply connected 7-dimensional spin manifold with regular 3-Sasaki structure is isometric to S^7 or the Aloff-Wallach space $N(1,1) = \mathrm{SU}(3)/S_1$ (see [FK90] or [BFGK91]). By now, it is possible to list all homogeneous 3-Sasaki manifolds:

Theorem 2.3 ([BGM94]). *A homogeneous* 3-*Sasaki manifold is isometric to one of the following:*

(1) *Four families:* $\frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)} \cong S^{4n+3}$, $S^{4n+3}/\mathbb{Z}_2 \cong \mathbb{RP}^{4n+3}$, $\frac{\mathrm{SU}(m)}{\mathrm{S}\left(\mathrm{U}(m-2)\times \mathrm{U}(1)\right)}$ *for* $m \geq 3$, $\frac{\mathrm{SO}(k)}{\mathrm{SO}(k-4)\times \mathrm{Sp}(1)}$ *for* $k \geq 7$.
(2) *Five exceptional spaces:* $G_2/\mathrm{Sp}(1)$, $F_4/\mathrm{Sp}(3)$, $E_6/\mathrm{SU}(10)$, $E_7/\mathrm{Spin}(12)$, E_8/E_7.

All these spaces fibre over a quaternionic Kähler manifold; the fibre is $\mathrm{Sp}(1)$ *for* S^{4n+3} *and* $\mathrm{SO}(3)$ *in all other cases.*

Many non-homogeneous examples have also been constructed. The analogy between 3-Sasaki manifolds and HKT-manifolds breaks down when one starts looking at connections, however. For an arbitrary 3-Sasaki structure with 1-forms η_i $(i = 1, 2, 3)$, each of the three underlying Sasaki structures yields one possible choice of a metric connection ∇^i with $\nabla^i \eta_i = 0$ and torsion $T^i = \eta_i \wedge d\eta_i$ as detailed in Theorem 2.2. *However, these three connections do not coincide*; hence, the 3-Sasaki structure itself is not preserved by any metric connection with skew-symmetric torsion. A detailed discussion of these three connections and their spinorial properties in dimension 7 can be found in [AF04a]. Nevertheless, 3-Sasaki manifolds have recently appeared and been investigated in the context of the AdS/CFT-correspondence by Martelli, Sparks and Yau [MSY06].

2.7. **Metric connections on surfaces.** Classical topics of surface theory like the Mercator projection can be understood in a different light with the help of metric connections with torsion.

In [Car23, § 67, p. 408–409], Cartan describes the two-dimensional sphere with its flat metric connection, and observes (without proof) that "on this manifold, the straight lines are the *loxodromes*, which intersect the meridians at a constant

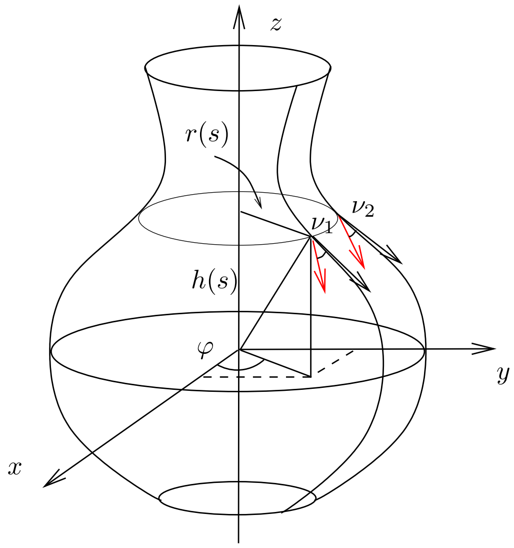

FIGURE 1. Surface of revolution generated by a curve α.

angle. The only straight lines realizing shortest paths are those normal to the torsion at every point: *these are the meridians*[5]".

This suggests that there exists a class of metric connections on surfaces of revolution whose geodesics admit a generalization of Clairaut's theorem, yielding loxodromes in the case of the flat connection. Furthermore, it is well known that the Mercator projection maps loxodromes to straight lines in the plane (i. e., Levi-Civita geodesics of the Euclidian metric), and that this mapping is conformal. Theorem 2.4 provides the right setting to understand both effects:

Theorem 2.4 ([AT04])**.** *Let σ be a function on the Riemannian manifold (M, g), ∇ the metric connection with vectorial torsion defined by $V = -\mathrm{grad}(\sigma)$, and consider the conformally equivalent metric $\tilde{g} = e^{2\sigma} g$. Then:*

(1) *Any ∇-geodesic $\gamma(t)$ is, up to a reparametrization τ, a $\nabla^{\tilde{g}}$-geodesic, and the function τ is the unique solution of the differential equation $\ddot{\tau} + \dot{\tau}\dot{\sigma} = 0$, where we set $\sigma(t) := \sigma \circ \gamma \circ \tau(t)$;*
(2) *If X is a Killing field for the metric $\tilde{g}$, the function $e^{\sigma} g(\dot{\gamma}, X)$ is a constant of motion for the ∇-geodesic $\gamma(t)$.*
(3) *The connection forms of ∇ and $\nabla^{\tilde{g}}$ coincide; in particular, they have the same curvature.*

[5]≪Sur cette variété, les lignes droites sont les *loxodromies*, qui font un angle constant avec les méridiennes. Les seules lignes droites qui réalisent les plus courts chemins sont celles qui sont normales en chaque point à la torsion : *ce sont les méridiennes.*≫

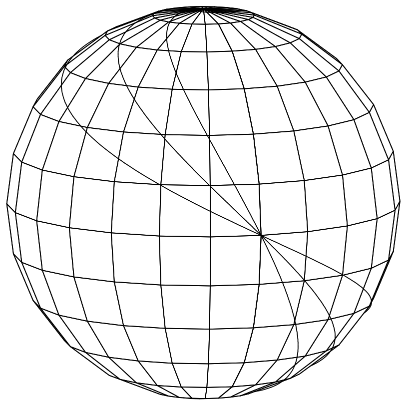

FIGURE 2. Loxodromes on the sphere.

We discuss Cartan's example in the light of Theorem 2.4. Let $\alpha = \big(r(s), h(s)\big)$ be a curve parametrized by arclength, and $M(s,\varphi) = \big(r(s)\cos\varphi, r(s)\sin\varphi, h(s)\big)$ the surface of revolution generated by it. The first fundamental form is $g = \mathrm{diag}\big(1, r^2(s)\big)$, so we can choose the orthonormal frame $e_1 = \partial_s$, $e_2 = (1/r)\partial_\varphi$ with dual 1-forms $\sigma^1 = ds$, $\sigma^2 = r\,d\varphi$. We define a connection ∇ by calling two tangential vectors v_1 and v_2 parallel if the angles ν_1 and ν_2 with the meridian through their foot point coincide (see Figure 1). Hence $\nabla e_1 = \nabla e_2 = 0$, and the connection ∇ is flat. But for a flat connection, the torsion T is can be derived from $d\sigma^i(e_j, e_k) = \sigma^i\big(T(e_j, e_k)\big)$. Since $d\sigma^1 = 0$ and $d\sigma^2 = (r'/r)\sigma^1 \wedge \sigma^2$, one obtains

$$T(e_1, e_2) = \frac{r'(s)}{r(s)}\, e_2 \quad \text{and} \quad V = \frac{r'(s)}{r(s)}\, e_1 = -\mathrm{grad}\big(-\ln r(s)\big)\,.$$

Thus, the metric connection ∇ with vectorial torsion T is determined by the gradient of the function $\sigma := -\ln r(s)$. By Theorem 2.4, we conclude that its geodesics are the Levi-Civita geodesics of the conformally equivalent metric $\tilde g = e^{2\sigma} g = \mathrm{diag}(1/r^2, 1)$. This coincides with the standard Euclidian metric by changing variables $x = \varphi$, $y = \int ds/r(s)$. For example, the sphere is obtained for $r(s) = \sin s, h(s) = \cos s$, hence $y = \int ds/\sin s = \ln\tan(s/2)$ $(|s| < \pi/2)$, and this is precisely the coordinate change of the Mercator projection. Furthermore, $X = \partial_\varphi$ is a Killing vector field for $\tilde g$, hence the second part of Theorem 2.4 yields for a ∇-geodesic γ the invariant of motion

$$\mathrm{const} = e^\sigma g(\dot\gamma, X) = \frac{1}{r(s)} g(\dot\gamma, \partial_\varphi) = g(\dot\gamma, e_2)\,,$$

i.e. the cosine of the angle between γ and a parallel circle. This shows that γ is a loxodrome on M, as claimed (see Figure 2 for loxodromes on the sphere). In the same way, one obtains a "generalized Clairaut theorem" for any gradient vector

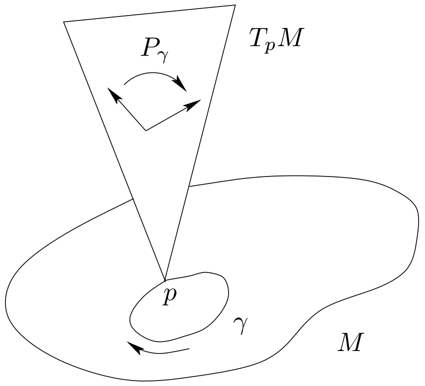

FIGURE 3. Schematic concept of holonomy.

field on a surface of revolution. For the pseudosphere, one chooses

$$r(s) = e^{-s}\,, \quad h(s) = \operatorname{arctanh}\sqrt{1-e^{-2s}} - \sqrt{1-e^{-2s}}\,,$$

hence $V = -e_1$ and $\nabla V = 0$, in accordance with the results by [TV83] cited before. Notice that X is also a Killing vector field for the metric g and *does* commute with V; nevertheless, $g(\dot\gamma, X)$ is *not* an invariant of motion.

The catenoid is another interesting example: since it is a minimal surface, the Gauss map to the sphere is conformal, hence it maps loxodromes to loxodromes. Thus, Beltrami's theorem ("If a portion of a surface S can be mapped LC-geodesically onto a portion of a surface S^* of constant Gaussian curvature, the Gaussian curvature of S must also be constant", see for example [Kre91, §95]) does *not* hold for metric connections with vectorial torsion — the sphere is a surface of constant Gaussian curvature, but the catenoid is not.

Remark 2.1. The unique flat metric connection ∇ does not have to be of vectorial type. For example, on the compact Lie group $\mathrm{SO}(3)$ the torsion is a 3-form: Fix an orthonormal basis e_1, e_2, e_3 with commutator relations $[e_1, e_2] = e_3, [e_2, e_3] = e_1$ and $[e_1, e_3] = -e_2$. Cartan's structural equations then read $d\sigma^1 = \sigma^2 \wedge \sigma^3, d\sigma^2 = -\sigma^1 \wedge \sigma^3, d\sigma^3 = \sigma^1 \wedge \sigma^2$, from which we deduce $T = 2A = \sigma^1 \wedge \sigma^2 \wedge \sigma^3$. In particular, ∇ has the same geodesics as ∇^g.

2.8. **Holonomy theory.** Let (M^n, g) be a (connected) Riemannian manifold equipped with *any* connection ∇. For a curve $\gamma(s)$ from p to q, parallel transport along γ is the linear mapping $P_\gamma : T_pM \to T_qM$ such that the vector field $\mathcal{V}$ defined by

$$\mathcal{V}(q) := P_\gamma \mathcal{V}(p) \quad \text{along} \quad \gamma$$

is parallel along γ, $\nabla\mathcal{V}(s)/ds = \nabla_{\dot\gamma}\mathcal{V} = 0$. P_γ is always an invertible endomorphism, hence, for a closed loop γ through $p \in M$, it can be viewed as an element of $\mathrm{GL}(n, \mathbb{R})$ (after choice of some basis). Consider the *loop space* $C(p)$ of all closed, piecewise smooth curves through p, and therein the subset $C_0(p)$ of curves that are homotopic to the identity. The set of parallel translations along loops in $C(p)$ or $C_0(p)$ forms a group acting on $\mathbb{R}^n \cong T_pM$, called the *holonomy group* $\mathrm{Hol}(p;\nabla)$ *of* ∇ or the *restricted holonomy group* $\mathrm{Hol}_0(p;\nabla)$ *of* ∇ at the point p. Let us now change the point of view from p to q, γ a path joining them; then $\mathrm{Hol}(q;\nabla) = P_\gamma \mathrm{Hol}(p;\nabla)P_\gamma^{-1}$ and similarly for $\mathrm{Hol}_0(p;\nabla)$. Hence, all holonomy groups are isomorphic, so we drop the base point from now on. Customary notation for them is $\mathrm{Hol}(M;\nabla)$ and $\mathrm{Hol}_0(M;\nabla)$. Their action on $R^n \cong T_pM$ shall be called the (*restricted*) *holonomy representation.*

In general, it is only known that ([KN63, Thm IV.4.2])

(1) $\mathrm{Hol}(M;\nabla)$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$,
(2) $\mathrm{Hol}_0(p)$ is the connected component of the identity of $\mathrm{Hol}(M;\nabla)$.

If one assumes in addition — as we will do through this text — that ∇ be metric, parallel transport becomes an isometry: for any two parallel vector fields $\mathcal{V}(s)$ and $\mathcal{W}(s)$, being metric implies

$$\frac{d}{ds} g\big(\mathcal{V}(s),\mathcal{W}(s)\big) = g\big(\frac{\nabla\mathcal{V}(s)}{ds},\mathcal{W}(s)\big) + \big(\mathcal{V}(s),\frac{\nabla\mathcal{W}(s)}{ds}\big) = 0\,.$$

Hence, $\mathrm{Hol}(M;\nabla) \subset \mathrm{O}(n)$ and $\mathrm{Hol}_0(M;\nabla) \subset \mathrm{SO}(n)$. For convenience, we shall henceforth speak of the *Riemannian (restricted) holonomy group* if ∇ is the Levi-Civita connection, to distinguish it from holonomy groups in our more general setting.

Example 2.6. This is a good moment to discuss Cartan's first example of a space with torsion (see [Car22, p. 595]). Consider $\mathbb{R}^3$ with its usual Euclidean metric and the connection

$$\nabla_X Y = \nabla^g_X Y - X\times Y\,,$$

corresponding, of course, to the choice $T = -2\cdot e_1\wedge e_2\wedge e_3$. Cartan observed correctly that this connection has same geodesics than ∇^g, but induces a different parallel transport[6]. Indeed, consider the z-axis $\gamma(t) = (0,0,t)$, a geodesic, and the vector field V which, in every point $\gamma(t)$, consists of the vector $(\cos t,\sin t,0)$. Then one checks immediately that $\nabla^g_{\dot\gamma}V = \dot\gamma\times V$, that is, the vector V is parallel transported according to a helicoidal movement. If we now transport the vector along the edges of a closed triangle, it will be rotated around three linearly independent axes, hence the holonomy algebra is $\mathfrak{hol}(\nabla) = \mathfrak{so}(3)$.

Example 2.7 (Holonomy of naturally reductive spaces)**.** Consider a naturally reductive space $M^n = G/H$ as in Example 2.2 with its canonical connection ∇^1, whose torsion is $T^1(X,Y) := -[X,Y]_\mathfrak{m}$. Recall that $\mathrm{ad}:\mathfrak{h}\to\mathfrak{so}(\mathfrak{m})$ denotes its isotropy representation. The holonomy algebra $\mathfrak{hol}(\nabla^1)$ is the Lie subalgebra of

[6]"Deux trièdres [...] de $\mathcal{E}$ seront parallèles lorsque les trièdres correspondants de E [l'espace euclidien classique] pourront se déduire l'un de l'autre par un déplacement hélicoïdal de pas donné, de sens donné[...]. L'espace $\mathcal{E}$ ainsi défini admet un groupe de transformations à 6 paramètres : ce serait notre espace ordinaire vu par des observateurs dont toutes les perceptions seraient tordues." loc.cit.

$\mathrm{ad}(\mathfrak{h}) \subset \mathfrak{so}(\mathfrak{m})$ generated by the images under ad of all projections of commutators $[X,Y]_\mathfrak{h}$ on $\mathfrak{h}$ for $X, Y \in \mathfrak{m}$,

$$\mathfrak{hol}(\nabla^1) = \mathrm{Lie}\big(\mathrm{ad}([X,Y]_\mathfrak{h})\big) \subset \mathrm{ad}(\mathfrak{h}) \subset \mathfrak{so}(\mathfrak{m})\,.$$

For all other connections ∇^t in this family, the general expression for the holonomy is considerably more complicated [KN69, Thm. X.4.1].

Remark 2.2 (Holonomy & contact properties)**.** As we observed earlier, all contact structures are non-integrable and therefore not covered by Berger's holonomy theorem. Via the cone construction, it is nevertheless possible to characterize them by a Riemannian holonomy property (see [BGM94], [BG99]). Consider a Riemannian manifold (M^n, g) and its cone over the positive real numbers $N := \mathbb{R}_+ \times M^n$ with the warped product metric $g_N := dr^2 + r^2 g$. Then, (M^n, g) is

(1) *Sasakian* if and only if $\mathrm{Hol}(N; \nabla^g) \subset \mathrm{U}(\frac{n+1}{2})$, that is, its positive cone is Kähler,
(2) *Einstein-Sasakian* if and only if $\mathrm{Hol}(N; \nabla^g) \subset \mathrm{SU}(\frac{n+1}{2})$, that is, its positive cone is a (non-compact) Calabi-Yau manifold,
(3) 3-*Sasakian* if and only if $\mathrm{Hol}(N; \nabla^g) \subset \mathrm{Sp}(\frac{n+1}{4})$, that is, its positive cone is hyper-Kähler.

A holonomy group can be determined by computing curvature.

Theorem 2.5 (Ambrose-Singer, 1953 [AS53])**.** *For any connection ∇ on the tangent bundle of a Riemannian manifold (M, g), the Lie algebra $\mathfrak{hol}(p)$ of $\mathrm{Hol}(p)$ in $p \in M$ is exactly the subalgebra of $\mathfrak{so}(T_pM)$ generated by the elements*

$$P_\gamma^{-1} \circ \mathcal{R}(P_\gamma V, P_\gamma W) \circ P_\gamma$$

where $V, W \in T_pM$ and γ runs through all piecewise smooth curves starting from p.

Yet, the practical use of this result is severely restricted by the fact that the properties of the curvature transformation of a metric connection with torsion are more complicated than the Riemannian ones. For example, $\mathcal{R}(U,V)$ is still skew-adjoint with respect to the metric g,

$$g\big(\mathcal{R}(U,V)W_1, W_2\big) = -g\big(\mathcal{R}(U,V)W_2, W_1\big)\,,$$

but there is in general no relation between $g\big(\mathcal{R}(U_1,U_2)\,W_1, W_2\big)$ and $g\big(\mathcal{R}(W_1,W_2)\,U_1, U_2\big)$ (but see Remark 2.3 below); in consequence, the Bianchi identities are less tractable, the Ricci tensor is not necessarily symmetric etc. As an example of these complications, we cite (see [IP01] for the case of skew-symmetric torsion):

Theorem 2.6 (First Bianchi identity)**.**

(1) *A metric connection ∇ with vectorial torsion $V \in TM^n$ satisfies*

$$\overset{X,Y,Z}{\mathfrak{S}}\ \mathcal{R}(X,Y)Z = \overset{X,Y,Z}{\mathfrak{S}}\ dV(X,Y)Z\,.$$

(2) *A metric connection ∇ with skew-symmetric torsion $T \in \Lambda^3(M^n)$ satisfies*

$$\overset{X,Y,Z}{\mathfrak{S}}\ \mathcal{R}(X,Y,Z,V) = dT(X,Y,Z,V) - \sigma_T(X,Y,Z,V) + (\nabla_V T)(X,Y,Z)\,,$$

where σ_T is a 4-form that is quadratic in T defined by $2\,\sigma_T = \sum_{i=1}^{n}(e_i \lrcorner T) \wedge (e_i \lrcorner T)$ for any orthonormal frame $e_1, \ldots, e_n$.

Remark 2.3. Consequently, if the torsion $T \in \Lambda^3(M^n)$ of a metric connection with skew-symmetric torsion happens to be ∇-parallel, $\overset{X,Y,Z}{\mathfrak{S}} \mathcal{R}(X,Y,Z,V)$ is a 4-form and thus antisymmetric. Since the cyclic sum over all four arguments of any 4-form vanishes, we obtain

$$\overset{X,Y,Z,V}{\mathfrak{S}} \Big[\overset{X,Y,Z}{\mathfrak{S}} \mathcal{R}(X,Y,Z,V)\Big] = 2\,\mathcal{R}(Z,X,Y,V) - 2\,\mathcal{R}(Y,V,Z,X) = 0\,,$$

as for the Levi-Civita connection. Thus the ∇-curvature tensor is invariant under swaps of the first and second pairs of arguments.

Extra care has to be taken when asking which properties of the Riemannian holonomy group are preserved:

Remark 2.4 (The holonomy representation may not be irreducible)**.** In fact, there are many instances of irreducible manifolds with metric connections whose holonomy representation is *not* irreducible — for example, the 7-dimensional Aloff-Wallach space $N(1,1) = \mathrm{SU}(3)/S^1$ or the 5-dimensional Stiefel manifolds $V_{4,2} = \mathrm{SO}(4)/\mathrm{SO}(2)$ (see [BFGK91], [Agr03] and [AF04a]). This sheds some light on why parallel objects are — sometimes — easier to find for such connections. This also implies that no analogue of de Rham's splitting theorem can hold ("A complete simply connected Riemannian manifold with reducible holonomy representation is a Riemannian product").

Remark 2.5 (The holonomy group may not be closed)**.** For the Riemannian restricted holonomy group, the argument goes as follows: By de Rham's Theorem, one can assume that the holonomy group acts irreducibly on each tangent space; but *any* connected subgroup G of $\mathrm{O}(n)$ with this property has to be closed and hence compact. A counter-example for a torsion-free non-metric connection due to Ozeki can be found in [KN63, p. 290]; similar (quite pathological) examples can be given for metric connections with torsion, although they seem to be not too interesting. It suffices to say that *there is no theoretical argument ensuring the closure of the restricted holonomy group of a metric connection with torsion.*

We are particularly interested in the vector bundle of (r,s)-tensors $T^{r,s}M$ over M^n, that of differential k-forms $\Lambda^k M$ and its spinor bundle ΣM (assuming that M is spin, of course). At some point $p \in M$, the fibers are just $(T_pM)^r \otimes (T_p^*M)^s$, $\Lambda^k T_p^* M$ or Δ_n, the n-dimensional spin representation (which has dimension $2^{[n/2]}$); an element of the fibre at some point will be called an *algebraic tensor, form, spinor* or just *algebraic vector* for short. Then, on any of these bundles,

(1) the holonomy representation induces a representation of $\mathrm{Hol}(M;\nabla)$ on each fibre (the "lifted holonomy representation");
(2) the metric connection ∇ induces a connection (again denoted by ∇) on these vector bundles (the "lifted connection") compatible with the induced

metric (for tensors) or the induced Hermitian scalar product (for spinors), it is thus again metric;

(3) in particular, there is a notion of "lifted parallel transport" consisting of isometries, and its abstract holonomy representation on the fibres coincides with the lifted holonomy representation.

We now formulate the general principle underlying our study.

Theorem 2.7 (General Holonomy Principle). *Let M be a differentiable manifold and E a (real or complex) vector bundle over M endowed with (any!) connection ∇. The following three properties are equivalent:*

(1) *E has a global section α invariant under parallel transport, i.e. $\alpha(q) = P_\gamma\big(\alpha(p)\big)$ for any path γ from p to q;*
(2) *E has a parallel global section α, i.e. $\nabla\alpha = 0$;*
(3) *at some point $p \in M$, there exists an algebraic vector $\alpha_0 \in E_p$ which is invariant under the holonomy representation on the fibre.*

Proof. The proof is almost philosophical. To begin, the first and last conditions are equivalent: if α is a section invariant under parallel transport $P_\gamma : E_p \to E_q$, then, for a closed curve γ through p, $\alpha(p) = P_\gamma\big(\alpha(p)\big)$ and hence α is invariant under all holonomy transformations in p.

Conversely, let $\alpha_0 \in E_p$ be a holonomy invariant algebraic vector. We then define $\alpha(q) := P_\gamma(\alpha_0)$ for $q \in M$ and any path γ from p to q. This definition is in fact path independent because, for any other path γ' from q to p, their concatenation is a closed loop, and α_0 is, by assumption, invariant under parallel transport along closed curves.

Finally, let X be a vector field, γ one of its integral curves going from p to q. Then obviously $\nabla_{\dot\gamma}\alpha = 0$ is equivalent to $\alpha(q) = P_\gamma\big(\alpha(p)\big)$, showing the equivalence of (1) and (2). □

The following two consequences are immediate, but of the utmost importance.

Corollary 2.2.

(1) *The number of parallel global sections of E coincides with the number of trivial representations occuring in the holonomy representation on the fibres.*
(2) *The holonomy group* $\mathrm{Hol}(\nabla)$ *is a subgroup of the isotropy group $G_\alpha := \{g \in \mathrm{O}(n) \ : \ g^*\alpha = \alpha\}$ of any parallel global section α of E.*

This is a powerful tool for (dis-)proving existence of parallel objects. For example, the following is a well-known result from linear algebra:

Lemma 2.4. *The determinant is an* $\mathrm{SO}(n)$*-invariant element in* $\Lambda^n(\mathbb{R}^n)$ *which is not* $\mathrm{O}(n)$*-invariant.*

Corollary 2.3. *A Riemannian manifold (M^n, g) is orientable if and only if the holonomy* $\mathrm{Hol}(M;\nabla)$ *of* any metric connection ∇ *is a subgroup of* $\mathrm{SO}(n)$*, and the volume form is then ∇-parallel.*

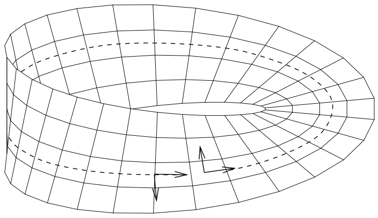

FIGURE 4. An orthonormal frame that is parallel transported along the drawn curve reverses its orientation.

Proof. One knows that (M^n, g) is orientable if and only if it admits a nowhere vanishing differential form dM^n of degree n. Then pick an orthonormal frame $e_1, \ldots, e_n$ in p with dual 1-forms $\sigma_1, \ldots, \sigma_n$. Set $dM^n_p := \sigma_1 \wedge \ldots \wedge \sigma_n$ and extend it to M^n by parallel transport. Now everything follows from the General Holonomy Principle. □

Remark 2.6. The property $\nabla dM^n = 0$ for $dM^n = \sigma_1 \wedge \ldots \wedge \sigma_n$ can also be seen directly from the formula

$$\nabla_X(dM^n)(e_1, \ldots, e_n) = X(1) - \sum_{i=1}^{n} dM^n(e_1, \ldots, \nabla_X e_i, \ldots, e_n)\,,$$

since a metric connection satisfies $g(\nabla e_i, e_j) + g(e_i, \nabla e_j) = 0$, so in particular ∇e_i has no e_i-component and all summands on the right hand side vanish.

Remark 2.7. In fact, an arbitrary connection ∇ admits a ∇-parallel n-form (possibly with zeroes) if and only if

$$\sum_{i=1}^{n} g(\mathcal{R}(U, V)e_i, e_i) = 0$$

for any orthonormal frame $e_1, \ldots, e_n$. This property is weaker than the skew-adjointness of $\mathcal{R}(U, V)$ that holds for all metric connections; the holonomy is a subgroup of $\mathrm{SL}(n, \mathbb{R})$. In 1924, J. A. Schouten called such connections "inhalt-streue Übertragungen" (*volume-preserving connections*), see [Sch24, p. 89]. This terminology seems not to have been used anymore afterwards[7].

[7]A *D'Atri space* is a Riemannian manifold whose local geodesic symmetries are volume-preserving. Although every naturally reductive space is a D'Atri space [Atr75], the two notions are only loosely related.

The existence of parallel objects imposes restrictions on the curvature of the connection. For example, if a connection ∇ admits a parallel spinor ψ, we obtain by contracting the identity

$$0 = \nabla\nabla\psi = \sum_{i,j=1}^{n} \mathcal{R}^{\nabla}(e_i, e_j) e_i \cdot e_j \cdot \psi$$

the following integrability condition (the Riemannian case has first been proved by Bonan in [Bon66]):

Proposition 2.2. *Let (M^n, g) be a Riemannian spin manifold, ∇ a metric connection with torsion $T \in \Lambda^3(M^n)$. A ∇-parallel spinor ψ satisfies*

$$\Big[\frac{1}{2} X \lrcorner\, dT + \nabla_X T\Big] \cdot \psi = \mathrm{Ric}^{\nabla}(X) \cdot \psi$$

In particular, the existence of a ∇^g-parallel spinor $(T = 0)$ implies Ricci-flatness.

Before considering general metric connections with torsion on manifolds, it is worthwile to investigate the flat case $\mathbb{R}^n$ endowed with its standard Euclidean metric and metric connections with constant torsion, for it exhibits already some characteristic features of the more general situation. Unless otherwise stated, these results can be found in [AF04a].

The exterior algebra $\Lambda(\mathbb{R}^n)$ and the Clifford algebra $\mathrm{Cl}(\mathbb{R}^n)$ are — as vector spaces — equivalent $\mathrm{SO}(n)$-representations, and they both act on the complex vector space Δ_n of n-dimensional spinors. The Clifford algebra is an associative algebra with an underlying Lie algebra structure,

$$[\alpha, \beta] = \alpha \cdot \beta - \beta \cdot \alpha\,, \quad \alpha, \beta \in \mathrm{Cl}(\mathbb{R}^n)\,.$$

We denote the corresponding Lie algebra by $\mathfrak{cl}(\mathbb{R}^n)$. The Lie algebra $\mathfrak{so}(n)$ of the special orthogonal group is a subalgebra of $\mathfrak{cl}(\mathbb{R}^n)$,

$$\mathfrak{so}(n) = \mathrm{Lin}\big\{X \cdot Y : X\,, Y \in \mathbb{R}^n \text{ and } \langle X, Y\rangle = 0\big\} \subset \mathfrak{cl}(\mathbb{R}^n)\,.$$

Consider an algebraic k-form $T \in \Lambda^k(\mathbb{R}^n)$ and denote by G_T the group of all orthogonal transformations of $\mathbb{R}^n$ preserving the form T, by $\mathfrak{g}_T$ its Lie algebra. As described in Example 2.3, we consider the spin connection acting on spinor fields $\psi : \mathbb{R}^n \to \Delta_n$ by the formula

$$\nabla_X \psi := \nabla^g_X \psi + \frac{1}{2}\,(X \lrcorner\, \mathrm{T}) \cdot \psi\,.$$

For a 3-form $T \in \Lambda^3(\mathbb{R}^n)$, the spinorial covariant derivative ∇ is induced by the linear metric connection with torsion tensor T. For a general exterior form T, we introduce a new Lie algebra $\mathfrak{g}^*_T$ that is a subalgebra of $\mathfrak{cl}(\mathbb{R}^n)$.

Definition 2.3. Let T be an exterior form on $\mathbb{R}^n$. The Lie algebra $\mathfrak{g}^*_T$ is the subalgebra of $\mathfrak{cl}(\mathbb{R}^n)$ generated by all elements $X \lrcorner\, T$, where $X \in \mathbb{R}^n$ is a vector.

The Lie algebra $\mathfrak{g}^*_T$ is invariant under the action of the isotropy group G_T. The derived algebra $\big[\mathfrak{g}^*_T, \mathfrak{g}^*_T\big]$ is the Lie algebra generated by all curvature transformations of the spinorial connection ∇. It is the Lie algebra of the infinitesimal holonomy group of the spinorial covariant derivative ∇^{T} (see [KN63, Ch. II.10]):

Definition 2.4. Let T be an exterior form on $\mathbb{R}^n$. The Lie algebra $\mathfrak{h}_T^* := \left[\mathfrak{g}_T^*, \mathfrak{g}_T^*\right] \subset \mathfrak{cl}(\mathbb{R}^n)$ is called the *infinitesimal holonomy algebra* of the exterior form T. It is invariant under the action of the isotropy group G_T.

For a 3-form T, the Lie algebras $\mathfrak{g}_T^*, \mathfrak{h}_T^* \subset \mathfrak{so}(n)$ are subalgebras of the orthogonal Lie algebra, reflecting the fact that the corresponding spinor derivative is induced by a linear metric connection. In fact, this result still holds for k-forms satisfying $k + \binom{k-1}{2} \equiv 0 \bmod 2$. Furthermore, the General Holonomy Principle (Theorem 2.7) implies:

Proposition 2.3. *There exists a non-trivial ∇-parallel spinor field $\psi : \mathbb{R}^n \to \Delta_n$ if and only if there exists a constant spinor $\psi_0 \in \Delta_n$ such that $\mathfrak{h}_{\mathrm{T}}^* \cdot \psi_0 = 0$. In particular, any ∇-parallel spinor field is constant for a perfect Lie algebra $\mathfrak{g}_T^*$ ($\mathfrak{g}_T^* = \mathfrak{h}_T^*$).*

Example 2.8. Any 2-form $\mathrm{T} \in \Lambda^2(\mathbb{R}^n)$ of rank $2k$ is equivalent to $A_1 \cdot e_{12} + \cdots + A_k \cdot e_{2k-1,2k}$. The Lie algebra $\mathfrak{g}_{\mathrm{T}}^*$ is generated by the elements $e_1, e_2, \cdots, e_{2k-1}, e_{2k}$. It is isomorphic to the Lie algebra $\mathfrak{spin}(2k+1)$. In particular, if $n = 8$ then $\Delta_8 = \mathbb{R}^{16}$ is a real, 16-dimensional and the spinorial holonomy algebra of a generic 2-form in eight variables is the unique 16-dimensional irreducible representation of $\mathfrak{spin}(9)$.

Example 2.9. Consider the 4-forms $T_1 := e_{1234} \in \Lambda^4(\mathbb{R}^n)$ for $n \geq 4$ and $T_2 = e_{1234} + e_{3456} \in \Lambda^4(\mathbb{R}^m)$ for $m \geq 6$. A straightforward computation yields that $\mathfrak{g}_{T_1}^*$ and $\mathfrak{g}_{T_2}^*$ are isomorphic to the pseudo-orthogonal Lie algebra $\mathfrak{so}(4,1)$ embedded in a non-standard way and the Euclidean Lie algebra $\mathfrak{e}(6)$, respectively.

Example 2.10. Consider the volume form $T = e_{123456}$ in $\mathbb{R}^n$ for $n \geq 6$. The subalgebra $\mathfrak{g}_T^*$ of $\mathrm{Cl}(\mathbb{R}^n)$ is isomorphic to the compact Lie algebra $\mathfrak{spin}(7)$.

If T is a 3-form, more can be said. For example, $\mathfrak{g}_T^*$ is always semisimple and the following proposition shows that it cannot be contained in the unitary Lie algebra $\mathfrak{u}(k) \subset \mathfrak{so}(2k)$. This latter result is in sharp contrast to the situation on arbitrary manifolds, where such 3-forms occur for almost Hermitian structures.

Proposition 2.4. *Let* T *be a* 3*-form in* $\mathbb{R}^{2k}$ *and suppose that there exists a* 2*-form* Ω *such that* $\Omega^k \neq 0$ *and* $[\mathfrak{g}_T^*, \Omega] = 0$. *Then T is zero, $T = 0$.*

Moreover, only constant spinors are parallel:

Theorem 2.8. *Let $T \in \Lambda^3(\mathbb{R}^n)$ be a 3-form. If there exists a non trivial spinor $\psi \in \Delta_n$ such that $\mathfrak{g}_T^* \cdot \psi = 0$, then $T = 0$. In particular, ∇-parallel spinor fields are ∇^g-parallel and thus constant.*

Proof. The proof is remarkable in as much as it is of purely algebraic nature. Indeed, it is a consequence of the following formulas concerning the action of exterior forms of different degrees on spinors (see Appendix A for the first, the other two are simple calculations in the Clifford algebra):

$$2\sigma_T = -T^2 + \|T\|^2 , \quad 2\sigma_T \cdot \psi = \sum_{k=1}^{n} (e_k \lrcorner T) \cdot (e_k \lrcorner T) + 3\|T\|^2 , \quad 3T = \sum_{k=1}^{n} e_i \cdot (e_i \lrcorner T) .$$

For they imply that a spinor ψ with $(e_i \lrcorner T) \cdot \psi = 0$ for all e_i has to satisfy $\|T\|^2\psi = 0$, so T must vanish. $\square$

This result also applies to flat tori $\mathbb{R}^n/\mathbb{Z}^n$, as the torsion form T is assumed to be constant. Later, we shall prove a suitable generalization on compact spin manifolds with $\mathrm{scal}^g \leq 0$, see Theorem 5.4.

3. Geometric stabilizers

By the General Holonomy Principle, geometric representations with invariant objects are a natural source for parallel objects. This leads to the systematic investigation of *geometric stabilizers*, which we shall now discuss.

3.1. $\mathrm{U}(n)$ **and** $\mathrm{SU}(n)$ **in dimension** $2n$**.** A Hermitian metric $h(V,W) = g(V,W) - ig(JV,W)$ is invariant under $A \in \mathrm{End}(\mathbb{R}^{2n})$ if and only if A preserves the Riemannian metric g and the Kähler form $\Omega(V,W) := g(JV,W)$. Thus $\mathrm{U}(n)$ is embedded in $\mathrm{SO}(2n)$ as

$$\mathrm{U}(n) = \{A \in \mathrm{SO}(2n) \mid A^*\Omega = \Omega\}\,.$$

To fix ideas, choose a skew-symmetric endomorphism J of $\mathbb{R}^{2n}$ with square -1 in the normal form

$$J = \mathrm{diag}\big(j, j, j, \dots\big) \text{ with } j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then a complex $(n \times n)$-matrix $A = (a_{ij}) \in \mathrm{U}(n)$ is realized as a real $(2n \times 2n)$-matrix with (2×2)-blocks $\begin{bmatrix} \mathrm{Re}\, a_{ij} & -\mathrm{Im}\, a_{ij} \\ \mathrm{Im}\, a_{ij} & \mathrm{Re}\, a_{ij} \end{bmatrix}$. An adapted orthonormal frame is one such that J has the given normal form; $\mathrm{U}(n)$ consists then exactly of those endomorphisms transforming adapted orthonormal frames into adapted orthonormal frames. Allowing now complex coefficients, one obtains an $(n,0)$-form Ψ by declaring

$$\Psi := (e_1 + ie_2) \wedge \ldots \wedge (e_{n-1} + ie_{2n}) =: \Psi^+ + i\Psi^-$$

in the adapted frame above. An element $A \in \mathrm{U}(n)$ acts on Ψ by multiplication with $\det A$.

Lemma 3.1. *Under the restricted action of* $\mathrm{U}(n)$*,* $\Lambda^{2k}(\mathbb{R}^{2n})$ *contains the trivial representation once; it is generated by* $\Omega, \Omega^2, \dots, \Omega^n$*.*

The action of $\mathrm{U}(n) \subset \mathrm{O}(2n)$ cannot be lifted to an action of $\mathrm{U}(n)$ inside $\mathrm{Spin}(2n)$ — reflecting the fact that not every Kähler manifold is spin. For the following arguments though, it is enough to consider $\mathfrak{u}(n)$ inside $\mathfrak{spin}(2n)$. It then appears that $\mathfrak{u}(n)$ has no invariant spinors, basically because $\mathfrak{u}(n)$ has a one-dimensional center, generated precisely by Ω after identifying $\Lambda^2(\mathbb{R}^{2n}) \cong \mathfrak{so}(2n)$. Hence one-dimensional $\mathfrak{u}(n)$-representations are usually not trivial. More precisely, the complex $2n$-dimensional spin representation Δ_{2n} splits into two irreducible components $\Delta^{\pm}_{2n}$ described in terms of eigenspaces of $\Omega \in \mathfrak{u}(n)$. Set (see [Fri00] and [Kir86] for details on this decomposition of spinors)

$$S_r = \{\psi \in \Delta_{2n} \ :\ \Omega\psi = i(n-2r)\psi\}\,, \quad \dim S_r = \binom{n}{r}, \quad 0 \leq r \leq n\,.$$

S_r is isomorphic to the space of $(0,r)$-forms with values in S_0 (which explains the dimension),

$$S_r \cong \overline{\Lambda^{0,r}} \otimes S_0 \,.$$

Since the spin representations decompose as

$$\Delta_{2n}^+\big|_{\mathfrak{u}(n)} \;\cong\; S_n \oplus S_{n-2} \oplus \ldots, \quad \Delta_{2n}^-\big|_{\mathfrak{u}(n)} \;\cong\; S_{n-1} \oplus S_{n-3} \oplus \ldots$$

we conclude immediately that they cannot contain a trivial $\mathfrak{u}(n)$-representation for n odd. For $n = 2k$ even, Ω has eigenvalue zero on S_k, but this space is an irreducible representation of dimension $\binom{2k}{k} \neq 1$, hence not trivial either. The representations S_0 and S_n are one-dimensional, but again not trivial under $\mathfrak{u}(n)$. If one restricts further to $\mathfrak{su}(n)$, they are indeed:

Lemma 3.2. *The spin representations $\Delta_{2n}^\pm$ contain no $\mathfrak{u}(n)$-invariant spinor. If one restricts further to $\mathfrak{su}(n)$, there are exactly two invariant spinors (both in Δ_{2n}^+ for n even, one in each $\Delta_{2n}^\pm$ for n odd).*

All other spinors in $\Delta_{2n}^\pm$ have geometric stabilizer groups that do not act irreducibly on the tangent representation $\mathbb{R}^{2n}$. They can be described explicitly in a similar way; By de Rham's splitting theorem, they do not appear in the Riemannian setting.

To finish, we observe that the almost complex structure J (and hence Ω) can be recovered from the invariant spinor $\psi^+ \in \Delta_{2n}^+$ by $J(X)\psi^+ := iX \cdot \psi^+$ $(X \in TM)$, a formula well known from the investigation of Killing spinors on 6-dimensional nearly Kähler manifolds (see [Gru90] and [BFGK91, Section 5.2]).

Remark 3.1. The discussion of geometric stabilizers would not be complete without the explicit realization of these subalgebras inside $\mathfrak{so}(n)$ or $\mathfrak{spin}(n)$. We illustrate this by describing $\mathfrak{u}(n)$ inside $\mathfrak{so}(2n)$. Writing elements $\omega \in \mathfrak{so}(2n)$ as 2-forms with respect to some orthonormal and J-adapted basis, $\omega = \sum \omega_{ij} e_i \wedge e_j$ for $1 \leq i < j \leq 2n$, the defining equations for $\mathfrak{u}(n)$ inside $\mathfrak{so}(2n)$ translate into the conditions

$$\omega_{2i-1,2j-1} = \omega_{2i,2j} \quad \text{and} \quad \omega_{2i-1,2j} = -\omega_{2i,2j-1} \quad \text{for} \quad 1 \leq i < j \leq n\,.$$

The additional equation picking out $\mathfrak{su}(n) \subset \mathfrak{u}(n)$ is

$$\omega_{12} + \omega_{34} + \ldots + \omega_{2n-1,2n} = 0\,.$$

Of course, the equations get more involved for complicated embeddings of higher codimension (see for example [AF04a] for the 36-dimensional $\mathfrak{spin}(9)$ inside the 120-dimensional $\mathfrak{so}(16)$), but they can easily be mastered with the help of standard linear algebra computer packages.

Remark 3.2. The group $\mathrm{Sp}(n) \subset \mathrm{SO}(4n)$ can be deduced from the previous discussion: $\mathrm{Sp}(n)$ with quaternionic entries $a+bj$ is embedded into $\mathrm{SU}(2n)$ by (2×2)-blocks $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$, and $\mathrm{SU}(2n)$ sits in $\mathrm{SO}(4n)$ as before. We shall not treat $\mathrm{Sp}(n)$- and quaternionic geometries in this expository article (but see [Sw89], [Sw91], [AMP98], [GP00], [Ale03], [PS03], [AC05], [MCS04] for a first acquaintance).

3.2. $\mathrm{U}(n)$ **and** $\mathrm{SU}(n)$ **in dimension** $2n+1$**.** These G-structures arise from contact structures and are remarkable inasmuch they manifest a genuinely non-integrable behaviour—they do not occur in Berger's list because the action of $\mathrm{U}(n)$ on $\mathbb{R}^{2n+1}$ is not irreducible, hence any manifold with this action as Riemannian holonomy representation splits by de Rham's theorem. Given an almost contact metric manifold $(M^{2n+1}, g, \xi, \eta, \varphi)$, we may construct an adapted local orthonormal frame by choosing any $e_1 \in \xi^\perp$ and setting $e_2 = \varphi(e_1)$ (as well as fixing $e_{2n+1} = \xi$ once and for all); now choose any e_3 perpendicular to e_1, e_2, e_{2n+1} and set again $e_4 = \varphi(e_3)$ etc. With respect to such a basis, φ is given by

$$\varphi = \mathrm{diag}(j, j, \dots, j, 0) \quad \text{with} \quad j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and we conclude that the structural group of M^{2n+1} is reducible to $\mathrm{U}(n) \times \{1\}$. If we denote the fundamental form by F (see Example 2.5), then

$$\mathrm{U}(n) \times \{1\} = \{A \in \mathrm{SO}(2n+1) \mid A^* F = F\}\,.$$

The $\mathrm{U}(n) \times \{1\}$-action on $\mathbb{R}^{2n+1}$ inherits invariants from the $\mathrm{U}(n)$-action on $\mathbb{R}^{2n}$ in a canonical way; one then just needs to check that no new one appears. Hence, we can conclude:

Lemma 3.3. *Under the action of* $\mathrm{U}(n)$*,* $\Lambda^{2k}(\mathbb{R}^{2n+1})$ *contains the trivial representation once; it is generated by* $F, F^2, \dots, F^n$*.*

The action of $\mathrm{U}(n) \subset \mathrm{O}(2n+1)$ cannot, in general, be lifted to an action of $\mathrm{U}(n)$ inside $\mathrm{Spin}(2n+1)$. As in the almost Hermitian case, let's thus study the $\mathfrak{u}(n)$ action on Δ_{2n+1}. The irreducible $\mathrm{Spin}(2n+1)$-module Δ_{2n+1} splits into $\Delta_{2n}^+ \oplus \Delta_{2n}^-$ under the restricted action of $\mathrm{Spin}(2n)$, and it decomposes accordingly into

$$\begin{aligned} &\Delta_{2n+1} = S_0 \oplus \dots S_n\,, \quad S_r = \{\psi \in \Delta_{2n+1} \ : \ F\psi = i(n-2r)\psi\}\,, \\ &\dim S_r = \binom{n}{r}, \quad 0 \le r \le n\,. \end{aligned}$$

Hence, Δ_{2n+1} can be identified with $\Delta_{2n}^+ \oplus \Delta_{2n}^-$, yielding finally the following result:

Lemma 3.4. *The spin representation* Δ_{2n+1} *contains no* $\mathfrak{u}(n)$*-invariant spinor. If one restricts further to* $\mathfrak{su}(n)$*, there are precisely two invariant spinors.*

3.3. G_2 **in dimension 7.** While invariant 2-forms exist in all even dimensions and lead to the rich variety of almost Hermitian structures, the geometry of 3-forms played a rather exotic role in classical Riemannian geometry until the nineties, as it occurs only in apparently random dimensions, most notably dimension seven. That G_2 is the relevant simple Lie group is a classical, although unfortunately not so well-known result from invariant theory. A mere dimension count shows already this effect (see Table 3): the stabilizer of a generic 3-form ω^3

$$G^n_{\omega^3} := \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \omega^3 = A^* \omega^3\}$$

n	$\dim \mathrm{GL}(n,\mathbb{R}) - \dim \Lambda^3\mathbb{R}^n$	$\dim \mathrm{SO}(n)$
3	$9-1=8$	3
4	$16-4=12$	6
5	$25-10=15$	10
6	$36-20=16$	15
7	$49-35=14$	21
8	$64-56=8$	28

TABLE 3. Dimension count for possible geometries defined by 3-forms.

cannot be contained in the orthogonal group for $n \leq 6$, it must lie in some group between $\mathrm{SO}(n)$ and $\mathrm{SL}(n,\mathbb{R})$ (for $n=3$, we even have $G^3_{\omega^3} = \mathrm{SL}(3,\mathbb{R})$). The case $n=7$ is the first dimension where $G^n_{\omega^3}$ can sit in $\mathrm{SO}(n)$. That this is indeed the case was shown as early as 1907 in the doctoral dissertation of Walter Reichel in Greifswald, supervised by F. Engel ([Reich07]). More precisely, he computed a system of invariants for a 3-form in seven variables and showed that there are exactly two open $\mathrm{GL}(7,\mathbb{R})$-orbits of 3-forms. The stabilizers of any representatives ω^3 and $\tilde{\omega}^3$ of these orbits are 14-dimensional simple Lie groups of rank two, one compact and the other non-compact:

$$G^7_{\omega^3} \cong G_2 \ \subset \mathrm{SO}(7), \quad G^7_{\tilde{\omega}^3} \cong \ G_2^* \subset \ \mathrm{SO}(3,4)\,.$$

Reichel also showed the corresponding embeddings of Lie algebras by explicitly writing down seven equations for the coefficients of $\mathfrak{so}(7)$ resp. $\mathfrak{so}(3,4)$ (see Remark 3.4). As in the case of almost Hermitian geometry, every author has his or her favourite normal 3-form with isotropy group G_2, for example,

$$\omega^3 := e_{127} + e_{347} - e_{567} + e_{135} - e_{245} + e_{146} + e_{236}\,.$$

An element of the second orbit with stabilizer the split form G_2^* of G_2 may be obtained by reversing any of the signs in ω^3.

Lemma 3.5. *Under G_2, one has the decomposition $\Lambda^3(\mathbb{R}^7) \cong \mathbb{R} \oplus \mathbb{R}^7 \oplus S_0(\mathbb{R}^7)$, where $\mathbb{R}^7$ denotes the 7-dimensional standard representation given by the embedding $G_2 \subset \mathrm{SO}(7)$ and $S_0(\mathbb{R}^7)$ denotes the traceless symmetric endomorphisms of $\mathbb{R}^7$ (of dimension 27).*

Now let's consider the spinorial picture, as G_2 can indeed be lifted to a subgroup of $\mathrm{Spin}(7)$. From a purely representation theoretic point of view, this case is trivial: $\dim \Delta_7 = 8$ and the only irreducible representations of G_2 of dimension ≤ 8 are the trivial and the 7-dimensional representations. Hence $8 = 1 + 7$ yields:

Lemma 3.6. *Under the restricted action of G_2, the 7-dimensional spin representation Δ_7 decomposes as $\Delta_7 \cong \mathbb{R} \oplus \mathbb{R}^7$.*

This Lemma has an important consequence: the 'spinorial' characterization of G_2-manifolds.

Corollary 3.1. *Let (M^7, g) be a Riemannian manifold, ∇ a metric connection on its spin bundle. Then there exists a ∇-parallel spinor if and only if* $\mathrm{Hol}(\nabla) \subset G_2$.

One direction follows from the fact that G_2 is the stabilizer of an algebraic spinor, the converse from Lemma 3.6.

In fact, the invariant 3-form and the invariant algebraic spinor ψ are equivalent data. They are related (modulo an irrelevant constant) by

$$\omega^3(X, Y, Z) = \langle X \cdot Y \cdot Z \cdot \psi, \psi \rangle \, . \tag{8}$$

We now want to ask which subgroups $G \subset G_2$ admit other invariant algebraic spinors. Such a subgroup has to appear on Berger's list and its induced action on $\mathbb{R}^7$ (viewed as a subspace of Δ_7) has to contain one or more copies of the trivial representation. Thus, the only possibilities are $\mathrm{SU}(3)$ with $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ (standard $\mathrm{SU}(3)$-action on $\mathbb{C}^3$) and $\mathrm{SU}(2)$ with $\mathbb{R}^7 \cong 3 \cdot \mathbb{R} \oplus \mathbb{C}^2$ (standard $\mathrm{SU}(2)$-action on $\mathbb{C}^2$). Both indeed occur, with a total of 2 resp. 4 invariant spinors.

Remark 3.3. A modern account of Reichel's results can be found in the article [We81] by R. Westwick; it is interesting (although it seems not to have had any further influence) that J. A. Schouten also rediscovered these results in 1931 [Sch31]. A classification of 3-forms is still possible in dimensions 8 ([Gu35], [Gu35], [Djo83]) and 9 ([VE88]), although the latter one is already of inexorable complexity. Based on these results, J. Bureš and J. Vanžura started recently the investigation of so-called *multisymplectic structures* ([Van01], [BV03], [Bu04]).

Remark 3.4. $\mathfrak{g}_2$ inside $\mathfrak{spin}(7)$ is a good example for illustrating how to obtain the defining equations of stabilizer subalgebras with the aid of the computer (see Remark 3.1); one has just to be aware that they depend not only on the orthonormal basis but also on a choice of spin representation. To this aim, fix a realization of the spin representation Δ_n and a representative ψ of the orbit of spinors whose stabilizer is the group G we are interested in. As usual, identify the Lie algebra $\mathfrak{spin}(n)$ with the elements of the form $\omega = \sum_{i<j} \omega_{ij} e_i \cdot e_j$ inside the Clifford algebra $\mathrm{Cl}(n)$. Replacing e_i, e_j by the chosen representative matrices, $\omega \cdot \psi = 0$ is equivalent to a set of equations for the coefficients ω_{ij}; see for example [FKMS97, p. 261] for an explicit realization of $\mathfrak{g}_2 \subset \mathfrak{spin}(7)$.

3.4. $\mathrm{Spin}(7)$ **in dimension** 8**.** As we just learned from G_2 geometry, $\mathrm{Spin}(7)$ has an irreducible 8-dimensional representation isomorphic to Δ_7, hence it can be viewed as a subgroup of $\mathrm{SO}(8)$, and it does lift to $\mathrm{Spin}(8)$. By restricting to $\mathrm{SO}(7)$, $\mathrm{Spin}(7)$ certainly also has a 7-dimensional representation. What is so special in this dimension is that $\mathrm{Spin}(7)$ has *two* conjugacy classes in $\mathrm{SO}(8)$ that are interchanged by means of the triality automorphism; hence the decomposition of the spin representation depends on the (arbitrary) choice of one of these classes.

Lemma 3.7. *Under the restricted action of* $\mathrm{Spin}(7)$*, the* 8*-dimensional spin representations decompose as* $\Delta_8^+ \cong \mathbb{R}^8 \cong \Delta_7$ *and* $\Delta_8^- \cong \mathbb{R} \oplus \mathbb{R}^7$ *for one choice of* $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$*; the other choice swaps* Δ_8^+ *and* Δ_8^-.

In particular, there is exactly one invariant spinor in Δ_8^+. Again, it corresponds one-to-one to an invariant form, of degree 4 in this case:

$$\beta^4(X,Y,Z,V) = \langle X\cdot Y\cdot Z\cdot V\cdot\psi,\psi\rangle\,.$$

Yet, Spin(7)-geometry in dimension eight is not just an enhanced version of G_2-geometry in dimension seven. Because $\dim \mathrm{GL}(8,\mathbb{R}) = 64 < 70 = \dim\Lambda^4(\mathbb{R}^8)$, there are no dense open orbits under the action of $\mathrm{GL}(8,\mathbb{R})$. Thus, there is no result in invariant theory similar to that of Reichel for G_2 in the background.

Let's fix the first choice for embedding Spin(7) in SO(8) made in Lemma 3.7. A second invariant spinor can either be in Δ_8^+ or in Δ_8^-. If it is in Δ_8^+, we are asking for a subgroup $G\subset \mathrm{Spin}(7)$ whose action on $\mathbb{R}^7$ contains the trivial representation once — obviously, G_2 is such a group. Under G_2, Δ_8^+ and Δ_8^- are isomorphic,

$$\Delta_8^+\big|_{G_2}\cong\Delta_8^-\big|_{G_2}\cong\mathbb{R}\oplus\mathbb{R}^7\,.$$

Thus, $\mathrm{SU}(3)\subset G_2\subset\mathrm{Spin}(7)$ and $\mathrm{SU}(2)\subset G_2\subset\mathrm{Spin}(7)$ are two further admissible groups with $2+2$ and $4+4$ invariant spinors. On the other hand, if we impose a second invariant spinor to live in Δ_8^-, we need a subgroup $G\subset\mathrm{Spin}(7)$ that has partially trivial action on $\mathbb{R}^7$, but not on $\Delta_8^+\cong\mathbb{R}^8$. A straightforward candidate is $G=\mathrm{Spin}(6)$ with its standard embedding and $\mathbb{R}^7=\mathbb{R}^6\oplus\mathbb{R}$; the classical isomorphism $\mathrm{Spin}(6)=\mathrm{SU}(4)$ shows that G acts irreducibly on $\Delta_8^+\cong\mathbb{C}^4$. The group SU(4) in turn has subgroups $\mathrm{Sp}(2)=\mathrm{Spin}(5)$ and $\mathrm{SU}(2)\times\mathrm{SU}(2)=\mathrm{Spin}(4)$ that still act irreducibly on Δ_8^+, and act on Δ_8^- by

$$\Delta_8^-\big|_{\mathrm{Sp}(2)}\cong 3\cdot\mathbb{R}\oplus\mathbb{R}^5\,,\quad \Delta_8^-\big|_{\mathrm{SU}(2)\times\mathrm{SU}(2)}\cong 4\cdot\mathbb{R}\oplus\mathbb{R}^4\,.$$

The results are summarized in Table 4. The resemblance between Tables 4 and 2 in Section 1.4 is no coincidence. A convenient way to describe Spin(9, 1) is to start with Spin(10) generated by elements $e_1,\ldots,e_{10}$ and acting irreducibly on Δ_{10}^+. The vector spaces Δ_{10}^+ and $\Delta_{9,1}^+$ can be identified, and Spin(9, 1) can be generated by $e_i^*:=e_i$ for $i=1,\ldots,9$ and $e_{10}^*:=i\,e_{10}$. Elements $\omega\in\mathfrak{spin}(9,1)$ can thus be written

$$\omega=\sum_{1\le i<j\le 10}\omega_{ij}\,\omega_{ij}e_i^*\wedge e_j^* \;=\; \sum_{i<j\le 9}\omega_{ij}\,e_i\wedge e_j + i\sum_{k\le 9}\omega_{k,10}\,e_k\wedge e_{10}$$

and we conclude that $\mathfrak{spin}(9,1)$ can be identified with $\mathfrak{spin}(9)\ltimes\mathbb{R}^9$. A spinor $\psi\in\Delta_{9,1}^+$ is stabilized by an element $\omega\in\mathfrak{spin}(9,1)$ if and only if

$$0=\sum_{1\le i<j\le 10}\omega_{ij}\,e_i^*\cdot e_j^*\cdot\psi=\sum_{i<j\le 9}\omega_{ij}\,e_i\cdot e_j\cdot\psi + i\sum_{k\le 9}\omega_{k,10}\,e_k\cdot e_{10}\cdot\psi\,.$$

In this last expression, both real and imaginary part have to vanish simultaneously, leading to 16 equations. A careful look reveals that they define $\mathfrak{spin}(7)\ltimes\mathbb{R}^8$, and the statements from Table 4 imply those from Table 2 since $\Delta_{9,1}^+\cong\Delta_8^+\oplus\Delta_8^-$.

Remark 3.5 (Weak PSU(3)-structures)**.** Recently, Hitchin observed that 3-forms can be of interest in 8-dimensional geometry as well ([Hit01]). The canonical 3-form on the Lie algebra $\mathfrak{su}(3)$ spans an open orbit under $\mathrm{GL}(8,\mathbb{R})$, and the corresponding 3-form on SU(3) is parallel with respect to the Levi-Civita connection

group	inv. spinors in Δ_8^+	inv. spinors in Δ_8^-
Spin(7)	0	1
$\mathrm{SU}(4) \cong \mathrm{Spin}(6)$	0	2
$\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$	0	3
$\mathrm{SU}(2) \times \mathrm{SU}(2) \cong \mathrm{Spin}(4)$	0	4
G_2	1	1
SU(3)	2	2
SU(2)	4	4
$\{e\}$	8	8

Table 4. Possible stabilizers of invariant spinors in dimension 8.

of the biinvariant metric. The Riemannian holonomy reduces to $\mathrm{SU}(3)/\mathbb{Z}_2 =: \mathrm{PSU}(3)$. More generally, manifolds modelled on this group lead to the investigation of closed and coclosed 3-forms that are not parallel, see also [Wi06].

Sorting out the technicalities that we purposely avoided, one obtains Wang's classification of Riemannian parallel spinors. By de Rham's theorem, only irreducible holonomy representations occur for the Levi-Civita connection. From Proposition 2.2, we already know that these manifolds are Ricci-flat.

Theorem 3.1 (Wang's Theorem, [McKW89])**.** *Let (M^n, g) be a complete, simply connected, irreducible Riemannian manifold of dimension n. Let N denote the dimension of the space of parallel spinors with respect to the Levi-Civita connection. If (M^n, g) is non-flat and $N > 0$, then one of the following holds:*

(1) *$n = 2m$ $(m \geq 2)$, the holonomy representation is the vector representation of* $\mathrm{SU}(m)$ *on $\mathbb{C}^m$, and $N = 2$ ("Calabi-Yau case").*
(2) *$n = 4m$ $(m \geq 2)$, the holonomy representation is the vector representation of* $\mathrm{Sp}(m)$ *on $\mathbb{C}^{2m}$, and $N = m + 1$ ("hyper-Kähler case").*
(3) *$n = 7$, the holonomy representation is the unique 7-dimensional representation of G_2, and $N = 1$ ("parallel G_2- or Joyce case").*
(4) *$n = 8$, the holonomy representation is the spin representation of* Spin(7), *and $N = 1$ ("parallel* Spin(7)*- or Joyce case").*

4. A unified approach to non-integrable geometries

4.1. **Motivation.** For G-structures defined by some tensor $\mathcal{T}$, it has been for a long time customary to classify the possible types of structures by the isotypic decomposition under G of the covariant derivative $\nabla^g \mathcal{T}$. The integrable case is described by $\nabla^g \mathcal{T} = 0$, all other classes of non-integrable G-structures correspond to combinations of non-vanishing contributions in the isotypic decomposition and are described by some differential equation in $\mathcal{T}$. This was carried out in detail for example for almost Hermitian manifolds (Gray/Hervella [GH80]), for G_2-structures in dimension 7 (Fernández/Gray [FG82]), for Spin(7)-structures in dimension 8 (Fernández [Fer86]) and for almost contact metric structures (Chinea/Gonzales [ChG90]).

In this section, we shall present a simpler and unified approach to non-integrable geometries. The theory of principal fibre bundles suggests that the difference Γ between the Levi-Civita connection and the canonical G-connection induced on the G-structure is a good measure for how much the given G-structure fails to be integrable. By now, Γ is widely known as the *intrinsic torsion* of the G-structure (see Section 1.5 for references). Although this is a "folklore" approach, it is still not as popular as it could be. Our presentation will follow the main lines of [Fri03b]. We will see that it easily reproduces the classical results cited above with much less computational work whilst having the advantage of being applicable to geometries not defined by tensors. Furthermore, it allows a uniform and clean description of those classes of geometries admitting G-connections with totally skew-symmetric torsion, and led to the discovery of new interesting geometries.

4.2. **G-structures on Riemannian manifolds.** Let $G \subset \mathrm{SO}(n)$ be a closed subgroup of the orthogonal group and decompose the Lie algebra $\mathfrak{so}(n)$ into the Lie algebra $\mathfrak{g}$ of G and its orthogonal complement $\mathfrak{m}$, i.e. $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$. Denote by $\mathrm{pr}_{\mathfrak{g}}$ and $\mathrm{pr}_{\mathfrak{m}}$ the projections onto $\mathfrak{g}$ and $\mathfrak{m}$, respectively. Consider an oriented Riemannian manifold (M^n, g) and denote its frame bundle by $\mathcal{F}(M^n)$; it is a principal $\mathrm{SO}(n)$-bundle over M^n. By definition, a G-structure on M^n is a reduction $\mathcal{R} \subset \mathcal{F}(M^n)$ of the frame bundle to the subgroup G. The Levi-Civita connection is a 1-form Z on $\mathcal{F}(M^n)$ with values in the Lie algebra $\mathfrak{so}(n)$. We restrict the Levi-Civita connection to $\mathcal{R}$ and decompose it with respect to the splitting $\mathfrak{g} \oplus \mathfrak{m}$:

$$Z\big|_{T(\mathcal{R})} := Z^* \oplus \Gamma \,.$$

Then, Z^* is a connection in the principal G-bundle $\mathcal{R}$ and Γ is a tensorial 1-form of type Ad, i.e. a 1-form on M^n with values in the associated bundle $\mathcal{R} \times_G \mathfrak{m}$. By now, it has become standard to call Γ the *intrinsic torsion* of the G-structure (see Section 1.5 for references). The G-structure $\mathcal{R}$ on (M^n, g) is called *integrable* if Γ vanishes, for this means that it is preserved by the Levi-Civita connection and that $\mathrm{Hol}(\nabla^g)$ is a subgroup of G. All G-structures with $\Gamma \neq 0$ are called *non-integrable*; the basic classes of non-integrable G-structures are defined — via the decomposition of Γ — as the irreducible G-components of the representation $\mathbb{R}^n \otimes \mathfrak{m}$. For an orthonormal frame $e_1, \dots, e_n$ adapted to the reduction $\mathcal{R}$, the connection forms $\omega_{ij} := g(\nabla^g e_i, e_j)$ of the Levi-Civita connection define a 1-form $\Omega := (\omega_{ij})$ with values in the Lie algebra $\mathfrak{so}(n)$ of all skew-symmetric matrices. The form Γ can then be computed as the $\mathfrak{m}$-projection of Ω,

$$\Gamma = \mathrm{pr}_{\mathfrak{m}}(\Omega) = \mathrm{pr}_{\mathfrak{m}}(\omega_{ij}) \,.$$

Interesting is the case in which G happens to be the isotropy group of some tensor $\mathcal{T}$. Suppose that there is a faithful representation $\varrho : \mathrm{SO}(n) \to \mathrm{SO}(V)$ and a tensor

$\mathcal{T} \in V$ such that

$$G = \big\{ g \in \mathrm{SO}(n) : \varrho(g)\mathcal{T} = \mathcal{T} \big\} .$$

The Riemannian covariant derivative of $\mathcal{T}$ is then given by the formula

$$\nabla^g \mathcal{T} = \varrho_*(\Gamma)(\mathcal{T}) ,$$

where $\varrho_* : \mathfrak{so}(n) \to \mathfrak{so}(V)$ is the differential of the representation. As a tensor, $\nabla^g \mathcal{T}$ is an element of $\mathbb{R}^n \otimes V$. The algebraic G-types of $\nabla^g \mathcal{T}$ define the algebraic G-types of Γ and vice versa. Indeed, we have

Proposition 4.1 ([Fri03b, Prop. 2.1.]). *The G-map*

$$\mathbb{R}^n \otimes \mathfrak{m} \longrightarrow \mathbb{R}^n \otimes \mathrm{End}(V) \longrightarrow \mathbb{R}^n \otimes V$$

given by $\Gamma \to \rho_*(\Gamma)(\mathcal{T})$ is injective.

An easy argument in representation theory shows that for $G \neq \mathrm{SO}(n)$, the G-representation $\mathbb{R}^n$ does always appear as summand in the G-decomposition of $\mathbb{R}^n \otimes \mathfrak{m}$. Geometrically, this module accounts precisely for conformal transformations of G-structures. Let $(M^n, g, \mathcal{R})$ be a Riemannian manifold with a fixed geometric structure and denote by $\hat{g} := e^{2f} \cdot g$ a conformal transformation of the metric. There is a natural identification of the frame bundles

$$\mathcal{F}(M^n, g) \cong \mathcal{F}(\hat{M}^n, \hat{g})$$

and a corresponding G-structure $\hat{\mathcal{R}}$. At the infinitesimal level, the conformal change is defined by the 1-form df, corresponding to an $\mathbb{R}^n$-part in Γ.

We shall now answer the question under which conditions a given G-structure admits a metric connection ∇ with skew-symmetric torsion preserving the structure. For this, consider for any orthonormal basis e_i of $\mathfrak{m}$ the G-equivariant map

$$\Theta : \Lambda^3(\mathbb{R}^n) \longrightarrow \mathbb{R}^n \otimes \mathfrak{m} , \quad \Theta(T) := \sum_i (e_i \lrcorner T) \otimes e_i .$$

Theorem 4.1 ([FI02, Prop. 4.1]). *A G-structure $\mathcal{R} \subset \mathcal{F}(M^n)$ of a Riemannian manifold admits a connection ∇ with skew-symmetric torsion if and only if the 1-form Γ belongs to the image of Θ,*

$$2\,\Gamma = -\Theta(T) \quad \textit{for some} \quad T \in \Lambda^3(\mathbb{R}^n) .$$

In this case the 3-form T is the torsion form of the connection.

Definition 4.1. A metric G-connection ∇ with torsion T as in Theorem 4.1 will be called a *characteristic connection* and denoted by ∇^c, $T =: T^c$ is called the *characteristic torsion*. By construction, the holonomy $\mathrm{Hol}(\nabla^c)$ is a subgroup of G.

Thus, not every G-structure admits a characteristic connection. If that is the case, T^c is unique for all geometries we have investigated so far, and it can easily be expressed in terms of the geometric data (almost complex structure etc.). Henceforth, we shall just speak of *the* characteristic connection. Due to its properties, it is an excellent substitute for the Levi-Civita connection, which in these situations is not adapted to the underlying geometric structure.

Remark 4.1. The canonical connection ∇^c of a naturally reductive homogeneous space is an example of a characteristic connection that satisfies in addition $\nabla^c T^c = \nabla^c \mathcal{R}^c = 0$; in this sense, geometric structures admitting a characteristic connection such that $\nabla^c T^c = 0$ constitute a natural generalization of naturally reductive homogeneous spaces. As a consequence of the General Holonomy Principle (Corollary 2.2), $\nabla^c T^c = 0$ implies that the holonomy group $\mathrm{Hol}(\nabla^c)$ lies in the stabilizer G_{T^c} of T^c.

With this technique, we shall now describe special classes of non-integrable geometries, some new and others previously encountered. We order them by increasing dimension.

4.3. **Almost contact metric structures.** At this stage, almost contact metric structures challenge any expository paper because of the large number of classes. Qualitatively, the situation is as follows. The first classifications of these structures proceeded in analogy to the Gray-Hervella set-up for almost Hermitian manifolds (see Section 4.5) by examining the space of tensors with the same symmetry properties as the covariant derivative of the fundamental form F (see Section 2.5) and decomposing it under the action of the structure group $G = \mathrm{U}(n) \times \{1\}$ using invariant theory. Because the G-action on $\mathbb{R}^{2n+1}$ is already not irreducible, this space decomposes into four G-irreducibles for $n = 1$, into 10 summands for $n = 2$ and into 12 for $n \geq 3$, leading eventually to 2^4, 2^{10} and 2^{12} possible classes of almost contact metric structures ([AG86], [ChG90], [ChM92]). Obviously, most of these classes do not carry names and are not studied, and the result being what it is, the investigation of such structures is burdened by technical details and assumptions. From the inner logic of non-integrable geometries, it makes not so much sense to base their investigation on the covariant derivative $\nabla^g F$ of F or some other fundamental tensor, as the Levi-Civita connection does not preserve the geometric structure. This accounts for the technical complications that one faces when following this approach.

For this section, we decided to restrict our attention to dimension five, this being the most relevant for the investigation of non-integrable geometries (in dimension seven, it is reasonable to study contact structures simultaneously with G_2-structures). Besides, this case will illustrate the power of the intrinsic torsion concept outlined above. We shall use throughout that $\mathbb{R}^5 = \mathbb{R}^4 \oplus \mathbb{R}$ with standard $\mathrm{U}(2)$-action on the first term and trivial action on the second term. Let us look at the decompositions of the orthogonal Lie algebras in dimension 4 and 5. First, we have

$$\mathfrak{so}(4) = \Lambda^2(\mathbb{R}^4) = \mathfrak{u}(2) \oplus \mathfrak{n}^2 .$$

Here, $\mathfrak{n}^2$ is $\mathrm{U}(2)$-irreducible, while $\mathfrak{u}(2)$ splits further into $\mathfrak{su}(2)$ and the span of Ω. Combining this remark with the characterization of these subspaces via the complex structure J defining $\mathfrak{u}(2)$, we obtain

$$\mathfrak{u}(2) = \{\omega \in \Lambda^2(\mathbb{R}^4) \, : \, J\omega = \omega\} = \mathfrak{su}(2) \oplus \mathbb{R} \cdot \Omega \,, \quad \mathfrak{n}^2 = \{\omega \in \Lambda^2(\mathbb{R}^4) \, : \, J\omega = -\omega\} \,.$$

In particular, $\Lambda^2(\mathbb{R}^4)$ is the sum of three U(2)-representations of dimensions 1, 2 and 3. For $\mathfrak{so}(5)$, we deduce immediatly

$$\mathfrak{so}(5) = \Lambda^2(\mathbb{R}^4 \oplus \mathbb{R}) = \Lambda^2(\mathbb{R}^4) \oplus \mathbb{R}^4 = \mathfrak{u}(2) \oplus (\mathbb{R}^4 \oplus \mathfrak{n}^2) =: \mathfrak{u}(2) \oplus \mathfrak{m}^6\,.$$

Thus, the intrinsic torsion Γ of a 5-dimensional almost metric contact structure is an element of the representation space

$$\mathbb{R}^5 \otimes \mathfrak{m}^6 = (\mathbb{R}^4 \oplus \mathbb{R}) \otimes (\mathbb{R}^4 \oplus \mathfrak{n}^2) = \mathfrak{n}^2 \oplus \mathbb{R}^4 \oplus (\mathbb{R}^4 \otimes \mathfrak{n}^2) \oplus (\mathbb{R}^4 \otimes \mathbb{R}^4)\,.$$

The last term splits further into trace-free symmetric, trace and antisymmetric part, written for short as

$$\mathbb{R}^4 \otimes \mathbb{R}^4 = S^2_0(\mathbb{R}^4) \oplus \mathbb{R} \oplus \Lambda^2(\mathbb{R}^4)\,.$$

The 9-dimensional representation $S^2_0(\mathbb{R}^4)$ is again a sum of two irreducible ones of dimensions 3 and 6, but we do not need this here. To decompose the representation $\mathbb{R}^4 \otimes \mathfrak{n}^2$, we observe that the U(2)-equivariant map $\Theta : \Lambda^3(\mathbb{R}^4) \to \mathbb{R}^4 \otimes \mathfrak{n}^2$ (see Section 4.2) has 4-dimensional irreducible image isomorphic to $\Lambda^3\mathbb{R}^4$ (which is again of dimension 4); its complement is an inequivalent irreducible U(2)-representation of dimension 4 which we call V_4. Consequently,

$$\mathbb{R}^5 \otimes \mathfrak{m}^6 = \mathbb{R} \oplus \mathfrak{n}^2 \oplus \mathbb{R}^4 \oplus S^2_0(\mathbb{R}^4) \oplus \Lambda^2(\mathbb{R}^4) \oplus \Lambda^3(\mathbb{R}^4) \oplus V_4\,.$$

Taking into account the further splitting of $S^2_0(\mathbb{R}^4) \oplus \Lambda^2(\mathbb{R}^4)$, this is the sum of 10 irreducible U(2)-representations as claimed. On the other side,

$$\Lambda^3(\mathbb{R}^5) = \Lambda^3(\mathbb{R}^4 \oplus \mathbb{R}) = \Lambda^2(\mathbb{R}^4) \oplus \Lambda^3(\mathbb{R}^4)\,.$$

We found a unique copy of this 10-dimensional space in the 30-dimensional space $\mathbb{R}^5 \otimes \mathfrak{m}^6$. Thus, we conclude from Theorem 4.1:

Proposition 4.2. *A 5-dimensional almost metric contact structure $(M^5, g\xi, \eta, \varphi)$ admits a unique characteristic connection if and only if its intrinsic torsion is of class $\Lambda^2(\mathbb{R}^4) \oplus \Lambda^3(\mathbb{R}^4)$.*

In dimension 5, skew-symmetry of the Nijenhuis tensor N implies that it has to be zero, hence in the light of the more general Theorem 2.5, the almost metric contact manifolds of class $\Lambda^2(\mathbb{R}^4)\oplus\Lambda^3(\mathbb{R}^4)$ should coincide with the almost metric contact structures with $N = 0$ and ξ a Killing vector field. That this is indeed the case follows from the classifications cited above. This class includes for example all *quasi-Sasakian manifolds* ($N = 0$ and $dF = 0$), see [KR02].

Example 4.1. Consider $\mathbb{R}^5$ with 1-forms

$$2e_1 = dx_1\,,\;\; 2e_2 = dy_1\,,\;\; 2e_3 = dx_2\,,\;\; 2e_4 = dy_2\,,\;\; 4e_5 = 4\eta = dz - y_1dx_1 - y_2dx_2\,,$$

metric $g = \sum_i e_i \otimes e_i$, and almost complex structure φ defined in $\langle\xi\rangle^\perp$ by

$$\varphi(e_1) = e_2\,,\quad \varphi(e_2) = -e_1\,,\quad \varphi(e_3) = e_4\,,\quad \varphi(e_4) = -e_3\,,\quad \varphi(e_5) = 0\,.$$

Then $(\mathbb{R}^5, g, \eta, \varphi)$ is a Sasakian manifold, and the torsion of its characteristic connection is of type $\Lambda^2(\mathbb{R}^4)$ ([Fin94, Example 3.D]) and explicitly given by

$$T^c = \eta \wedge d\eta = 2(e_1 \wedge e_2 + e_3 \wedge e_4) \wedge e_5\,.$$

This example is in fact a left-invariant metric on a 5-dimensional Heisenberg group with $\mathrm{scal}^g = -4$ and $\mathrm{scal}^{\nabla^c} = \mathrm{scal}^g - 3\|T\|^2/2 = -16$ (see Theorem A.1). In a left-invariant frame, spinors are simply functions $\psi : \mathbb{R}^5 \to \Delta_5$ with values in the 5-dimensional spin representation. In [FI03a], it is shown that there exist two ∇^c-parallel spinors ψ_i with the additional property $F \cdot \psi_i = 0$ $(i = 1, 2)$. This implies $T \cdot \psi_i = 0$, an equation of interest in superstring theory (see Section 5.5). It turns out that these spinors are constant, hence the same result holds for all compact quotients $\mathbb{R}^5/\Gamma$ (Γ a discrete subgroup).

We recommend the articles [Fin94] and [Fin95] for a detailed investigation of the representation theory of almost metric contact structures (very much in the style of the book [Sal01]) — in particular, the decomposition of the space of possible torsion tensors $\mathcal{T}$ of metric connections (see Proposition 2.1) under $\mathrm{U}(n)$ is being related to the possible classes for the intrinsic torsion.

4.4. $\mathrm{SO}(3)$**-structures in dimension** 5**.** These structures were discovered by Th. Friedrich in a systematic investigation of possible G-structures for interesting non-integrable geometries (see [Fri03b]); until that moment, it was generally believed that contact structures were the only remarkable G-structures in dimension 5.

The group $\mathrm{SO}(3)$ has a unique, real, irreducible representation in dimension 5. We consider the corresponding non-standard embedding $\mathrm{SO}(3) \subset \mathrm{SO}(5)$ as well as the decomposition

$$\mathfrak{so}(5) = \mathfrak{so}(3) \oplus \mathfrak{m}^7 .$$

It is well known that the $\mathrm{SO}(3)$-representation $\mathfrak{m}^7$ is the unique, real, irreducible representation of $\mathrm{SO}(3)$ in dimension 7. We decompose the tensor product into irreducible components

$$\mathbb{R}^5 \otimes \mathfrak{m}^7 = \mathbb{R}^3 \oplus \mathbb{R}^5 \oplus \mathfrak{m}^7 \oplus \mathrm{E}^9 \oplus \mathrm{E}^{11} .$$

There are five basic types of $\mathrm{SO}(3)$-structures on 5-dimensional Riemannian manifolds. The symmetric spaces $\mathrm{SU}(3)/\mathrm{SO}(3)$ and $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ are examples of 5-dimensional Riemannian manifolds with an integrable $\mathrm{SO}(3)$-structure ($\Gamma = 0$). On the other hand, 3-forms on $\mathbb{R}^5$ decompose into

$$\Lambda^3(\mathbb{R}^5) = \mathbb{R}^3 \oplus \mathfrak{m}^7 .$$

In particular, a conformal change of an $\mathrm{SO}(3)$-structure does *not* preserve the property that the structure admits a connection with totally skew-symmetric torsion.

M. Bobienski and P. Nurowski investigated $\mathrm{SO}(3)$-structures in their articles [BN05] and [Bob06]. In particular, they found a ternary symmetric form describing the reduction to $\mathrm{SO}(3)$ and constructed many examples of non-integrable $\mathrm{SO}(3)$-structures with non-vanishing intrinsic torsion. Recently, P. Nurowski suggested a link to Cartan's work on isoparametric surfaces in spheres, and predicted the existence of similar geometries in dimensions 8, 14 and 26; we refer the reader to [Nu06] for details. The case of $\mathrm{SO}(3)$-structures illustrates that new classes of non-integrable geometries are still to be discovered beyond the well-established ones, and that their study reveals deeper connections between areas which used to be far from each other.

4.5. **Almost Hermitian manifolds in dimension 6.** We begin with the Gray-Hervella classification of almost Hermitian manifolds and the consequences for the characteristic connection. Although most of these results hold in all even dimensions, we shall henceforth restrict our attention to the most interesting case, namely dimension 6.

Let us consider a 6-dimensional almost Hermitian manifold (M^6, g, J), corresponding to a U(3)-structure inside SO(6). We decompose the Lie algebra into $\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{m}$ and remark that the U(3)-representation in $\mathbb{R}^6$ is the real representation underlying $\Lambda^{1,0}$. Similarly, $\mathfrak{m}$ is the real representation underlying $\Lambda^{2,0}$. We decompose the complexification under the action of U(3):

$$\Big(\mathbb{R}^6 \otimes \mathfrak{m}\Big)^{\mathbb{C}} = \Big(\Lambda^{1,0} \otimes \Lambda^{2,0} \oplus \Lambda^{1,0} \otimes \Lambda^{0,2}\Big)^{\mathbb{C}}_{\mathbb{R}} .$$

The symbol $(\ldots)^{\mathbb{C}}_{\mathbb{R}}$ means that we understand the complex representation as a real representation and complexify it. Next we split the complex U(3)-representations

$$\Lambda^{1,0} \otimes \Lambda^{2,0} = \mathbb{C}^3 \otimes \Lambda^2(\mathbb{C}^3) = \Lambda^{3,0} \oplus \mathrm{E}^8 ,$$

$$\Lambda^{1,0} \otimes \Lambda^{0,2} = \mathbb{C}^3 \otimes \Lambda^2(\overline{\mathbb{C}}^3) = \mathbb{C}^3 \otimes \Lambda^2(\mathbb{C}^3)^* = (\mathbb{C}^3)^* \oplus \mathrm{E}^6 .$$

E^6 and E^8 are irreducible U(3)-representations of complex dimensions 6 and 8, respectively. Finally we obtain

$$\mathbb{R}^6 \otimes \mathfrak{m} = \Lambda^{3,0} \oplus \mathrm{E}^8 \oplus \mathrm{E}^6 \oplus (\mathbb{C}^3)^* =: \mathcal{W}_1^{(2)} \oplus \mathcal{W}_2^{(16)} \oplus \mathcal{W}_3^{(12)} \oplus \mathcal{W}_4^{(6)} .$$

Consequently, $\mathbb{R}^6 \otimes \mathfrak{m}$ splits into four irreducible representations of real dimensions $2, 16, 12$ and 6, that is, there are four basic classes and a total of 16 classes of U(3)-structures on 6-dimensional Riemannian manifolds, a result known as Gray/Hervella-classification ([GH80]). Recently, F. Martín Cabrera established the defining differential equations for these classes solely in terms of the intrinsic torsion (see [MC05]), as we shall state them for G_2-manifolds in the next section. In case we restrict the structure group to SU(3), the orthogonal complement $\mathfrak{su}(3)^{\perp}$ is now 7- instead of 6-dimensional, and we obtain

$$\mathbb{R}^6 \otimes \mathfrak{su}(3)^{\perp} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 ,$$

where $\mathcal{W}_5$ is isomorphic to $\mathcal{W}_4 \cong (\mathbb{C}^3)^*$. Furthermore, $\mathcal{W}_1$ and $\mathcal{W}_2$ are not irreducible anymore, but they split into $\mathcal{W}_1 = \mathcal{W}_1^+ \oplus \mathcal{W}_1^- = \mathbb{R} \oplus \mathbb{R}$ and $\mathcal{W}_2 = \mathcal{W}_2^+ \oplus \mathcal{W}_2^- = \mathfrak{su}(3) \oplus \mathfrak{su}(3)$ (see [CS02], [MC05]). Table 5 summarizes some remarkable classes of U(3)-structures in dimension 6. Most of these have by now well-established names, while there is still some confusion for others; these can be recognized by the parentheses indicating the different names to be found in the literature. In the last column, we collected characterizations of these classes (where several are listed, these are to be understood as equivalent characterizations, not as simultaneous requirements). Observe that we included in the last line a remarkable class of SU(3)-structures, the so called *half-flat* SU(3)-structures (Ψ^+ is the real part of the $(3,0)$-form defined by J, see Section 3.1). The name is chosen

name	class	characterization
nearly Kähler manifold	$\mathcal{W}_1$	a) $(\nabla^g_X J)(X) = 0$ b) N skew-sym. and $\tau^2(d\Omega) = -9d\Omega$ c) $\exists$ real Killing spinor
almost Kähler manifold	$\mathcal{W}_2$	$d\Omega = 0$
balanced (almost Hermitian) or (Hermitian) semi-Kähler m.	$\mathcal{W}_3$	$N = 0$ and $\delta\Omega = 0$
locally conformally Kähler m.	$\mathcal{W}_4$	$N = 0$ and $d\Omega = \Omega \wedge \theta$ (θ: Lee form)
quasi-Kähler manifold	$\mathcal{W}_1 \oplus \mathcal{W}_2$	$\nabla^g_X\Omega(Y,Z) + \nabla^g_{JX}\Omega(JY,Z) = 0$
Hermitian manifold	$\mathcal{W}_3 \oplus \mathcal{W}_4$	a) $N = 0$, b) $\tau^2(d\Omega) = -d\Omega$
(almost-)semi-Kähler or (almost) cosymplectic m.	$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$	a) $\delta\Omega = 0$, b) $\Omega \wedge d\Omega = 0$
KT- or $\mathcal{G}_1$-manifold	$\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	a) N is skew-symmetric b) $\exists$ char. connection ∇^c
half-flat SU(3)-manifold	$\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$	$\Omega \wedge d\Omega = 0$ and $d\Psi^+ = 0$

TABLE 5. Some types of U(3)- and SU(3)-structures in dimension six.

in order to suggest that half of all $\mathcal{W}$-components vanish for these structures. Relying on results of [Hit01], S. Chiossi and S. Salamon described in [CS02] explicit metrics with Riemannian holonomy G_2 on the product of any half-flat SU(3)-manifold with a suitable interval. A construction of half-flat SU(3)-manifolds as T^2-principal fibre bundles over Kählerian 4-manifolds goes back to Goldstein and Prokushkin [GP02] and was generalized by Li, Fu and Yau [LY05], [FY05]. We refer to Section 5.2 for examples of half-flat SU(3)-structures on nilmanifolds.

Theorem 4.2. *An almost Hermitian* 6*-manifold* (M^6, g, J) *admits a characteristic connection* ∇^c *if and only if it is of class* $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$*, i.e. if its Nijenhuis tensor* N *is skew-symmetric. Furthermore,* ∇^c *is unique and given by the expression*

$$g(\nabla^c_X Y, Z) := g(\nabla^g_X Y, Z) + \frac{1}{2}\left[N(X,Y,Z) + d\Omega(JX,JY,JZ)\right]$$

Proof. In order to apply Theorem 4.1, we need the decomposition of 3-forms into isotypic U(3)-representations,

$$\Lambda^3(\mathbb{R}^6) \;=\; \Lambda^{3,0} \oplus \mathrm{E}^6 \oplus (\mathbb{C}^3)^* \;=\; \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4.$$

Therefore, the image of Θ consists of the sum $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ and Θ is injective, i. e., there exists at most one characteristic torsion form. From the Gray-Hervella classification, we know that almost Hermitian manifolds without $\mathcal{W}_2$-part (so-called $\mathcal{G}_1$*-manifolds* or *KT-manifolds*, standing for 'Kähler with torsion', although not Kähler) are characterized by the property that their Nijenhuis tensor N is skew-symmetric. In Lemma 2.3, it was shown that the stated connection fulfills all requirements, hence by uniqueness it coincides with the characteristic connection. $\square$

$\Lambda^3(\mathbb{R}^6)$ admits another decomposition. The map

$$\tau : \Lambda^3(\mathbb{R}^6) \longrightarrow \Lambda^3(\mathbb{R}^6)\,, \quad \tau(T) := \sum_{i=1}^{6} (e_i \lrcorner\, \Omega) \wedge (e_i \lrcorner\, T)$$

is U(3)-equivariant. Its square τ^2 is diagonalizable with eigenspaces $\mathcal{W}_1$ (eigenvalue -9) and $\mathcal{W}_3 \oplus \mathcal{W}_4$ (eigenvalue -1). This explains the second characterization of these two classes in Table 5.

Remark 4.2 (Parallel torsion)**.** In Example 2.3, it had been observed that the characteristic torsion of nearly Kähler manifolds is always parallel (Kirichenko's Theorem). Another interesting class of almost Hermitian $\mathcal{G}_1$-manifolds with this property are the so-called *generalized Hopf structures*, that is, locally conformally Kähler manifolds (class $\mathcal{W}_4$, sometimes abbreviated lcK-manifolds) with parallel *Lee form* $\theta := \delta\Omega \circ J \neq 0$ (in fact, $\nabla^g \theta = 0$ and $\nabla^c T^c = 0$ are equivalent conditions for $\mathcal{W}_4$-manifolds). Besides the classical Hopf manifolds, they include for example total spaces of flat principal S^1-bundles over compact 5-dimensional Sasaki manifolds (see [Vai76], [Vai79] for details); generalized Hopf structures are never Einstein. We recommend the book by S. Dragomir and L. Ornea [DO98] as a general reference and the articles [Bel00], [FP05] for complex lcK-surfaces.

In his thesis, N. Schoemann investigates almost hermitian structures with parallel skew-symmetric torsion in dimension 6. A full classification of the possible algebraic types of the torsion form is worked out, and based on this a systematic description of the possible geometries is given. In addition numerous new examples are constructed (and, partially, classified) on naturally reductive spaces (including compact spaces with closed torsion form) and on nilmanifolds (see [AFS05] and [Sch06]).

Remark 4.3 (Almost Kähler manifolds)**.** The geometry of almost Kähler manifolds is strongly related to famous problems in differential geometry. W. Thurston was the first to construct an explicit compact symplectic manifold with $b_1 = 3$, hence that does not admit a Kähler structure [Thu76]. E. Abbena generalized this example and gave a natural associated metric which makes it into an almost Kähler non-Kähler manifold [Ab84]; many more examples of this type have been constructed since then.

In 1969, S. I. Goldberg conjectured that a compact almost Kähler-Einstein manifold is Kähler [Gol69]. In this generality, the conjecture is still open. K. Sekigawa proved it under the assumption of non-negative scalar curvature [Sek87], and it is known that the conjecture is false for non-compact manifolds: P. Nurowski and M. Przanowski gave the first example of a 4-dimensional Ricci-flat almost-Kähler non-Kähler manifold [NP97], J. Armstrong showed some non-existence results [Arm98], while V. Apostolov, T. Drăghici and A. Moroianu constructed non-compact counterexamples to the conjecture in dimensions ≥ 6 [ADM01]. Different partial results with various additional curvature assumptions are now available. The integrability conditions for almost Kähler manifolds were studied in full generality in [AAD02] and [Kir05].

Remark 4.4 (Nearly Kähler manifolds)**.** We close this section with some additional remarks on nearly Kähler manifolds. Kirichenko's Theorem ($\nabla^c T^c = 0$) implies that $\mathrm{Hol}(\nabla^c) \subset \mathrm{SU}(3)$, that the first Chern class of the tangent bundle $c_1(TM^6, J)$ vanishes, M^6 is spin and that the metric is Einstein. The only known examples are homogeneous metrics on $S^6, \mathbb{CP}^3, S^3 \times S^3$ and on the flag manifold $F(1,2) = \mathrm{U}(3)/\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$, although (many?) more are expected to exist. It was shown that these exhaust all nearly Kähler manifolds that are locally homogeneous (see [Bu05]) or satisfying $\mathrm{Hol}(\nabla^c) \neq \mathrm{SU}(3)$ (see [BM01]). A by now classical result asserts that a 6-dimensional spin manifold admits a real Killing spinor if and only if it is nearly Kähler (see [FG85], [Gru90] and [BFGK91]). Finally, more recent structure theorems justify why nearly Kähler manifolds are only interesting in dimension 6: any complete simply connected nearly Kähler manifold is locally a Riemannian product of Kähler manifolds, twistor spaces over Kähler manifolds and 6-dimensional nearly Kähler manifolds (see [Na02a], [Na02b]).

4.6. **G_2-structures in dimension 7.** We consider 7-dimensional Riemannian manifolds equipped with a G_2-structure. Since G_2 is the isotropy group of a 3-form ω of general type, a G_2-structure is a triple (M^7, g, ω) consisting of a 7-dimensional Riemannian manifold and a 3-form ω of general type at any point. We decompose the G_2-representation (see [FI02])

$$\mathbb{R}^7 \otimes \mathfrak{m} = \mathbb{R} \oplus \Lambda^2_{14} \oplus \Lambda^3_{27} \oplus \mathbb{R}^7 =: \mathcal{X}_1^{(1)} \oplus \mathcal{X}_2^{(14)} \oplus \mathcal{X}_3^{(27)} \oplus \mathcal{W}_4^{(7)} ,$$

and, consequently, there are again four basic classes and a total of 16 classes G_2-structures (namely, parallel G_2-manifolds and 15 non-integrable G_2-structures). This result is known as the Fernández/Gray-classification of G_2-structures (see [FG82]); some important classes are again summarized in tabular form, see Table 6. The different classes of G_2-structures can be characterized by differential equations. They can be written in a unified way as

$$d\omega = \lambda \cdot *\omega + \frac{3}{4}\theta \wedge \omega + *\tau_3 , \quad \delta\omega = - * d * \omega = - * (\theta \wedge *\omega) + *(\tau_2 \wedge \omega) ,$$

where λ is a scalar function corresponding to the $\mathcal{X}_1$-part of the intrinsic torsion Γ, τ_2, τ_3 are 2- resp. 3-forms corresponding to its $\mathcal{X}_2$ resp. $\mathcal{X}_3$-part and θ is a 1-form describing its $\mathcal{X}_4$-part, which one sometimes calls the *Lee form* of the G_2-structure. This accounts for some of the characterizations listed in Table 6. For example, a G_2-structure is of type $\mathcal{X}_1$ (*nearly parallel* G_2-structure) if and only if there exists a number λ (it has to be constant in this case) such that $d\omega = \lambda * \omega$ holds. Again, this condition is equivalent to the existence of a real Killing spinor and the metric has to be Einstein [FK90]; more recently, the Riemannian curvature properties of arbitrary G_2-manifolds have been discussed in detail by R. Cleyton and S. Ivanov [CI06a]. G_2-structures of type $\mathcal{X}_1 \oplus \mathcal{X}_3$ (*cocalibrated* G_2-structures) are characterized by the condition that the 3-form is coclosed, $\delta\omega^3 = 0$. Under

name	class	characterization
nearly parallel G_2-manifold	$\mathcal{X}_1$	a) $d\omega = \lambda * \omega$ for some $\lambda \in \mathbb{R}$ b) $\exists$ real Killing spinor
almost parallel or closed (or calibrated symplectic) G_2-m.	$\mathcal{X}_2$	$d\omega = 0$
balanced G_2-manifold	$\mathcal{X}_3$	$\delta\omega = 0$ and $d\omega \wedge \omega = 0$
locally conformally parallel G_2-m.	$\mathcal{X}_4$	$d\omega = \frac{3}{4}\theta \wedge \omega$ and $d * \omega = \theta \wedge *\omega$ for some 1-form θ
cocalibrated (or semi-parallel or cosymplectic) G_2-manifold	$\mathcal{X}_1 \oplus \mathcal{X}_3$	$\delta\omega = 0$
locally conformally (almost) parallel G_2-manifold	$\mathcal{X}_2 \oplus \mathcal{X}_4$	$d\omega = \frac{3}{4}\theta \wedge \omega$
G_2T-manifold	$\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$	a) $d * \omega = \theta \wedge *\omega$ for some 1-form θ b) $\exists$ char. connection ∇^c

TABLE 6. Some types of G_2-structures in dimension seven.

the restricted action of G_2, one obtains the following isotypic decomposition of 3-forms on $\mathbb{R}^7$:

$$\Lambda^3(\mathbb{R}^7) = \mathbb{R} \oplus \Lambda^3_{27} \oplus \mathbb{R}^7 = \mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4 \,.$$

This explains the first part of the following theorem and the acronym 'G_2T-manifolds' for this class: it stands for 'G_2 with (skew) torsion'. The explicit formula for the characteristic torsion may be derived directly from the properties of ∇^c.

Theorem 4.3 ([FI02, Thm. 4.8])**.** *A 7-dimensional manifold (M^7, g, ω) with a fixed G_2-structure $\omega \in \Lambda^3(M^7)$ admits a characteristic connection ∇^c if and only if it is of class $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$, i.e. if there exists a 1-form θ such that $d*\omega = \theta \wedge *\omega$. Furthermore, ∇^c is unique and given by the expression*

$$\nabla^c_X Y := \nabla^g_X Y + \frac{1}{2}\Big[- *d\omega - \frac{1}{6} g(d\omega, *\omega)\omega + *(\theta \wedge \omega)\Big] \,.$$

∇^c admits (at least) one parallel spinor.

This last remarkable property is a direct consequence of our investigation of geometric stabilizers, as explained in Corollary 3.1. For a nearly parallel G_2-manifold, the ∇^c-parallel spinor coincides with the Riemannian Killing spinor and the manifold turns out to be ∇^c-Einstein [FI02]. Some subtle effects occur when more spinors enter the play, as we shall now explain. First, we recall the fundamental theorem on Killing spinors in dimension 7:

Theorem 4.4 ([FK90], [BFGK91], [FKMS97])**.** *A 7-dimensional simply connected compact Riemannian spin manifold (M^7, g) admits*

(1) *one real Killing spinor if and only if it is a nearly parallel G_2-manifold;*
(2) *two real Killing spinors if and only if it is a Sasaki-Einstein manifold;*
(3) *three real Killing spinors if and only if it is a 3-Sasaki manifold.*

Furthermore, 3 is also the maximal possible number of Killing spinors for $M^7 \neq S^7$. On the other side, the characteristic connection ∇^c of a G_2T-manifold has 2 resp. 4 parallel spinors if its holonomy reduces further to SU(3) resp. SU(2). But there is no general argument identifying Killing spinors with parallel spinors: the characteristic connection of a Sasaki-Einstein manifold does not necessarily admit parallel spinors (see [FI02], [FI03a]), a 3-Sasaki manifold does not even admit a characteristic connection in any reasonable sense (each Sasaki structure has a characteristic connection, but it does not preserve the other two Sasaki structures), see Section 2.6 and [AF04a]. This reflects the fact that Sasaki-Einstein and 3-Sasaki manifolds do not fit too well into the general framework of G-structures.

Remark 4.5 (Parallel torsion)**.** For a nearly parallel G_2-manifold, the explicit formula from Theorem 4.3 implies that the characteristic torsion T^c is proportional to ω, hence it is trivially ∇^c-parallel. For the larger class of cocalibrated G_2-manifolds (class $\mathcal{X}_1 \oplus \mathcal{X}_3$), the case of parallel characteristic torsion has been investigated systematically by Th. Friedrich (see [Fri06]). Again, many formerly unknown examples have been constructed, for example, from deformations of η-Einstein Sasaki manifolds, from S^1-principal fibre bundles over 6-dimensional Kähler manifolds or from naturally reductive spaces.

4.7. Spin(7)**-structures in dimension** 8**.** Let us consider Spin(7)-structures on 8-dimensional Riemannian manifolds. The subgroup Spin(7) $\subset$ SO(8) is the real Spin(7)-representation $\Delta_7 = \mathbb{R}^8$, the complement $\mathfrak{m} = \mathbb{R}^7$ is the standard 7-dimensional representation and the Spin(7)-structures on an 8-dimensional Riemannian manifold M^8 correspond to the irreducible components of the tensor product

$$\mathbb{R}^8 \otimes \mathfrak{m} = \mathbb{R}^8 \otimes \mathbb{R}^7 = \Delta_7 \otimes \mathbb{R}^7 = \Delta_7 \oplus \mathrm{K} = \mathbb{R}^8 \oplus \mathrm{K}\,,$$

where K denotes the kernel of the Clifford multiplication $\Delta_7 \otimes \mathbb{R}^7 \to \Delta_7$. It is well known that K is an irreducible Spin(7)-representation, i.e. there are two basic classes of Spin(7)-structures (the Fernández classification of Spin(7)-structures, see [Fer86]). For 3-forms, we find the isotypic decomposition

$$\Lambda^3(\mathbb{R}^8) = \Delta_7 \oplus \mathrm{K}\,,$$

showing that $\Lambda^3(\mathbb{R}^8)$ and $\mathbb{R}^8 \otimes \mathfrak{m}$ are isomorphic. Theorem 4.1 yields immediately that *any* Spin(7)*-structure on an* 8*-dimensional Riemannian manifold admits a unique connection with totally skew-symmetric torsion.* The explicit formula for its characteristic torsion may be found in [Iv04].

5. Weitzenböck formulas for Dirac operators with torsion

5.1. **Motivation.** The question whether or not the characteristic connection of a G-structure admits parallel tensor fields differs radically from the corresponding problem for the Levi-Civita connection. In particular, one is interested in the existence of parallel spinor fields, interpreted in superstring theory as supersymmetries of the model. The main analytical tool for the investigation of parallel spinors is the Dirac operator and several remarkable identities for it. We discuss two identities for the square of the Dirac operator. While the first one is straightforward and merely of computational difficulty, the second relies on comparing the Dirac operator corresponding to the connection with torsion T with the spinorial Laplace operator corresponding to the connection with torsion $3\,T$. Such an argument has been used in the literature at several places. The first was probably S. Slebarski ([Sle87a], [Sle87b]) who noticed that on a naturally reductive space, the connection with torsion one-third that of the canonical connection behaves well under fibrations; S. Goette applied this property to the computation of the η-invariant on homogeneous spaces [Goe99]. J.-P. Bismut used such a rescaling for proving an index theorem for Hermitian manifolds [Bis89]. It is implicit in Kostant's work on a 'cubic Dirac operator', which can be understood as an identity in the Clifford algebra for the symbol of the Dirac operator of the rescaled canonical connection on a naturally reductive space ([Kos99], [Agr03]).

5.2. **The square of the Dirac operator and parallel spinors.** Consider a Riemannian spin manifold (M^n, g, T) with a 3-form $T \in \Lambda^3(M^n)$ as well as the one-parameter family of linear metric connections with skew-symmetric torsion $(s \in \mathbb{R})$,

$$\nabla^s_X Y := \nabla^g_X Y + 2s\,T(X,Y,-)\,.$$

In particular, the superscript $s = 0$ corresponds to the Levi-Civita connection and $s = 1/4$ to the connection with torsion T considered before. As before, we shall also sometimes use the superscript g to denote the Riemannian quantities corresponding to $s = 0$. These connections can all be lifted to connections on the spinor bundle ΣM^n, where they take the expression

$$\nabla^s_X \psi := \nabla^g_X \psi + s(X \lrcorner T)\cdot\psi\,.$$

Two important elliptic operators may be defined on ΣM^n, namely, the Dirac operator and the spinor Laplacian associated with the connection ∇^s:

$$D^s := \sum_{k=1}^n e_k\cdot\nabla^s_{e_k} = D^0 + 3sT\,, \quad \Delta^s(\psi) = (\nabla^s)^*\nabla^s\psi = -\sum_{k=1}^n \nabla^s_{e_k}\nabla^s_{e_k}\psi + \nabla^s_{\nabla^g_{e_i}e_i}\psi\,.$$

By a result of Th. Friedrich and S. Sulanke [FS79], the Dirac operator D^∇ associated with any metric connection ∇ is formally self-adjoint if and only if the ∇-divergence $\mathrm{div}^\nabla(X) := \sum_i g(\nabla_{e_i}X, e_i)$ of any vector field X coincides with its Riemannian ∇^g-divergence. Writing $\nabla = \nabla^g + A$, this is manifestly equivalent to $\sum_i g(A(e_i,X),e_i) = 0$ and trivially satisfied for metric connections with totally skew-symmetric torsion[8].

Shortly after P. Dirac introduced the Dirac operator, E. Schrödinger noticed the existence of a remarkable formula for its square [Schr32]. Of course, since

[8]One checks that it also holds for metric connections with vectorial torsion, but not for connections of Cartan type $\mathcal{A}'$.

the concept of spin manifold had not yet been established, all arguments of that time were of local nature, but contained already all important ingredients that would be established in a more mathematical way later. By the sixties and the seminal work of Atiyah and Singer on index theory for elliptic differential operators, Schrödinger's article was almost forgotten and the formula rediscovered by A. Lichnerowicz [Li63]. In our notation, the *Schrödinger-Lichnerowicz formula* states that

$$(D^0)^2 = \Delta^0 + \frac{1}{4}\mathrm{scal}^0 .$$

Our goal is to derive useful relations for the square of D^s. In order to state the first formula, let us introduce the first order differential operator

$$\mathcal{D}^s\psi := \sum_{k=1}^{n}(e_k \lrcorner \mathrm{T}) \cdot \nabla^s_{e_k}\psi = \mathcal{D}^0\psi + s\sum_{k=1}^{n}(e_k \lrcorner \mathrm{T}) \cdot (e_k \lrcorner \mathrm{T}) \cdot \psi . \tag{9}$$

Theorem 5.1 ([FI02, Thm 3.1, 3.3])**.** *Let (M^n, g, ∇^s) be an n-dimensional Riemannian spin manifold with a metric connection ∇^s of skew-symmetric torsion $4s \cdot T$. Then, the square of the Dirac operator D^s associated with ∇^s acts on an arbitrary spinor field ψ as*

$$(D^s)^2\psi = \Delta^s(\psi) + 3s\, dT \cdot \psi - 8s^2\,\sigma_T \cdot \psi + 2s\,\delta T \cdot \psi - 4s\,\mathcal{D}^s\psi + \frac{1}{4}\,\mathrm{scal}^s \cdot \psi . \tag{10}$$

Furthermore, the anticommutator of D^s and T is

$$D^s \circ T + T \circ D^s = dT + \delta T - 8s\,\sigma_T - 2\,\mathcal{D}^s . \tag{11}$$

scal^s denotes the scalar curvature of the connection ∇^s. Remark that $\mathrm{scal}^0 = \mathrm{scal}^g$ is the usual scalar curvature of the underlying Riemannian manifold (M^n, g) and that the relation $\mathrm{scal}^s = \mathrm{scal}^0 - 24s^2\|T\|^2$ holds. Moreover, the divergence δT can be taken with respect to any connection ∇^s from the family, hence we do not make a notational difference between them (see Proposition A.2).

This formula for $(D^s)^2$ has the disadvantage of still containing a first order differential operator with uncontrollable spectrum as well as several 4-forms that are difficult to treat algebraically, hence it is not suitable for deriving vanishing theorems. It has however a nice application in the study of ∇^s-parallel spinors for *different* values of s. As motivation, let's consider the following example:

Example 5.1. Let G be a simply connected Lie group, g a biinvariant metric and consider the torsion form $T(X, Y, Z) := g([X, Y], Z)$. The connections $\nabla^{\pm 1/4}$ are flat [KN69], hence they both admit non-trivial parallel spinor fields.

Such a property for the connections with torsions $\pm T$ is required in some superstring models. Theorem 5.1 now implies that there cannot be many values s admitting ∇^s-parallel spinors.

Theorem 5.2 ([AF04a, Thm. 7.1.])**.** *Let (M^n, g, T) be a compact spin manifold, ∇^s the family of metric connections defined by T as above. For any ∇^s-parallel spinor ψ, the following formula holds:*

$$64\, s^2 \int_{M^n} \langle \sigma_T \cdot \psi,\, \psi\rangle + \int_{M^n} \mathrm{scal}^s \cdot \|\psi\|^2 = 0 .$$

If the mean value of $\langle \sigma_T \cdot \psi, \psi\rangle$ does not vanish, the parameter s is given by

$$s = \frac{1}{8}\int_{M^n} \langle dT\cdot\psi,\psi\rangle \Big/ \int_{M^n} \langle \sigma_T\cdot\psi,\psi\rangle \,.$$

If the mean value of $\langle \sigma_T \cdot \psi, \psi\rangle$ vanishes, the parameter s depends only on the Riemannian scalar curvature and on the length of the torsion form,

$$0 = \int_{M^n} \mathrm{Scal}^s = \int_{M^n} \mathrm{scal}^g - 24 s^2 \int_{M^n} \|T\|^2 \,.$$

Finally, if the 4-forms dT and σ_T are proportional (for example, if $\nabla^{1/4}T = 0$), there are at most three parameters with ∇^s-parallel spinors.

Remark 5.1. The property that dT and σ_T are proportional is more general than requiring parallel torsion. For example, it holds for the whole family of connections ∇^t on naturally reductive spaces discussed in Sections 2.2 and 5.3, but its torsion is ∇^t-parallel only for $t = 1$.

Example 5.2. On the 7-dimensional Aloff-Wallach space $N(1,1) = \mathrm{SU}(3)/S^1$, one can construct a non-flat connection such that ∇^{s_0} *and* ∇^{-s_0} admit parallel spinors for suitable s_0, hence showing that both cases from Theorem 5.2 can actually occur in non-trivial situations. On the other hand, it can be shown that on a 5-dimensional Sasaki manifold, only the characteristic connection ∇^c can have parallel spinors [AF04a].

Inspired by the homogeneous case (see Section 5.3), we were looking for an alternative comparison of $(D^s)^2$ with the Laplace operator of some *other* connection $\nabla^{s'}$ from the same family. Since $(D^s)^2$ is a symmetric second order differential operator with metric principal symbol, a very general result by P. B. Gilkey claims that there exists a connection ∇ and an endomorphism E such that $(D^s)^2 = \nabla^*\nabla + E$ [Gil75]. Based on the results of Theorem 5.1, one shows:

Theorem 5.3 (Generalized Schrödinger-Lichnerowicz formula, [AF04a, Thm. 6.2])**.** *The spinor Laplacian Δ^s and the square of the Dirac operator $D^{s/3}$ are related by*

$$(D^{s/3})^2 = \Delta^s + s\, dT + \frac{1}{4}\mathrm{scal}^g - 2s^2\,\|T\|^2 \,.$$

We observe that $D^{s/3}$ appears basically by quadratic completion. A first consequence is a non linear version of Theorem 2.8.

Theorem 5.4 ([AF04a, Thm. 6.3])**.** *Let (M^n, g, T) be a compact, Riemannian spin manifold of non positive scalar curvature,* $\mathrm{scal}^g \leq 0$. *If there exists a solution $\psi \neq 0$ of the equations*

$$\nabla_X \psi = \nabla^g_X\psi + \frac{1}{2}(X \,\lrcorner\, T)\cdot\psi = 0\,, \quad \langle dT\cdot\psi,\psi\rangle \ \leq\ 0\,,$$

the 3-form and the scalar curvature vanish, $T = 0 = \mathrm{scal}^g$, and ψ is parallel with respect to the Levi-Civita connection.

Theorem 5.4 applies, in particular, to Calabi-Yau or Joyce manifolds, where we know that ∇^g-parallel spinors exist by Wang's Theorem (Theorem 3.1). Let us perturb the connection ∇^g by a suitable 3-form (for example, a closed one). Then the new connection ∇ does not admit ∇-parallel spinor fields: the Levi-Civita connection and its parallel spinors are thus, in some sense, rigid. Nilmanifolds are a second family of examples where the theorem applies. A further family of examples arises from certain naturally reductive spaces with torsion form T proportional to the torsion form of the canonical connection, see [Agr03]. From the high energy physics point of view, a parallel spinor is interpreted as a supersymmetry transformation. Hence the physical problem behind the above question (which in fact motivated our investigations) is really whether a free "vacuum solution" can also carry a non-vacuum supersymmetry, and how the two are related.

Naturally, Theorem 5.4 raises the question to which extent compactness is really necessary. We shall now show that it is by using the equivalence between the inclusion $\mathrm{Hol}(\nabla) \subset G_2$ and the existence of a ∇-parallel spinor for a metric connection with skew-symmetric torsion known from Corollary 3.1. For this, it is sufficient to find a 7-dimensional Riemannian manifold (M^7, g) whose Levi-Civita connection has a parallel spinor (hence is Ricci-flat, in particular), but also admits a ∇-parallel spinor for some other metric connection with skew-symmetric torsion.

Gibbons et al. produced non-complete metrics with Riemannian holonomy G_2 in [GLPS02]. Those metrics have among others the interesting feature of admitting a 2-step nilpotent isometry group N acting on orbits of codimension one. By [ChF05] such metrics are locally conformal to homogeneous metrics on rank-one solvable extension of N, and the induced SU(3)-structure on N is half-flat. In the same paper all half-flat SU(3) structures on 6-dimensional nilpotent Lie groups whose rank-one solvable extension is endowed with a conformally parallel G_2 structure were classified. Besides the torus, there are exactly six instances, which we considered in relation to the problem posed. It turns out that four metrics of the six only carry integrable G_2 structures, thus reproducing the pattern of the compact situation, whilst one admits complex solutions, a physical interpretation for which is still lacking. The remaining solvmanifold (Sol, g)—which has *exact* Riemannian holonomy G_2—provides a positive answer to both questions posed above, hence becoming the most interesting. The Lie algebra associated to this solvmanifold has structure equations

$$\begin{aligned}&[e_i, e_7] = \tfrac{3}{5} m e_i,\ i = 1,2,5, \quad [e_j, e_7] = \tfrac{6}{5} m e_j,\ j = 3,4,6,\\ &[e_1, e_5] = -\tfrac{2}{5} m e_3, \quad [e_2, e_5] = -\tfrac{2}{5} m e_4, \quad [e_1, e_2] = -\tfrac{2}{5} m e_6\,.\end{aligned}$$

The homogeneous metric it bears can be also seen as a G_2 metric on the product $\mathbb{R} \times \mathbb{T}$, where $\mathbb{T}$ is the total space of a T^3-bundle over another 3-torus. For the sake of an easier formulation of the result, we denote by ∇^T the metric connection with torsion T.

Theorem 5.5 ([ACF05, Thm. 4.1.])**.** *The equation* $\nabla^T \Psi = 0$ *admits* 7 *solutions for some* 3*-form, namely:*

a) *A two-parameter family of pairs* $(T_{r,s}, \Psi_{r,s}) \in \Lambda^3(\mathrm{Sol}) \times \Sigma(\mathrm{Sol})$ *such that* $\nabla^{T_{r,s}}\Psi_{r,s} = 0$*;*
for $r = s$ *the torsion* $T_{r,r} = 0$ *and* $\Psi_{r,r}$ *is a multiple of the* ∇^g*-parallel spinor.*

b) *Six 'isolated' solutions occuring in pairs,* $(T_i^\varepsilon, \Psi_i^\varepsilon) \in \Lambda^3(\mathrm{Sol}) \times \Sigma(\mathrm{Sol})$ *for* $i = 1, 2, 3$ *and* $\varepsilon = \pm$.

All these G_2 *structures admit exactly one parallel spinor, and for*

$|r| \neq |s|$*:* $\omega_{r,s}$ *is of general type* $\mathbb{R} \oplus S_0^2\mathbb{R}^7 \oplus \mathbb{R}^7$*,*

$r = s$*:* $\omega_{r,r}$ *is* ∇^g*-parallel,*

$r = -s$*: the* G_2 *class has no* $\mathbb{R}$*-part.*

Here, $\omega_{r,s}$ *denotes the defining* 3*-form of the* G_2*-structure, see eq.* (8)*.*

Remark 5.2. A routine computation establishes that $\langle dT \cdot \Psi, \Psi\rangle < 0$ for all solutions found in Theorem 5.5, except for the integrable case $r = s$ of solution a) where it vanishes trivially since $T = 0$.

Remark 5.3. The interaction between explicit Riemannian metrics with holonomy G_2 on non-compact manifolds and the non-integrable G_2-geometries as investigated with the help of connections with torsion was up-to-now limited to "cone-type arguments", i.e. a non-integrable structure on some manifold was used to construct an integrable structure on a higher dimensional manifold (like its cone, an so on). It is thus a natural question whether the same Riemannian manifold (M, g) can carry structures of both type *simultaneously.* This appears to be a remarkable property, of which the above example is the only known instance. To emphasize this, consider that the projective space $\mathbb{C}P^3$ with the well-known Kähler-Einstein structure and the nearly Kähler one inherited from triality does not fit the picture, as they refer to different metrics.

5.3. **Naturally reductive spaces and Kostant's cubic Dirac operator.** On arbitrary manifolds, only Weitzenböck formulas that express D^2 through the Laplacian are available. On homogeneous spaces, it makes sense to look for expressions for D^2 of *Parthasarathy type*, that is, in terms of Casimir operators. Naturally reductive spaces $M^n = G/H$ with their family of metric connections $(X, Y \in \mathfrak{m})$

$$\nabla^t_X Y := \nabla^g_X Y - \frac{t}{2}[X, Y]_{\mathfrak{m}}$$

were in fact investigated prior to the more general case described in the previous section. As symmetric spaces are good toy models for integrable geometries, homogeneous non-symmetric spaces are a very useful field for 'experiments' in non-integrable geometry. Furthermore, many examples of such geometries are in fact homogeneous. We will show that the main achievement in [Kos99] was to realize that, for the parameter value $t = 1/3$, the square of D^t may be expressed in a very simple way in terms of Casimir operators and scalars only ([Kos99, Thm 2.13], [Ste99, 10.18]). It is a remarkable generalization of the classical Parthasarathy formula for D^2 on symmetric spaces (formula (1) in this article, see [Par72]). We shall speak of the *generalized Kostant-Parthasarathy formula* in the sequel. S. Slebarski used the connection $\nabla^{1/3}$ to prove a "vanishing theorem" for the kernel of the twisted Dirac operator, which can be easily recovered from Kostant's formula (see [Lan00, Thm 4]). His articles [Sle87a] and [Sle87b] contain several formulas of Weitzenböck type for D^2, but none of them is of Parthasarathy type.

In order to exploit the full power of harmonic analysis, it is necessary to extend the naturally reductive metric $\langle\ ,\ \rangle$ on $\mathfrak{m}$ to the whole Lie algebra $\mathfrak{g}$ of G. By a classical theorem of B. Kostant, there exists a unique $\mathrm{Ad}(G)$ invariant, symmetric, non degenerate, bilinear form Q on $\mathfrak{g}$ such that

$$Q(\mathfrak{h}\cap\mathfrak{g},\mathfrak{m}) = 0 \quad \text{and} \quad Q|_{\mathfrak{m}} = \langle\ ,\ \rangle$$

if G acts effectively on M^n and $\mathfrak{g} = \mathfrak{m} + [\mathfrak{m},\mathfrak{m}]$, which we will tacitly assume from now on [Kos56]. In general, Q does not have to be positive definite; if it is, the metric is called *normal homogeneous*. Assume furthermore that there exists a homogeneous spin structure on M, i. e., a lift $\widetilde{\mathrm{Ad}}\,:\ H \to \mathrm{Spin}(\mathfrak{m})$ of the isotropy representation such that the diagram

$$\begin{tikzcd} & \mathrm{Spin}(\mathfrak{m}) \\ H \arrow[ur, "\widetilde{\mathrm{Ad}}"] \arrow[r, "\mathrm{Ad}"] & \mathrm{SO}(\mathfrak{m}) \arrow[u, "\lambda"'] \end{tikzcd}$$

commutes, where λ denotes the spin covering. Moreover, denote by $\widetilde{\mathrm{ad}}$ the corresponding lift into $\mathfrak{spin}(\mathfrak{m})$ of the differential $\mathrm{ad}\,:\ \mathfrak{h} \to \mathfrak{so}(\mathfrak{m})$ of Ad. Let $\kappa\,:\ \mathrm{Spin}(\mathfrak{m}) \to \mathrm{GL}(\Delta_{\mathfrak{m}})$ be the spin representation, and identify sections of the spinor bundle $\Sigma M^n = G \times_{\widetilde{\mathrm{Ad}}} \Delta_{\mathfrak{m}}$ with functions $\psi:\ G \to \Delta_{\mathfrak{m}}$ satisfying

$$\psi(gh) = \kappa\big(\widetilde{\mathrm{Ad}}\,(h^{-1})\big)\psi(g)\,.$$

The Dirac operator takes for $\psi \in \Sigma M^n$ the form

$$D^t\psi = \sum_{i=1}^{n} e_i(\psi) + \frac{1-t}{2}\, H\cdot\psi\,,$$

where H is the third degree element in the Clifford algebra $\mathrm{Cl}(\mathfrak{m})$ of $\mathfrak{m}$ induced from the torsion,

$$H := \frac{3}{2}\sum_{i<j<k} \langle [e_i,e_j]_{\mathfrak{m}}, e_k\rangle\ e_i\cdot e_j\cdot e_k\,.$$

This fact suggested the name "cubic Dirac operator" to B. Kostant. Two expressions appear over and over again for naturally reductive spaces: these are the $\mathfrak{m}$- and $\mathfrak{h}$-parts of the Jacobi identity,

$$\mathrm{Jac}_{\mathfrak{m}}(X,Y,Z) := [X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} + [Y,[Z,X]_{\mathfrak{m}}]_{\mathfrak{m}} + [Z,[X,Y]_{\mathfrak{m}}]_{\mathfrak{m}}\,,$$

$$\mathrm{Jac}_{\mathfrak{h}}(X,Y,Z) := [X,[Y,Z]_{\mathfrak{h}}]\ +\ [Y,[Z,X]_{\mathfrak{h}}]\ +\ [Z,[X,Y]_{\mathfrak{h}}]\,.$$

Notice that the summands of $\mathrm{Jac}_{\mathfrak{h}}(X,Y,Z)$ automatically lie in $\mathfrak{m}$ by the assumption that M is reductive. The Jacobi identity for $\mathfrak{g}$ implies $\mathrm{Jac}_{\mathfrak{m}}(X,Y,Z)+$

$\mathrm{Jac}_{\mathfrak{h}}(X,Y,Z)=0$. In fact, since the torsion is given by $T^t(X,Y)=-t[X,Y]_{\mathfrak{m}}$, one immediately sees that $\langle \mathrm{Jac}_{\mathfrak{m}}(X,Y,Z),V\rangle$ is just $-\sigma_{T^t}(X,Y,Z,V)$ as defined before. From the explicit formula for T^t and the property $\nabla^1 T^1=0$, it is a routine computation to show that (see [Agr03, Lemma 2.3, 2.5])

$$\nabla^t_Z T^t(X,Y)=\frac{1}{2}t(t-1)\mathrm{Jac}_{\mathfrak{m}}(X,Y,Z)\,,\ \ dT^t(X,Y,Z,V)=-2t\,\langle \mathrm{Jac}_{\mathfrak{m}}(X,Y,Z),V\rangle\ .$$

In particular, dT^t and σ_{T^t} are always proportional (see Remark 5.1). The first formula implies $X\,\lrcorner\,\nabla^t_X T^t=0$, hence $\delta^t T^t=0$ and it equals the Riemannian divergence $\delta^g T^t$ by Proposition A.2 of the Appendix. Since the $\mathrm{Ad}(G)$-invariant extension Q of $\langle\ ,\ \rangle$ is not necessarily positive definite when restricted to $\mathfrak{h}$, it is more appropriate to work with dual rather than with orthonormal bases. So pick bases x_i,y_i of $\mathfrak{h}$ wich are dual with respect to $Q_{\mathfrak{h}}$, i.e., $Q_{\mathfrak{h}}(x_i,y_j)=\delta_{ij}$. The (lift into the spin bundle of the) Casimir operator of the full Lie algebra $\mathfrak{g}$ is now the sum of a second order differential operator (its $\mathfrak{m}$-part) and a constant element of the Clifford algebra (its $\mathfrak{h}$-part)

$$\Omega_{\mathfrak{g}}(\psi)=-\sum_{i=1}^{n}e_i^2(\psi)-\sum_{j=1}^{\dim\mathfrak{h}}\widetilde{\mathrm{ad}}\,(x_j)\circ\widetilde{\mathrm{ad}}\,(y_j)\cdot\psi\quad\text{for}\quad \psi\in\Sigma M^n\,.$$

In order to prove the generalized Kostant-Parthasarathy formula for the square of D^t, similar technical prerequisites as in Section 5.2 are needed, but now expressed with respect to representation theoretical quantities instead of analytical ones. We refer to [Agr03] for details and will rather formulate the final result without detours. Observe that the dimension restriction below ($n\geq 5$) is not essential, for small dimensions a similar formula holds, but it looks slightly different.

Theorem 5.6 (Generalized Kostant-Parthasarathy formula, [Agr03, Thm 3.2])**.** *For $n\geq 5$, the square of D^t is given by*

$$\begin{aligned}(D^t)^2\psi=\Omega_{\mathfrak{g}}(\psi)&+\frac{1}{2}(3t-1)\sum_{i,j,k}\langle[e_i,e_j]_{\mathfrak{m}},e_k\rangle\, e_i\cdot e_j\cdot e_k(\psi)\\ &-\frac{1}{2}\sum_{i<j<k<l}\Big\langle e_i,\mathrm{Jac}_{\mathfrak{h}}(e_j,e_k,e_l)\\ &+\frac{9(1-t)^2}{4}\,\mathrm{Jac}_{\mathfrak{m}}(e_j,e_k,e_l)\Big\rangle\cdot e_i\cdot e_j\cdot e_k\cdot e_l\cdot\psi\\ &+\frac{1}{8}\sum_{i,j}Q_{\mathfrak{h}}\big([e_i,e_j],[e_i,e_j]\big)\psi+\frac{3(1-t)^2}{24}\sum_{i,j}Q_{\mathfrak{m}}\big([e_i,e_j],[e_i,e_j]\big)\psi.\end{aligned}$$

Qualitatively, this result is similar to equation (10) of Theorem 5.1, although one cannot be deduced directly from the other. Again, the square of the Dirac operator is written as the sum of a second order differential operator (the Casimir operator), a first order differential operator, a four-fold product in the Clifford algebra and a scalar part (recall that $\delta^t T^t=0$, hence this term has no counterpart here). An immediate consequence is the special case $t=1/3$:

Corollary 5.1 (The Kostant-Parthasarathy formula for $t = 1/3$). *For $n \geq 5$ and $t = 1/3$, the general formula for $(D^t)^2$ reduces to*

$$\begin{aligned}(D^{1/3})^2\psi &= \Omega_{\mathfrak{g}}(\psi) + \frac{1}{8}\Big[\sum_{i,j} Q_{\mathfrak{h}}([e_i,e_j],[e_i,e_j]) + \frac{1}{3}\sum_{i,j} Q_{\mathfrak{m}}([e_i,e_j],[e_i,e_j])\Big]\psi \\ &= \Omega_{\mathfrak{g}}(\psi) + \frac{1}{8}\Big[\mathrm{scal}^{1/3} + \frac{1}{9}\sum_{i,j} Q_{\mathfrak{m}}([e_i,e_j],[e_i,e_j])\Big]\psi\,.\end{aligned}$$

Remark 5.4. In particular, one immediately recovers the classical Parthasarathy formula for a symmetric space, since then all scalar curvatures coincide and $[e_i, e_j] \in \mathfrak{h}$. In fact, compared with Theorem 5.3, Corollary 5.1 has the advantage of containing no 4-form action on the spinor and the draw-back that the Casimir operator of a naturally reductive space is not necessarily a non-negative operator (see Section 5.4 for a detailed investigation of this point).

As in the classical Parthasarathy formula, the scalar term as well as the eigenvalues of $\Omega_{\mathfrak{g}}(\psi)$ may be expressed in representation theoretical terms if G (and hence M) is compact.

Lemma 5.1 ([Agr03, Lemma 3.6]). *Let G be compact, $n \geq 5$, and denote by $\varrho_{\mathfrak{g}}$ and $\varrho_{\mathfrak{h}}$ the half sum of the positive roots of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Then the Kostant-Parthasarathy formula for $(D^{1/3})^2$ may be restated as*

$$(D^{1/3})^2\psi = \Omega_{\mathfrak{g}}(\psi) + \big[Q(\varrho_{\mathfrak{g}},\varrho_{\mathfrak{g}}) - Q(\varrho_{\mathfrak{h}},\varrho_{\mathfrak{h}})\big]\psi = \Omega_{\mathfrak{g}}(\psi) + \langle \varrho_{\mathfrak{g}} - \varrho_{\mathfrak{h}}, \varrho_{\mathfrak{g}} - \varrho_{\mathfrak{h}}\rangle\,\psi\,.$$

In particular, the scalar term is positive independently of the properties of Q.

We can formulate our first conclusion from Corollary 5.1:

Corollary 5.2 ([Agr03, Cor. 3.1]). *Let G be compact. If the operator $\Omega_{\mathfrak{g}}$ is non-negative, the first eigenvalue $\lambda_1^{1/3}$ of the Dirac operator $D^{1/3}$ satisfies the inequality*

$$\big(\lambda_1^{1/3}\big)^2 \geq Q(\varrho_{\mathfrak{g}},\varrho_{\mathfrak{g}}) - Q(\varrho_{\mathfrak{h}},\varrho_{\mathfrak{h}})\,.$$

Equality occurs if and only if there exists an algebraic spinor in $\Delta_{\mathfrak{m}}$ which is fixed under the lift $\kappa(\widetilde{\mathrm{Ad}}\,H)$ of the isotropy representation.

Remark 5.5. This eigenvalue estimate is remarkable for several reasons. Firstly, for homogeneous non symmetric spaces, it is sharper than the classical Parthasarathy formula. For a symmetric space, one classically obtains $\lambda_1^2 \geq \mathrm{scal}/8$. But since the Schrödinger-Lichnerowicz formula yields immediately $\lambda_1^2 \geq \mathrm{scal}/4$, the lower bound in the classical Parthasarathy formula is never attained. In contrast, there exist many examples of homogeneous non symmetric spaces with constant spinors. Secondly, it uses a lower bound wich is *always* strictly positive; for many naturally reductive metrics with negative scalar curvature a pure curvature bound would be of small interest. Our previously discussed generalizations of the Schrödinger-Lichnerowicz yield no immediate eigenvalue estimate. S. Goette derived in [Goe99, Lemma 1.17] an eigenvalue estimate for normal homogeneous naturally reductive metrics, but it is also not sharp.

Remark 5.6. Since D^t is a G-invariant differential operator on M by construction, Theorem 5.6 implies that the linear combination of the first order differential operator and the multiplication by the element of degree four in the Clifford algebra appearing in the formula for $(D^t)^2$ is again G invariant for all t. Hence, the first order differential operator

$$\tilde{\mathcal{D}}\psi := \sum_{i,j,k} \langle [Z_i, Z_j]_{\mathfrak{m}}, Z_k \rangle\, Z_i \cdot Z_j \cdot Z_k(\psi)$$

has to be a G-invariant differential operator, a fact that cannot be seen directly by any simple arguments. It has no analogue on symmetric spaces and certainly deserves further separate investigations. Of course, it should be understood as a 'homogeneous cousin' of the more general operator $\mathcal{D}$ defined in equation (9).

5.4. **A Casimir operator for characteristic connections.** Typically, the canonical connection of a naturally reductive homogeneous space M can be given an alternative geometric characterization—for example, as the unique metric connection with skew-symmetric torsion preserving a given G-structure. Once this is done, $D^{1/3}$, scal^g and $\|\mathrm{T}\|^2$ are geometrically invariant objects, whereas $\Omega_{\mathfrak{g}}$ still heavily relies on the concrete realization of the homogeneous space M as a quotient. At the same time, the same interesting G-structures exist on many non-homogeneous manifolds. Hence it was our goal to find a tool similar to $\Omega_{\mathfrak{g}}$ which has more intrinsic geometric meaning and which can be used in both situations just described [AF04b].

We consider a Riemannian spin manifold (M^n, g, ∇) with a metric connection ∇ and skew-symmetric torsion T. Denote by Δ_T the spinor Laplacian of the connection ∇.

Definition 5.1. The *Casimir operator* of (M^n, g, ∇) is the differential operator acting on spinor fields by

$$\begin{aligned}\Omega :=& (D^{1/3})^2 + \frac{1}{8}\,(dT - 2\sigma_T) + \frac{1}{4}\,\delta(T) - \frac{1}{8}\,\mathrm{scal}^g - \frac{1}{16}\,\|\mathrm{T}\|^2 \\ =& \Delta_T + \frac{1}{8}\big(3\,dT - 2\sigma_T + 2\,\delta(T) + \mathrm{scal}\big)\,.\end{aligned}$$

Remark 5.7. A naturally reductive space $M^n = G/H$ endowed with its canonical connection satisfies $dT = 2\sigma_T$ and $\delta T = 0$, hence $\Omega = \Omega_{\mathfrak{g}}$ by Theorem 5.1. For connections with $dT \neq 2\sigma_T$ and $\delta T \neq 0$, the numerical factors are chosen in such a way to yield an overall expression proportional to the scalar part of the right hand side of equation (10).

Example 5.3. For the Levi-Civita connection ($T = 0$) of an arbitrary Riemannian manifold, we obtain

$$\Omega = (D^g)^2 - \frac{1}{8}\,\mathrm{scal}^g = \Delta^g + \frac{1}{8}\,\mathrm{scal}^g\,.$$

The second equality is just the classical Schrödinger-Lichnerowicz formula for the Riemannian Dirac operator, whereas the first one is — in case of a symmetric space — the classical Parthasarathy formula.

Example 5.4. Consider a 3-dimensional manifold of constant scalar curvature, a constant $a \in \mathbb{R}$ and the 3-form $T = 2\,a\,dM^3$. Then

$$\Omega = (D^g)^2 - a\,D^g - \frac{1}{8}\,\mathrm{scal}^g\,.$$

The kernel of the Casimir operator corresponds to eigenvalues $\lambda \in \mathrm{Spec}(D^g)$ of the Riemannian Dirac operator such that

$$8\,(\lambda^2 - a\lambda) - \mathrm{scal}^g = 0\,.$$

In particular, the kernel of Ω is in general larger then the space of ∇-parallel spinors. Indeed, such spinors exist only on space forms. More generally, fix a real-valued smooth function f and consider the 3-form $T := f \cdot dM^3$. If there exists a ∇-parallel spinor

$$\nabla^g_X\psi + (X \lrcorner\, T)\cdot\psi = \nabla^g_X\psi + f\cdot X\cdot\psi = 0\,,$$

then, by a theorem of A. Lichnerowicz (see [Li87]), f is constant and (M^3, g) is a space form.

Let us collect some elementary properties of the Casimir operator.

Proposition 5.1 ([AF04b, Prop. 3.1])**.** *The kernel of the Casimir operator contains all ∇-parallel spinors.*

Proof. By Theorem 5.1, one of the integrability conditions for a ∇-parallel spinor field ψ is

$$\big(3\,dT - 2\sigma_T + 2\,\delta(T) + \mathrm{scal}\big)\cdot\psi = 0\,. \qquad \square$$

If the torsion form T is ∇-parallel, the formulas for the Casimir operator simplify. Indeed, in this case we have (see the Appendix)

$$d\mathrm{T} = 2\sigma_{\mathrm{T}}\,, \quad \delta(\mathrm{T}) = 0\,,$$

and the Ricci tensor Ric of ∇ is symmetric. Using the formulas of Section 5.2 (in particular, Theorems 5.1 and 5.3), we obtain a simpler expressions for the Casimir operator.

Proposition 5.2 ([AF04b, Prop. 3.2])**.** *For a metric connection with parallel torsion ($\nabla T = 0$), the Casimir operator can equivalently be written as:*

$$\begin{aligned}\Omega &= (D^{1/3})^2 - \frac{1}{16}\big(2\,\mathrm{scal}^g + \|T\|^2\big) = \Delta_T + \frac{1}{16}\big(2\,\mathrm{scal}^g + \|T\|^2\big) - \frac{1}{4}\,T^2\\ &= \Delta_T + \frac{1}{8}\big(2\,dT + \mathrm{scal}\big)\,.\end{aligned}$$

Integrating these formulas, we obtain a vanishing theorem for the kernel of the Casimir operator.

Proposition 5.3 ([AF04b, Prop. 3.3])**.** *Assume that M is compact and that ∇ has parallel torsion T. If one of the conditions*

$$2\,\mathrm{scal}^g \leq -\|\mathrm{T}\|^2 \quad \text{or} \quad 2\,\mathrm{scal}^g \geq 4\mathrm{T}^2 - \|\mathrm{T}\|^2\,,$$

holds, the Casimir operator is non-negative in $\mathrm{L}^2(S)$.

Example 5.5. For a naturally reductive space $M = G/H$, the first condition can never hold, since by Lemma 5.1, $2\,\mathrm{scal}^g + \|\mathrm{T}\|^2$ is strictly positive. In concrete examples, the second condition typically singles out the normal homogeneous metrics among the naturally reductive ones.

Proposition 5.4 ([AF04b, Prop. 3.4])**.** *If the torsion form is ∇-parallel, the Casimir operator Ω and the square of the Dirac operator $(D^{1/3})^2$ commute with the endomorphism T,*

$$\Omega \circ \mathrm{T} = \mathrm{T} \circ \Omega\,, \quad (D^{1/3})^2 \circ \mathrm{T} = \mathrm{T} \circ (D^{1/3})^2\,.$$

The endomorphism T acts on the spinor bundle as a symmetric endomorphism with *constant* eigenvalues.

Theorem 5.7. *Let (M^n, g, ∇) be a compact Riemannian spin manifold equipped with a metric connection ∇ with parallel, skew-symmetric torsion, $\nabla T = 0$. The endomorphism T and the Riemannian Dirac operator D^g act in the kernel of the Dirac operator $D^{1/3}$. In particular, if, for all $\mu \in \mathrm{Spec}(T)$, the number $-\mu/4$ is not an eigenvalue of the Riemannian Dirac operator, then the kernel of $D^{1/3}$ is trivial.*

If ψ belongs to the kernel of $D^{1/3}$ and is an eigenspinor of the endomorphism T, we have $4 \cdot D^g\psi = -\mu \cdot \psi$, $\mu \in \mathrm{Spec}(T)$. Using the estimate of the eigenvalues of the Riemannian Dirac operator (see [Fri80]), we obtain an upper bound for the minimum $\mathrm{scal}^g_{\mathrm{min}}$ Riemannian scalar curvature in case that the kernel of the operator $D^{1/3}$ is non trivial.

Proposition 5.5. *Let (M^n, g, ∇) be a compact Riemannian spin manifold equipped with a metric connection ∇ with parallel, skew-symmetric torsion, $\nabla T = 0$. If the kernel of the Dirac operator $D^{1/3}$ is non trivial, then the minimum of the Riemannian scalar curvature is bounded by*

$$\max\left\{\mu^2 \,:\, \mu \in \mathrm{Spec}(T)\right\} \geq \frac{4n}{n-1}\,\mathrm{scal}^g_{\mathrm{min}}\,.$$

Remark 5.8. If $(n-1)\,\mu^2 = 4n\,\mathrm{scal}^g$ is in the spectrum of T and there exists a spinor field ψ in the kernel of $D^{1/3}$ such that $T \cdot \psi = \mu \cdot \psi$, then we are in the limiting case of the inequality in [Fri80]. Consequently, M^n is an Einstein manifold of non-negative Riemannian scalar curvature and ψ is a Riemannian Killing spinor. Examples of this type are 7-dimensional 3-Sasakian manifolds. The possible torsion forms have been discussed in [AF04a], Section 9.

We discuss in detail what happens for 5-dimensional Sasakian manifold. Let $(M^5, g,$
$\xi, \eta, \varphi)$ be a (compact) 5-dimensional Sasakian spin manifold with a fixed spin structure, ∇^c its characteristic connection. We orient M^5 by the condition that the differential of the contact form is given by $d\eta = 2(e_1 \wedge e_2 + e_3 \wedge e_4)$, and write henceforth $e_{ij\ldots}$ for $e_i \wedge e_j \wedge \ldots$. Then we know that

$$\nabla T^c = 0\,, \quad T^c = \eta \wedge d\eta = 2\,(e_{12} + e_{34}) \wedge e_5\,, \quad (T^c)^2 = 8 - 8\,e_{1234}$$

and

$$\Omega = (D^{1/3})^2 - \frac{1}{8}\,\mathrm{scal}^g - \frac{1}{2} = \Delta_{T^c} + \frac{1}{8}\,\mathrm{scal}^g - \frac{3}{2} + 2\,e_{1234}\,.$$

We study the kernel of the Dirac operator $D^{1/3}$. The endomorphism T^c acts in the 5-dimensional spin representation with eigenvalues $(-4, 0, 0, 4)$ and, according to Theorem 5.7, we have to distinguish two cases. If $D^{1/3}\psi = 0$ and $T^c \cdot \psi = 0$, the spinor field is harmonic and the formulas of Proposition 5.2 yield in the compact case the condition

$$\int_{M^5} \big(2\,\mathrm{scal}^g + 8\big)\|\psi\|^2 \leq 0\,.$$

Examples of that type are the 5-dimensional Heisenberg group with its left invariant Sasakian structure and its compact quotients (Example 4.1) or certain S^1-bundles over a flat torus. The space of all ∇-parallel spinors satisfying $T^c \cdot \psi_0 = 0$ is a 2-dimensional subspace of the kernel of the operator $D^{1/3}$ (see [FI02], [FI03a]). The second case for spinors in the kernel is given by $D^{1/3}\psi = 0$ and $T^c \cdot \psi = \pm 4\,\psi$. The spinor field is an eigenspinor for the Riemannian Dirac operator, $D^g\psi = \mp\psi$. The formulas of Proposition 5.2 and Proposition 5.5 yield in the compact case two conditions:

$$\int_{M^5} \big(\mathrm{scal}^g - 12\big)\|\psi\|^2 \leq 0 \quad \text{and} \quad 5\,\mathrm{scal}^g_{\min} \leq 16\,.$$

The paper [FK90] contains a construction of Sasakian manifolds admitting a spinor field of that algebraic type in the kernel of $D^{1/3}$. We describe the construction explicitly. Suppose that the Riemannian Ricci tensor Ric^g of a simply-connected, 5-dimensional Sasakian manifold is given by the formula

$$\mathrm{Ric}^g = -2 \cdot g + 6 \cdot \eta \otimes \eta\,.$$

Its scalar curvature equals $\mathrm{scal}^g = -4$. In the simply-connected and compact case, they are total spaces of S^1 principal bundles over 4-dimensional Calabi-Yau orbifolds (see [BG99]). There exist (see [FK90], Theorem 6.3) two spinor fields ψ_1, ψ_2 such that

$$\nabla^g_X\psi_1 = -\frac{1}{2}\,X \cdot \psi_1 + \frac{3}{2}\,\eta(X) \cdot \xi \cdot \psi_1\,, \quad \mathrm{T} \cdot \psi_1 = -4\psi_1\,,$$

$$\nabla^g_X\psi_2 = \frac{1}{2}\,X \cdot \psi_2 - \frac{3}{2}\,\eta(X) \cdot \xi \cdot \psi_2\,, \quad T^c \cdot \psi_2 = 4\,\psi_2\,.$$

In particular, we obtain

$$D^g\psi_1 = \psi_1\,, \quad T^c \cdot \psi_1 = -4\,\psi_1\,, \quad \text{and} \quad D^g\psi_2 = -\psi_2\,, \quad T^c \cdot \psi_2 = 4\psi_2\,,$$

and therefore the spinor fields ψ_1 and ψ_2 belong to the kernel of the operator $D^{1/3}$.

Next, we investigate the kernel of the Casimir operator. Under the action of the torsion form, the spinor bundle ΣM^5 splits into three subbundles $\Sigma M^5 = \Sigma_0 \oplus \Sigma_4 \oplus \Sigma_{-4}$ corresponding to the eigenvalues of T^c. Since $\nabla T^c = 0$, the connection ∇ preserves the splitting. The endomorphism e_{1234} acts by the formulas

$$e_{1234} = 1 \quad \text{on} \quad \Sigma_0\,, \quad e_{1234} = -1 \quad \text{on} \quad \Sigma_4 \oplus \Sigma_{-4}\,.$$

Consequently, the formula

$$\Omega = \Delta_{T^c} + \frac{1}{8}\,\mathrm{scal}^g - \frac{3}{2} + 2\,e_{1234}$$

shows that the Casimir operator splits into the sum $\Omega = \Omega_0 \oplus \Omega_4 \oplus \Omega_{-4}$ of three operators acting on sections in Σ_0, Σ_4 and Σ_{-4}. On Σ_0, we have

$$\Omega_0 = \Delta_{T^c} + \frac{1}{8}\,\mathrm{scal}^g + \frac{1}{2} = (D^{1/3})^2 - \frac{1}{8}\,\mathrm{scal}^g - \frac{1}{2}\,.$$

In particular, the kernel of Ω_0 is trivial if $\mathrm{scal}^g \neq -4$. The Casimir operator on $\Sigma_4 \oplus \Sigma_{-4}$ is given by

$$\Omega_{\pm 4} = \Delta_{T^c} + \frac{1}{8}\,\mathrm{scal}^g - \frac{7}{2} = (D^{1/3})^2 - \frac{1}{8}\,\mathrm{scal}^g - \frac{1}{2}$$

and a non trivial kernel can only occur if $-4 \leq \mathrm{scal}^g \leq 28$. A spinor field ψ in the kernel of the Casimir operator Ω satisfies the equations

$$(D^{1/3})^2 \cdot \psi = \frac{1}{8}\,(4 + \mathrm{scal}^g)\psi\,, \quad T^c \cdot \psi = \pm 4\,\psi\,.$$

In particular, we obtain

$$\int_{M^5} \big\langle (D^g \pm 1)^2 \psi, \psi \big\rangle = \frac{1}{8} \int_{M^5} \big(4 + \mathrm{scal}^g\big)\,\|\psi\|^2\,,$$

and the first eigenvalue of the operator $(D^g \pm 1)^2$ is bounded by the scalar curvature,

$$\lambda_1 (D^g \pm 1)^2 \leq \frac{1}{8}\big(4 + \mathrm{scal}^g_{\max}\big)\,.$$

Let us consider special classes of Sasakian manifolds. A first case is $\mathrm{scal}^g = -4$. Then the formula for the Casimir operator simplifies,

$$\Omega_0 = \Delta_{T^c} = (D^{1/3})^2\,, \quad \Omega_{\pm 4} = \Delta_{\mathrm{T}} - 4 = (D^{1/3})^2\,.$$

If M^5 is compact, the kernel of the operator Ω_0 coincides with the space of ∇-parallel spinors in the bundle S_0. A spinor field ψ in the kernel the operator $\Omega_{\pm 4}$ is an eigenspinor of the Riemannian Dirac operator,

$$D^g(\psi) = \mp \psi\,, \quad \mathrm{T} \cdot \psi = \pm 4\,\psi\,.$$

Compact Sasakian manifolds admitting spinor fields in the kernel of Ω_0 are quotients of the 5-dimensional Heisenberg group (see [FI03a], Theorem 4.1). Moreover, the 5-dimensional Heisenberg group and its compact quotients admit spinor fields in the kernel of $\Omega_{\pm 4}$, too.

A second case is $\mathrm{scal}^g = 28$. Then

$$\Omega_0 = \Delta_{T^c} + 4 = (D^{1/3})^2 - 4\,, \quad \Omega_{\pm 4} = \Delta_{T^c} = (D^{1/3})^2 - 4\,.$$

The kernel of Ω_0 is trivial and the kernel of $\Omega_{\pm 4}$ coincides with the space of ∇-parallel spinors in the bundle $\mathrm{S}_{\pm 4}$. Sasakian manifolds admitting spinor fields of that type have been described in [FI02], Theorem 7.3 and Example 7.4.

If $-4 < \mathrm{Scal}^g < 28$, the kernel of the operator Ω_0 is trivial and the kernel of $\Omega_{\pm 4}$ depends on the geometry of the Sasakian structure. Let us discuss Einstein-Sasakian manifolds. Their scalar curvature equals $\mathrm{scal}^g = 20$ and the Casimir operators are

$$\Omega_0 = \Delta_{T^c} + 3\,, \quad \Omega_{\pm 4} = \Delta_{T^c} - 1 = (D^{1/3})^2 - 3\,.$$

If M^5 is simply-connected, there exist two Riemannian Killing spinors (see [FK90])

$$\nabla^g_X \psi_1 = \frac{1}{2}\, X \cdot \psi_1\,, \quad D^g(\psi_1) = -\frac{5}{2}\,\psi_1, \quad T^c \cdot \psi_1 = 4\psi_1\,,$$
$$\nabla^g_X \psi_2 = -\frac{1}{2}\, X \cdot \psi_2\,, \quad D^g(\psi_2) = \ \frac{5}{2}\,\psi_2\,, \quad T^c \cdot \psi_2 = -4\,\psi_2\,.$$

We compute the Casimir operator

$$\Omega(\psi_1) = -\frac{3}{4}\,\psi_1\,, \quad \Omega(\psi_2) = -\frac{3}{4}\,\psi_2\,.$$

In particular, the Casimir operator of a Einstein-Sasakian manifold has *negative* eigenvalues. The Riemannian Killing spinors are parallel sections in the bundles $\Sigma_{\pm 4}$ with respect to the flat connections $\nabla^{\pm}$

$$\nabla^+_X \psi := \nabla^g_X \psi - \frac{1}{2}\, X \cdot \psi \quad \text{in} \quad \Sigma_4\,, \quad \nabla^-_X \psi := \nabla^g_X \psi + \frac{1}{2}\, X \cdot \psi \quad \text{in} \quad \Sigma_{-4}\,.$$

We compare these connections with our canonical connection ∇:

$$\big(\nabla^{\pm}_X - \nabla_X\big) \cdot \psi^{\pm} = \pm \frac{i}{2}\, g(X,\xi) \cdot \psi^{\pm}\,, \quad \psi^{\pm} \in \Sigma_{\pm 4}\,.$$

The latter equation means that the bundle $\Sigma_4 \oplus \Sigma_{-4}$ equipped with the connection ∇ is equivalent to the 2-dimensional trivial bundle with the connection form

$$\mathcal{A} = \frac{i}{2}\,\eta \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\,.$$

The curvature of ∇ on these bundles is given by the formula

$$\mathcal{R}^{\nabla} = \frac{i}{2}\, d\eta \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = i\,(e_1 \wedge e_2 + e_3 \wedge e_4) \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\,.$$

Since the divergence $\mathrm{div}(\xi) = 0$ of the Killing vector field vanishes, the Casimir operator on $\Sigma_4 \oplus \Sigma_{-4}$ is the following operator acting on pairs of functions:

$$\Omega_4 \oplus \Omega_{-4} = \Delta_{\mathrm{T}} - 1 = \Delta - \frac{3}{4} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\,\xi\,.$$

Here Δ means the usual Laplacian of M^5 acting on functions and ξ is the differentiation in direction of the vector field ξ. In particular, the kernel of Ω coincides with solutions $f : M^5 \to \mathbb{C}$ of the equation

$$\Delta(f) - \frac{3}{4}\, f \pm i\,\xi(f) = 0\,.$$

The L^2-symmetric differential operators Δ and $i\,\xi$ commute. Therefore, we can diagonalize them simultaneously. The latter equation is solvable if and only if there exists a common eigenfunction

$$\Delta(f) = \mu\, f\,, \quad i\,\xi(f) = \lambda\, f\,, \quad 4(\mu+\lambda) - 3 = 0\,.$$

The Laplacian Δ is the sum of the *non-negative* horizontal Laplacian and the operator $(i\,\xi)^2$. Now, the conditions

$$\lambda^2 \leq \mu\,, \quad 4(\mu+\lambda) - 3 = 0$$

restrict the eigenvalue of the Laplacian, $0 \leq \mu \leq 3$. On the other side, by the Lichnerowicz-Obata Theorem, we have $5 \leq \mu$, a contradiction. In particular, we proved

Theorem 5.8. *The Casimir operator of a compact* 5*-dimensional Sasaki-Einstein manifold has trivial kernel; in particular, there are no* ∇^c*-parallel spinors.*

The same argument estimates the eigenvalues of the Casimir operator. It turns out that the smallest eigenvalues of Ω is negative and equals $-3/4$. The eigenspinors are the Riemannian Killing spinors. The next eigenvalue of the Casimir operator is at least

$$\lambda_2(\Omega) \geq \frac{17}{4} - \sqrt{5} \approx\ 2.014\,.$$

In the literature, similar results for almost Hermitian 6-manifolds and G_2-manifolds admitting a characteristic connection can be found.

5.5. **Some remarks on the common sector of type II superstring theory.** The mathematical model discussed in the common sector of type II superstring theory (also sometimes referred to as type I supergravity) consists of a Riemannian manifold (M^n, g), a metric connection ∇ with totally skew-symmetric torsion T and a non-trivial spinor field Ψ. Putting the full Ricci tensor aside for starters and assuming that the dilaton is constant, there are three equations relating these objects:

$$(*) \qquad \nabla\Psi = 0\,, \quad \delta(T) = 0\,, \quad T\cdot\Psi = \mu\cdot\Psi\,.$$

The spinor field describes the supersymmetry of the model. It has been our conviction throughout this article that this is the most important of the equations, as non-existence of ∇-parallel spinors implies the breakdown of supersymmetry. Yet, interesting things can be said if looking at all equations simultaneously. Since ∇ is a metric connection with totally skew-symmetric torsion, the divergences $\delta^{\nabla}(T) = \delta^g(T)$ of the torsion form coincide (see Proposition A.2). We denote this unique 2-form simply by $\delta(T)$. The third equation is an algebraic link between the torsion form T and the spinor field Ψ. Indeed, the 3-form T acts as an endomorphism in the spinor bundle and the last equation requires that Ψ is an eigenspinor for this endomorphism. Generically, $\mu = 0$ in the physical model; but the mathematical analysis becomes more transparent if we first include this parameter. A priori, μ may be an arbitrary function. Since T acts on spinors as a symmetric endomorphism, μ has to be real. Moreover, we will see that only real, constant parameters μ are possible. Recall that the conservation law $\delta(T) = 0$

implies that the Ricci tensor Ric^∇ of the connection ∇ is symmetric, see the Appendix. Denote by scal^∇ the ∇-scalar curvature and by scal^g the scalar curvature of the Riemannian metric. Based on the results of Section 5.2, the existence of the ∇-parallel spinor field yields the so called integrability conditions, i.e. relations between μ, T and the curvature tensor of the connection ∇.

Theorem 5.9 ([AFNP05, Thm 1.1.]). *Let $(M^n, g, \nabla, T, \Psi, \mu)$ be a solution of $(*)$ and assume that the spinor field Ψ is non-trivial. Then the function μ is constant and we have*

$$\|T\|^2 = \mu^2 - \frac{\mathrm{scal}^\nabla}{2} \geq 0 \,, \quad \mathrm{scal}^g = \frac{3}{2}\,\mu^2 + \frac{\mathrm{scal}^\nabla}{4} \,.$$

Moreover, the spinor field Ψ is an eigenspinor of the endomorphism defined by the 4-form dT,

$$dT \cdot \Psi = -\frac{\mathrm{scal}^\nabla}{2} \cdot \Psi \,.$$

Since μ has to be constant, equation $T \cdot \Psi = \mu \cdot \Psi$ yields:

Corollary 5.3. *For all vectors X, one has*

$$(\nabla_X T) \cdot \Psi = 0 \,.$$

The set of equations $(*)$ is completed in the common sector of type II superstring theory by the condition $\mathrm{Ric}^\nabla = 0$ and the requirement $\mu = 0$. In [Agr03], it had been shown that the existence of a non-trivial solution of this system implies $T = 0$ on *compact* manifolds. Theorem 5.9 enables us to prove the same result without compactness assumption and under the much weaker curvature assumption $\mathrm{scal}^\nabla = 0$:

Corollary 5.4. *Assume that there exists a spinor field $\Psi \neq 0$ satisfying the equations $(*)$. If $\mu = 0$ and $\mathrm{scal}^\nabla = 0$, the torsion form T has to vanish.*

This result underlines the strength of the algebraic identities in Theorem 5.9. Physically, this result may either show that the dilaton is a necessary ingredient for $T \neq 0$ (while it is not for $T = 0$) or that the set of equations is too restrictive (it is derived from a variational principle).

Remark 5.9. In the common sector of type II string theories, the "Bianchi identity" $dT = 0$ is often required in addition. It does not affect the mathematical structure of the equations $(*)$, hence we do not include it into our discussion.

On a naturally reductive space, even more is true. The generalized Kostant-Parthasarathy formula implies for the family of connections ∇^t:

Theorem 5.10 ([Agr03, Thm. 4.3]). *If the operator $\Omega_{\mathfrak{g}}$ is non-negative and if ∇^t is not the Levi-Civita connection, there do not exist any non trivial solutions to the equations*

$$\nabla^t \psi = 0 \,, \quad T^t \cdot \psi = 0 \,.$$

The last equation in type II string theory deals with the Ricci tensor Ric^∇ of the connection. Usually one requires for constant dilaton that the Ricci tensor has to vanish (see [GMW03]). The result above, however, indicates that this condition may be too strong. Understanding this tensor as an energy-momentum tensor, it seems to be more convenient to impose a weaker condition, namely

$$\mathrm{div}(\mathrm{Ric}^\nabla) = 0\,.$$

A subtle point is however the fact that there are a priori two different divergence operators. The first operator div^g is defined by the Levi-Civita connection of the Riemannian metric, while the second operator div^∇ is defined by the connection ∇. By Lemma A.2, they coincide if Ric is symmetric, that is, if $\delta T = 0$. This is for example satisfied if $\nabla T = 0$. We can then prove:

Corollary 5.5. *Let $(M^n, g, \nabla, T, \Psi, \mu)$ be a a manifold with metric connection defined by T and assume that there exists a spinor $0 \neq \psi \in \Sigma M^n$ such that*

$$\nabla\psi = 0\,, \quad \nabla T = 0\,, \quad T\cdot\psi = \mu\cdot\psi\,.$$

Then all scalar curvatures are constant and the divergence of the Ricci tensor vanishes, $\mathrm{div}(\mathrm{Ric}^\nabla) = 0$.

This is one possible way to weaken the original set of equations in such a way that the curvature condition follows from the other ones, as it is the case for $T = 0$—there, the existence of a ∇^g-parallel spinor implies $\mathrm{Ric}^g = 0$. Of course, only physics can provide a definite answer whether these or other possible replacements are 'the right ones'.

Incorporating a non-constant dilaton $\Phi \in C^\infty(M^n)$ is more subtle. The full set of equations reads in this case

$$\mathrm{Ric}^\nabla + \frac{1}{2}\delta T + 2\nabla^g d\Phi = 0\,, \quad \delta T = 2\,\mathrm{grad}(\Phi)\,\lrcorner\, T, \quad \nabla\psi = 0\,, \quad (2\,d\Phi - T)\cdot\psi = 0\,.$$

In some geometries, it is possible to interpret it as a partial conformal change of the metric. In dimension 5, this allows the proof that Φ basically has to be constant:

Theorem 5.11 ([FI03a])**.** *Let $(M^5, g, \xi, \eta, \varphi)$ be a normal almost contact metric structure with Killing vector field ξ, ∇^c its characteristic connection and Φ a smooth function on M^5. If there exists a spinor field $\psi \in \Sigma M^5$ such that*

$$\nabla^c\psi = 0\,, \quad (2\,d\Phi - T)\cdot\psi = 0\,,$$

then the function Φ is constant.

In higher dimension, the picture is less clear, basically because a clean geometric interpretation of Φ is missing.

Note added in proof. In January 2007, I learned from P. Nurowski that E. Ferapontov discovered a non-integrable $\mathrm{GL}(2,\mathbb{R})$-geometry in dimension 5 through his investigations of non-linear partial differential equations in hydrodynamics. It turns out that this is a an analogue of the 5-dimensional SO(3)-geometry discussed in Section 4.4 with indefinite signature $(3,2)$, which furthermore includes the conformal invariance of the defining quantities. M. Godlinski and P. Nurowski then observed that such geometries can be constructed as solution spaces of certain 5th order ordinary differential equations modulo contact transformations, yielding in particular non-homogeneous examples. For further details, please consult the forthcoming publications by Ferapontov and Godlinski/Nurowski.

Appendix A. Compilation of remarkable identities for connections with skew-symmetric torsion

We collect in this appendix some more or less technical formulas that one needs in the investigation of metric connections with skew-symmetric torsion. In order to keep this exposition readable, we decided to gather them in a separate section.

We tried to provide at least one reference with full proofs for every stated result; however, no claim is made whether these are the articles where these identities appeared for the first time. In fact, many of them have been derived and rederived by authors when needed, some had been published earlier but the authors had not considered it worth to publish a proof etc.

In this section, the connection ∇ is normalized as

$$\nabla_X Y = \nabla^g_X Y + \frac{1}{2} T(X,Y,*)\,, \quad \nabla_X \psi = \nabla^g_X \psi + \frac{1}{4}(X \lrcorner T)\cdot\psi\,.$$

It then easily follows that the Dirac operators are related by $D^\nabla = D^g + (3/4)T$.

Definition A.1. Recall that for any 3-form T, an algebraic 4-form σ_T quadratic in T may be defined by $2\,\sigma_T = \sum_{i=1}^{n}(e_i \lrcorner T)\wedge(e_i \lrcorner T)$, where $e_1,\dots,e_n$ denotes an orthonormal frame. Alternatively, σ_T may be written without reference to an orthonormal frame as

$$\sigma_T(X,Y,Z,V) = g\big(T(X,Y),T(Z,V)\big) + g\big(T(Y,Z),T(X,V)\big) + g\big(T(Z,X),T(Y,V)\big)\,.$$

We first encountered σ_T in the first Bianchi identity for metric connections with torsion T (Theorem 2.6).

Proposition A.1 ([Agr03, Prop. 3.1.])**.** *Let T be a 3-form, and denote by the same symbol its associated $(2,1)$-tensor. Then its square inside the Clifford algebra has no contribution of degree 6 and 2, and its scalar and fourth degree part are given by*

$$T_0^2 = \frac{1}{6}\sum_{i,j=1}^{n}\|T(e_i,e_j)\|^2 =: \|T\|^2, \quad T_4^2 = -2\cdot\sigma_T\,.$$

Lemma A.1 ([Agr03, Lemma 2.4.])**.** *If ω is an r-form and ∇ any connection with torsion, then*

$$\begin{aligned}(d\omega)(X_0,\,\dots,X_r) = &\sum_{i=0}^{r}(-1)^i(\nabla_{X_i}\omega)(X_0,\dots,\hat{X}_i,\dots,X_r)\\ &-\sum_{0\le i<j\le r}(-1)^{i+j}\omega(T(X_i,X_j),X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_r)\,.\end{aligned}$$

Corollary A.1 ([IP01]). *For a metric connection ∇ with torsion T, the exterior derivative of T is given by*

$$dT(X,Y,Z,V) = \overset{X,Y,Z}{\sigma}\,[(\nabla_X T)(Y,Z,V)] - (\nabla_V T)(X,Y,Z) + 2\,\sigma_T(X,Y,Z,V).$$

In particular, $dT = 2\sigma_T$ if $\nabla T = 0$.

Proposition A.2 ([AF04a, Prop. 5.1.]). *Let ∇ be a connection with skew-symmetric torsion and define the ∇-divergence of a differential form ω as*

$$\delta^\nabla \omega := -\sum_{i=1}^{n} e_i \lrcorner \nabla_{e_i}\omega\,.$$

Then, for any exterior form ω, the following formula holds:

$$\delta^\nabla \omega = \delta^g \omega - \frac{1}{2}\cdot \sum_{i,j=1}^{n} (e_i \lrcorner e_j \lrcorner T) \wedge (e_i \lrcorner e_j \lrcorner \omega)\,.$$

In particular, for the torsion form itself, we obtain $\delta^\nabla T = \delta^g T =: \delta T$.

Corollary A.2. *If the torsion form T is ∇-parallel, then its divergence vanishes,*

$$\delta^g T = \delta^\nabla T = 0\,.$$

We define the divergence for a $(0,2)$-tensor S as $\mathrm{div}^\nabla(S)(X) := \sum_i (\nabla_{e_i} S)(X, e_i)$ and denote by div^g the divergence with respect to the Levi-Civita connection ∇^g. Then

$$\mathrm{div}^g(S)(X) - \mathrm{div}^\nabla(S)(X) = -\frac{1}{2}\sum_{i,j=1}^{n} S(e_i,e_j)\,\mathrm{T}(e_i,X,e_j) = 0$$

because S is symmetric while T is antisymmetric, and we conclude immediately:

Lemma A.2 ([AFNP05, Lemma 1.1]). *If ∇ is a metric connection with totally skew-symmetric torsion and S a symmetric 2-tensor, then*

$$\mathrm{div}^g(S) = \mathrm{div}^\nabla(S)\,.$$

Theorem A.1 ([IP01]). *The Riemannian curvature quantities and the ∇-curvature quantities are related by*

$$\begin{aligned}\mathcal{R}^g(X,Y,Z,V) &= \mathcal{R}^\nabla(X,Y,Z,V) - \frac{1}{2}(\nabla_X T)(Y,Z,V) + \frac{1}{2}(\nabla_Y T)(X,Z,V)\\ &\quad - \frac{1}{4}\,g\big(T(X,Y),T(Z,V)\big) - \frac{1}{4}\sigma_T(X,Y,Z,V)\\ \mathrm{Ric}^g(X,Y) &= \mathrm{Ric}^\nabla(X,Y) + \frac{1}{2}\delta T(X,Y) - \frac{1}{4}\sum_{i=1}^{\dim M} g\big(T(e_i,X),T(e_i,Y)\big)\mathrm{scal}^\nabla\\ &= \mathrm{scal}^g - \frac{3}{2}\,\|T\|^2\,.\end{aligned}$$

In particular, Ric^∇ is symmetric if and only if $\delta T = 0$,

$$\mathrm{Ric}^\nabla(X,Y) - \mathrm{Ric}^\nabla(Y,X) = -\delta T(X,Y)\,.$$

References

[Ab84] E. Abbena, *An example of an almost Kähler manifold which is not Kählerian*, Bolletino U. M. I. (6) 3 A (1984), 383–392.

[AGS00] E. Abbena, S. Gabiero, S. Salamon, *Almost Hermitian geometry on six dimensional nilmanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) Vol. XXX (2001), 147–170.

[Agr02] I. Agricola, *Connexions sur les espaces homogènes naturellement réductifs et leurs opérateurs de Dirac*, C. R. Acad. Sci. Paris Sér. I 335 (2002), 43–46.

[Agr03] ———, *Connections on naturally reductive spaces, their Dirac operator and homogeneous models in string theory*, Comm. Math. Phys. **232** (2003), 535–563.

[ACF05] I. Agricola, S. Chiossi, A. Fino, *Solvmanifolds with integrable and non-integrable G_2-structures*, math.DG/0510300, to appear in Differential Geom. Appl.

[AF01] I. Agricola and Th. Friedrich, *Global Analysis – Differential forms in Calculus, Geometry and Physics*, Graduate Studies in Mathematics, Publications of the AMS 2002, Providence, Rhode Island 2002.

[AF03] ———, *Killing spinors in supergravity with* 4*-fluxes*, Classical Quantum Gravity **20** (2003), 4707–4717.

[AF04a] ———, *On the holonomy of connections with skew-symmetric torsion*, Math. Ann. **328** (2004), 711–748.

[AF04b] ———, *The Casimir operator of a metric connection with totally skew-symmetric torsion*, J. Geom. Phys. **50** (2004), 188–204.

[AF05] ———, *Geometric structures of vectorial type*, math.DG/0509147, to appear in J. Geom. Phys.

[AFNP05] I. Agricola, T. Friedrich, P.-A. Nagy, C. Puhle, *On the Ricci tensor in the common sector of type II string theory*, Classical Quantum Gravity **22** (2005), 2569–2577.

[AT04] I. Agricola and Chr. Thier, *The geodesics of metric connections with vectorial torsion*, Ann. Global Anal. Geom. **26** (2004), 321–332.

[Ale68] D. V. Alekseevski, *Riemannian spaces with exceptional holonomy groups*, Func. Anal. Prilozh. **2** (1968), 1–10.

[AMP98] D. V. Alekseevsky, S. Marchiafava, M. Pontecorvo, *Compatible almost complex structures on quaternion Kähler manifolds*, Ann. Global Anal. Geom. **16** (1998), 419–444.

[AC05] D. V. Alekseevsky, V. Cortés, *Classification of pseudo-Riemannian symmetric spaces of quaternionic Kähler type*, Vinberg, Ernest (ed.), Lie groups and invariant theory. Providence, RI: American Mathematical Society 213 (AMS). Translations. Series 2. Adv. Math. Sci. **56** (2005), 33–62.

[AG86] V. Aleksiev, G. Ganchev, *On the classification of the almost contact metric manifolds*, Mathematics and education in mathematics, Proc. 15th Spring Conf., Sunny Beach/Bulg. 1986, 155–161.

[Ale03] B. Alexandrov, *$Sp(n)U(1)$-connections with parallel totally skew-symmetric torsion*, J. Geom. Phys. **57** (2006), 323–337, math.DG/0311248.

[Ale04] ———, *On weak holonomy*, Math. Scand. **96** (2005), 169–189.

[AFS05] B. Alexandrov, Th. Friedrich, N. Schoemann, *Almost Hermitian* 6*-manifolds revisited*, J. Geom. Phys. **53** (2005), 1–30.

[AI00] B. Alexandrov, S. Ivanov, *Dirac operators on Hermitian spin surfaces*, Ann. Global Anal. Geom. **18** (2000), 529–539.

[Ali01] T. Ali, *M-theory on seven manifolds with G-fluxes*, hep-th/0111220.

[AS53] W. Ambrose and I. M. Singer, *A theorem on holonomy*, Trans. Amer. Math. Soc. **75** (1953), 428–443.

[AS58] ———, *On homogeneous Riemannian manifolds*, Duke Math. J. **25** (1958), 647–669.

[ADM01] V. Apostolov, T. Drăghici, A. Moroianu, *A splitting theorem for Kähler manifolds whose Ricci tensors have constant eigenvalues*, Internat. J. Math. **12** (2001), 769–789.

[AAD02] V. Apostolov, J. Armstrong, T. Drăghici, *Local rigidity of certain classes of almost Kähler 4-manifolds*, Ann. Global Anal. Geom. **21** (2002), 151–176.

[Arm98] J. Armstrong, *Almost Kähler geometry*, Ph. D. Thesis, Oxford University, 1998.

[AS77] M. Atiyah and W. Schmid, *A geometric construction for the discrete series for semisimple Lie groups*, Invent. Math. **42** (1977), 1–62.

[AW01] M. Atiyah and E. Witten, *M-theory dynamics on a manifold of G_2 holonomy*, Adv. Theor. Math. Phys. **6** (2002), 1–106.

[Atr75] J. E. D'Atri, *Geodesic spheres and symmetries in naturally reductive spaces*, Michigan Math. J. **22** (1975), 71–76.

[AZ79] J. E. D'Atri and W. Ziller, *Naturally reductive metrics and Einstein metrics on compact Lie groups*, Mem. Amer. Math. Soc. **18** (1979).

[Bär93] Chr. Bär, *Real Killing spinors and holonomy*, Comm. Math. Phys. **154** (1993), 509–521.

[BS04] B. Banos, A. F. Swann, *Potentials for hyper-Kähler metrics with torsion*, Classical Quantum Gravity **21** (2004), 3127–3135.

[BFGK91] H. Baum, Th. Friedrich, R. Grunewald, I. Kath, *Twistors and Killing spinors on Riemannian manifolds*, Teubner-Texte zur Mathematik, Band 124, Teubner-Verlag Stuttgart/Leipzig, 1991.

[BJ03] K. Behrndt, C. Jeschek, *Fluxes in M-theory on 7-manifolds and G-structures*, hep-th/0302047.

[Bel00] F. A. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000), 1–40.

[BM01] F. A. Belgun and A. Moroianu, *Nearly Kähler 6-manifolds with reduced holonomy*, Ann. Global Anal. Geom. **19** (2001), 307–319.

[Ber55] M. Berger, *Sur les groupes d'holonomie des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France **83** (1955), 279–330.

[Ber61] M. Berger, *Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive*, Ann. Sc. Norm. Sup. Pisa **15** (1961), 179–246.

[BTVh95] J. Berndt, F. Tricerri, L. Vanhecke, *Generalized Heisenberg groups and Damek-Ricci harmonic spaces*, LNM 1598, Springer, 1995.

[Bes87] A. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete Bd. 10, Springer-Verlag Berlin-Heidelberg 1987.

[BDS01] A. Bilal, J.-P. Derendinger and K. Sfetsos, *Weak G_2-holonomy from self-duality, flux and supersymmetry*, Nuclear Phys. B **628** (2002), 112–132.

[Bis89] J. M. Bismut, *A local index theorem for non-Kählerian manifolds*, Math. Ann. **284** (1989), 681–699.

[Bla76] D. E. Blair, *Contact manifolds in Riemannian geometry*, LNM 509 (1976), Springer.

[Bla02] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics vol. 203, Birkhäuser, 2002.

[BlVh87] D. E. Blair and L. Vanhecke, *New characterization of φ-symmetric spaces*, Kodai Math. J. **10** (1987), 102–107.

[Bob06] M. Bobieński, *The topological obstructions to the existence of an irreducible* SO(3)*-structure on a five manifold*, math.DG/0601066.

[BN05] M. Bobieński and P. Nurowski, *Irreducible* SO(3)*-geometries in dimension five*, to appear in J. Reine Angew. Math.; math.DG/0507152.

[Bon66] E. Bonan, *Sur les variétés riemanniennes à groupe d'holonomie G_2 ou* Spin(7), C. R. Acad. Sc. Paris **262** (1966), 127–129.

[BG99] C. P. Boyer and K. Galicki, *3-Sasakian manifolds*, in *Essays on Einstein manifolds*, (ed. by C. LeBrun and M. Wang), International Press 1999.

[BG01] ———, *Einstein manifolds and contact geometry*, Proc. Amer. Math. Soc. **129** (2001), 2419–2430.

[BG07] ———, *Sasakian Geometry*, Oxford Mathematical Monographs, Oxford University Press, to appear 2007.

[BGM94] C. P. Boyer, K. Galicki, B. M. Mann, *The geometry and topology of 3-Sasakian manifolds*, J. Reine Angew. Math. **455** (1994), 183–220.

[BRX02] L. Brink, P. Ramond and X. Xiong, *Supersymmetry and Euler multiplets*, hep-th/0207253.

[BG72] R. B. Brown and A. Gray, *Riemannian manifolds with holonomy group* Spin(7), Differential Geometry in honor of K. Yano, Kinokiniya, Tokyo, 1972, 41–59.

[Br87] R. L. Bryant, *Metrics with exceptional holonomy*, Ann. of Math. **126** (1987), 525–576.

[Br96] ———, *Classical, exceptional, and exotic holonomies: a status report.* Actes de la Table ronde de Géométrie Différentielle en l'honneur de M. Berger. Collection SMF Séminaires et Congrès 1 (1996), 93–166.

[Br03] ———, *Some remarks on G_2-structures*, in Proceeding of the 2004 Gokova Conference on Geometry and Topology (May, 2003), math.DG/0305124.

[BrS89] R. L. Bryant, and S. M. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** (1989), 829–850.

[Bu04] J. Bureš, *Multisymplectic structures of degree three of product type on 6-dimensional manifolds*, Suppl. Rend. Circ. Mat. Palermo II, Ser. bf 72 (2004), 91–98.

[BV03] J. Bureš and J. Vanžura, *Multisymplectic forms of degree three in dimension seven*, Suppl. Rend. Circ. Mat. Palermo II, Ser. **71** (2003), 73–91.

[Bu05] J. B. Butruille, *Classification des variétés approximativement kähleriennes homogènes*, Ann. Global Anal. Geom. **27** (2005), 201–225.

[CP99] D. M. J. Calderbank and H. Pedersen, *Einstein-Weyl geometry*, Surveys in differential geometry: Essays on Einstein manifolds. Lectures on geometry and topology, J. Diff. Geom. Suppl. **6** (1999), 387–423.

[Car22] E. Cartan, *Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion*, C. R. Ac. Sc. **174** (1922), 593–595.

[Car23] ———, *Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie)*, Ann. Ec. Norm. Sup. **40** (1923), 325–412, part one.

[Car24a] ———, *Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie, suite)*, Ann. Ec. Norm. Sup. **41** (1924), 1–25, part one (continuation).

[Car24b] ———, *Les récentes généralisations de la notion d'espace*, Bull. Sc. Math. **48** (1924), 294–320.

[Car25] ———, *Sur les variétés à connexion affine et la théorie de la relativité généralisée (deuxième partie)*, Ann. Ec. Norm. Sup. **42** (1925), 17–88, part two. English transl. of both parts by A. Magnon and A. Ashtekar, *On manifolds with an affine connection and the theory of general relativity.* Napoli: Bibliopolis (1986).

[Cha70] I. Chavel, *A class of Riemannian homogeneous spaces*, J. Differential Geom. **4** (1970), 13–20.

[ChG90] D. Chinea and G. Gonzales, *A classification of almost contact metric manifolds*, Ann. Mat. Pura Appl. **156** (1990), 15–36.

[ChM92] D. Chinea and J. C. Marrero, *Classifications of almost contact metric structures*, Rev. Roumaine Math. Pures Appl. **37** (1992), 581–599.

[ChF05] S. G. Chiossi, A. Fino, *Conformally parallel G_2-structures on a class of solvmanifolds*, Math. Z. **252** (2006), 825–848.

[CS02] S. Chiossi and S. Salamon, *The intrinsic torsion of $SU(3)$ and G_2 structures*, in O. Gil-Medrano et. al. (eds.), Proc. Intern. Conf. Valencia, Spain, July 8-14, 2001, Singapore, World Scientific, 115–133 (2002).

[CI03] R. Cleyton and S. Ivanov, *On the geometry of closed G_2-structures*, math.DG/0306362.

[CI06a] ———, *Curvature decomposition of G_2 manifolds*, to appear.

[CI06b] ———, *Conformal equivalence between certain geometries in dimension* 6 *and* 7, math.DG/0607487.

[CS02] R. Cleyton and A. Swann, *Cohomogeneity-one G_2-structures*, J. Geom. Phys. **44** (2002), 202–220.

[CS04] ———, *Einstein metrics via intrinsic or parallel torsion*, Math. Z. **247** (2004), 513–528.

[CKL01] G. Curio, B. Körs and D. Lüst, *Fluxes and branes in type II vacua and M-theory geometry with G_2 and $Spin(7)$ holonomy*, hep-th/0111165.

[DI01] P. Dalakov and S. Ivanov, *Harmonic spinors of Dirac operators of connections with torsion in dimension* 4, Classical Quantum Gravity **18** (2001), 253–265.

[dWNW85] B. de Witt, H. Nicolai and N. P. Warner, *The embedding of gauged $n = 8$ supergravity into $d = 11$ supergravity*, Nuclear Phys. B **255** (1985), 29.

[dWSHD87] B. de Witt, D. J. Smit, and N. D. Hari Dass, *Residual supersymmetry of compactified $D = 10$ Supergravity*, Nuclear Phys. B **283** (1987), 165.

[Djo83] D. Ž. Djoković, *Classification of trivectors of an eight-dimensional real vector space*, Linear and Multilinear Algebra **13** (1983), 3–39.

[DF02] I. G. Dotti and A. Fino, *HyperKähler torsion structures invariant by nilpotent Lie groups*, Classical Quantum Gravity **19** (2002), 551–562.

[DO98] S. Dragomir, L. Ornea, *Locally conformal Kähler geometry*, Progr. Math. vol. 155, Birkhäuser Verlag, 1998.

[Duf02] M. J. Duff, *M-theory on manifolds of G_2-holonomy: the first twenty years*, hep-th/0201062.

[Fer86] M. Fernández, *A classification of Riemannian manifolds with structure group* Spin(7), Ann. Mat. Pura Appl. **143** (1986), 101–122.

[Fer87] ———, *An example of a compact calibrated manifold associated with the exceptional Lie group G_2*, J. Differential Geom. **26** (1987), 367–370.

[FG82] M. Fernández and A. Gray, *Riemannian manifolds with structure group* G_2, Ann. Mat. Pura Appl. **132** (1982), 19–45.

[FP02] J. Figueroa-O'Farrill, G. Papadopoulos, *Maximally supersymmetric solutions of ten- and eleven-dimensional supergravities*, hep-th/0211089.

[Fin94] A. Fino, *Almost contact homogeneous manifolds*, Riv. Mat. Univ. Parma (5) **3** (1994), 321–332.

[Fin95] ———, *Almost contact homogeneous structures*, Boll. Un. Mat. Ital. A 9 (1995), 299–311.

[Fin98] ———, *Intrinsic torsion and weak holonomy*, Math. J. Toyama Univ. **21** (1998), 1–22.

[Fin05] ———, *Almost Kähler* 4*-dimensional Lie groups with J-invariant Ricci tensor*, Differential Geom. Appl. 23 (2005), 26-37.

[FG03] A. Fino and G. Grantcharov, *Properties of manifolds with skew-symmetric torsion and special holonomy*, Adv. Math. **189** (2004), 439–450.

[FPS04] A. Fino, M. Parton, S. Salamon, *Families of strong KT structures in six dimensons*, Comment. Math. Helv. **79** (2004), 317–340.

[Fri80] Th. Friedrich, *Der erste Eigenwert des Dirac-Operators einer kompakten, Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung*, Math. Nachr. **97** (1980), 117–146.

[Fri00] ———, *Dirac operators in Riemannian geometry*, Grad. Stud. Math. vol. 25, 2000.

[Fr01] ———, *Weak* Spin(9)*-structures on* 16*-dimensional Riemannian manifolds*, Asian Math. J. **5** (2001), 129–160.

[Fr03a] ———, *$Spin(9)$-structures and connections with totally skew-symmetric torsion*, J. Geom. Phys. **47** (2003), 197–206.

[Fri03b] ———, *On types of non-integrable geometries*, Rend. Circ. Mat. Palermo (2) Suppl. **71** (2003), 99–113.

[Fri06] ———, *G_2-manifolds with parallel characteristic torsion*, math.DG/0604441, to appear in Differential Geom. Appl.

[FG85] Th. Friedrich and R. Grunewald, *On the first eigenvalue of the Dirac operator on* 6*-dimensional manifolds*, Ann. Global Anal. Geom. **3** (1985), 265–273.

[FI02] Th. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. **6** (2002), 303–336.

[FI03a] ———, *Almost contact manifolds, connections with torsion and parallel spinors*, J. Reine Angew. Math. 559 (2003), 217–236.

[FI03b] ———, *Killing spinor equations in dimension* 7 *and geometry of integrable* G_2*-manifolds*, J. Geom. Phys. **48** (2003), 1–11.

[FK89] Th. Friedrich and I. Kath, *Einstein manifolds of dimension five with small eigenvalues of the Dirac operator*, J. Differential Geom. **19** (1989), 263–279.

[FK90] ———, 7*-dimensional compact Riemannian manifolds with Killing spinors*, Comm. Math. Phys. **133** (1990), 543–561.

[FKMS97] Th. Friedrich, I. Kath, A. Moroianu and U. Semmelmann, *On nearly parallel* G_2*-structures*, J. Geom. Phys. **3** (1997), 256–286.

[FK00] Th. Friedrich and E. C. Kim, *The Einstein-Dirac equation on Riemannian spin manifolds*, J. Geom. Phys. **33** (2000), 128–172.

[FS79] Th. Friedrich and S. Sulanke, *Ein Kriterium für die formale Selbstadjungiertheit des Dirac-Operators*, Coll. Math. XL (1979), 239–247.

[FY05] J.-X. Fu and S.-T. Yau, *Existence of supersymmetric Hermitian metrics with torsion on non-Kähler manifolds*, hep-th/0509028.

[FP05] A. Fujiki and M. Pontecorvo, *On Hermitian geometry of complex surfaces*, in O. Kowalski et al. (ed.), *Complex, contact and symmetric manifolds. In honor of L. Vanhecke. Selected lectures from the international conference "Curvature in Geometry" held in Lecce, Italy, June 11-14, 2003.* Birkhäuser, Progr. Math. **234** (2005), 153–163.

[FI78] T. Fukami and S. Ishihara, *Almost Hermitian structure on* S^6, Hokkaido Math. J. **7** (1978), 206–213.

[GHR84] S. J. Gates, C. M. Hull, M. Rocek, *Twisted multiplets and new supersymmetric nonlinear sigma models*, Nuclear Phys. B **248** (1984), 157.

[GKMW01] J. Gauntlett, N. Kim, D. Martelli, D. Waldram, *Fivebranes wrapped on SLAG three-cycles and related geometry*, hep-th/0110034.

[GMW03] J. P. Gauntlett, D. Martelli and D. Waldram, *Superstrings with intrinsic torsion*, Phys. Rev. D (3) **69** (2004), 086002.

[Gau95] P. Gauduchon, *Structures de Weyl-Einstein, espaces de twisteurs et variétés de type* $S^1 \times S^3$, J. Reine Angew. Math. **469** (1995), 1–50.

[Gau97] ———, *Hermitian connections and Dirac operators*, Boll. Un. Mat. Ital. Ser. VII **2** (1997), 257–289.

[GT98] P. Gauduchon and K. P. Tod *Hyper-Hermitian metrics with symmetry*, J. Geom. Phys. **25** (1998), 291–304.

[Gil75] P. B. Gilkey, *The spectral geometry of a Riemannian manifold*, J. Differential Geom. **10** (1975), 601–618.

[GLPS02] G. W. Gibbons, H. Lü, C. N. Pope, and K. S. Stelle, *Supersymmetric domain walls from metrics of special holonomy*, Nuclear Phys. B **623** (2002), 3–46.

[GKN00] M. Godlinski, W. Kopczynski, P. Nurowski, *Locally Sasakian manifolds*, Classical Quantum Gravity **17** (2000), L105–L115.

[Goe99] S. Goette, *Equivariant* η*-invariants on homogeneous spaces*, Math. Z. **232** (1999), 1–42.

[Gol69] S. I. Goldberg, *Integrabilty of almost Kähler manifolds*, Proc. Amer. Math. Soc. **21** (1969), 96–100.

[GP02] E. Goldstein, S. Prokushkin, *Geometric model for complex non-Kähler manifolds with* $\mathrm{SU}(3)$*-structure*, Comm. Math. Phys. **251** (2004), 65–78.

[GZ84] C. Gordon and W. Ziller, *Naturally reductive metrics of nonpositive Ricci curvature*, Proc. Amer. Math. Soc. **91** (1984), 287–290.

[GP00] G. Grantcharov and Y. S. Poon, *Geometry of hyper-Kähler connections with torsion*, Comm. Math. Phys. **213** (2000), 19–37.

[Gra70] A. Gray, *Nearly Kähler manifolds*, J. Differential Geom. **4** (1970), 283–309.

[Gra71] ———, *Weak holonomy groups*, Math. Z. **123** (1971), 290–300.

[Gra76] ———, *The structure of nearly Kähler manifolds*, Math. Ann. **223** (1976), 233–248.

[GH80] A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. **123** (1980), 35–58.

[GSW87] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory. Volume 2: Loop amplitudes, anomalies and phenomenology.* Cambridge Monogr. Math. Phys. 1987.

[GKRS98] B. H. Gross, B. Kostant, P. Ramond, and S. Sternberg, *The Weyl character formula, the half-spin representations, and equal rank subgroups*, Proc. Natl. Acad. Sci. USA **95** (1998), no. 15, 8441–8442.

[Gru90] R. Grunewald, *Six-dimensional Riemannian manifolds with a real Killing spinor*, Ann. Glob. Anal. Geom. **8** (1990), 43–59.

[Gu35] G. B. Gurevich, *Classification of trivectors of rank* 8 (in Russian), Dokl. Akad. Nauk SSSR **2** (1935), 353–355.

[Gu48] ———, *Algebra of trivectors II* (in Russian), Trudy Sem. Vektor. Tenzor. Anal. **6** (1948), 28–124.

[HHKN76] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick, J. M. Nester *General relativity with spin and torsion: Foundations and prospects*, Rev. Modern Phys. **48** (1976), 393–416.

[HMMN95] F. W. Hehl, J. D. McCrea, E. W. Mielke, Y. Ne'eman, *Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance*, Phys. Rep. **258** (1995), 1–171.

[Hel78] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Pure Appl. Math. vol. 80, Acad. Press, New York, 1978.

[Hit74] N. Hitchin, *Harmonic spinors*, Adv. Math. **14** (1974), 1–55.

[Hit00] ———, *The geometry of three-forms in six and seven dimensions*, J. Differential Geom. **55** (2000), 547–576.

[Hit01] ———, *Stable forms and special metrics*, math.DG/0107101; Contemp. Math. **288** (2001), 70–89.

[HP87] P. S. Howe and G. Papadopoulos, *Ultraviolet behavior of two-dimensional supersymmetric nonlinear sigma models*, Nuclear Phys. B **289** (1987), 264–276.

[HP92] ———, *Finitness and anomalies in* $(4, 0)$ *supersymmetric sigma models*, Nuclear Phys. B **381** (1992), 360.

[HP96] ———, *Twistor spaces for hyper-Kähler manifolds with torsion*, Phys. Lett. B **379** (1996), 80–86.

[HP02] J.-S. Huang and P. Pandžić, *Dirac cohomology, unitary representations and a proof of a conjecture of Vogan*, J. Amer. Math. Soc. **15** (2002), 185–202.

[Hu86] C. M. Hull, *Lectures On Nonlinear Sigma Models And Strings*, PRINT-87-0480(Cambridge); Lectures given at Super Field Theories Workshop, Vancouver, Canada, July 25-Aug 6, 1986.

[Igu70] J.-I. Igusa,*A classification of spinors up to dimension twelve*, Amer. J. Math. **92** (1970), 997–1028.

[Ike75] A. Ikeda, *Formally self adjointness for the Dirac operator on homogeneous spaces*, Osaka J. Math. **12** (1975), 173–185.

[In62] L. Infeld (volume dedicated to), *Recent developments in General Relativity*, Oxford, Pergamon Press & Warszawa, PWN, 1962.

[Iv04] S. Ivanov, *Connections with torsion, parallel spinors and geometry of* Spin(7)*-manifolds*, Math. Res. Lett. **11** (2004), 171–186.

[IM04] S. Ivanov and I. Minchev, *Quaternionic Kähler and hyperKähler manifolds with torsion and twistor spaces*, J. Reine Angew. Math. **567** (2004), 215–233.

[IP01] S. Ivanov and G. Papadopoulos, *Vanishing theorems and string background*, Classical Quantum Gravity **18** (2001), 1089–1110.

[IPP05] S. Ivanov, M. Parton and P. Piccinni, *Locally conformal parallel* G_2*- and* Spin(7)*-structures*, math.DG/0509038, to appear in Math. Res. Lett. **13** (2006).

[Jel96] W. Jelonek, *Some simple examples of almost Kähler non-Kähler structures*, Math. Ann. **305** (1996), 639–649.

[Jen75] G. Jensen, *Imbeddings of Stiefel manifolds into Grassmannians*, Duke Math. J. **42** (1975), 397–407.

[Joy92] D. Joyce, *Compact hypercomplex and quaternionic manifolds*, J. Differential Geom. **35** (1992), 743–761.

[Joy96a] ———, *Compact Riemannian 7-manifolds with holonomy G_2. I*, J. Differential Geom. **43** (1996), 291–328.

[Joy96b] ———, *Compact Riemannian 7-manifolds with holonomy G_2. II*, J. Differential Geom. **43** (1996), 329–375.

[Joy96c] ———, *Compact 8-manifolds with holonomy* Spin(7), Invent. Math. **123** (1996), 507–552.

[Joy00] ———, *Compact manifolds with special holonomy*, Oxford Science Publ., 2000.

[Kap83] A. Kaplan, *On the geometry of groups of Heisenberg type*, Bull. London Math. Soc. **15** (1983), 35–42.

[Ka71] T. Kashiwada, *A note on a Riemannian space with Sasakian 3-structure*, Natur. Sci. Rep. Ochanomizu Univ. **22** (1971), 1–2.

[Ka01] ———, *On a contact 3-structure*, Math. Z. **238** (2001), 829–832.

[KS01] G. Ketsetzis and S. Salamon, *Complex structures on the Iwasawa manifold*, Adv. Geom. **4** (2004), 165–179.

[Kib61] T. W. B. Kibble, *Lorentz invariance and the gravitational field*, J. Math. Phys. **2** (1961), 212–221.

[Kl72] F. Klein, *Das Erlanger Programm*, Ostwalds Klassiker der exakten Wissenschaften Band 253, Verlag H. Deutsch, Frankfurt a. M., 1995.

[Kir86] K.-D. Kirchberg, *An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature*, Ann. Global Anal. Geom. **4** (1986), 291–325.

[Kir93] ———, *Killing spinors on Kähler manifolds*, Ann. Global Anal. Geom. **11** (1993), 141–164.

[Kir05] ———, *Integrability conditions for almost Hermitian and almost Kähler 4-manifolds*, math.DG/0605611.

[Kir77] V. F. Kirichenko, *K-spaces of maximal rank*, Mat. Zametki **22** (1977), 465–476.

[KR02] V. F. Kirichenko, A. R. Rustanov, *Differential geometry of quasi-Sasakian manifolds*, Sb. Math. **193** (2002), 1173-1201; translation from Mat. Sb. **193** (2002), 71–100.

[KN63] S. Kobayashi and K. Nomizu, *Foundations of differential geometry I*, Wiley Classics Library, Wiley Inc., Princeton, 1963, 1991.

[KN69] ———, *Foundations of differential geometry II*, Wiley Classics Library, Wiley Inc., Princeton, 1969, 1996.

[Kop73] W. Kopczyński, *An anisotropic universe with torsion*, Phys. Lett. A **43** (1973), 63–64.

[Kos56] B. Kostant, *On differential geometry and homogeneous spaces II*, Proc. N. A. S. **42** (1956), 354–357.

[Kos99] ———, *A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups*, Duke Math. J. **100** (1999), 447–501.

[Kost02] ———, *Dirac cohomology for the Cubic Dirac operator*, in: Joseph, Anthony (ed.) et al., Studies in memory of Issai Schur. Basel: Birkhäuser. Progr. Math. **210** (2003), 69–93.

[KM01] B. Kostant and P. Michor, *The generalized Cayley map from an algebraic group to its Lie algebra*, preprint (arXiv:math.RT/0109066v1, 10 Sep 2001), to appear in *The Orbit Method in Geometry and Physics* (A. A. Kirillov Festschrift), Progr. Math. (2003).

[Kov03] A. Kovalev, *Twisted connected sums and special Riemannian holonomy*, J. Reine Angew. Math. **565** (2003), 125–160.

[KVh83] O. Kowalski and L. Vanhecke, *Four-dimensional naturally reductive homogeneous spaces*, Differential geometry on homogeneous spaces, Conf. Torino/Italy 1983, Rend. Sem. Mat., Torino, Fasc. Spec. (1983), 223-232.

[KVh84] ———, *A generalization of a theorem on naturally reductive homogeneous spaces*, Proc. Amer. Math. Soc. **91** (1984), 433–435.

[KVh85] ———, *Classification of five-dimensional naturally reductive spaces*, Math. Proc. Cambridge Philos. Soc. **97** (1985), 445–463.

[KW87] O. Kowalski and S. Wegrzynowski, *A classification of* 5*-dimensional* φ*-symmetric spaces*, Tensor, N. S. **46** (1987), 379–386.

[Kre91] E. Kreyszig, *Differential geometry*, Dover Publ., inc., New York, 1991, unabridged republication of the 1963 printing.

[Lan00] G. Landweber, *Harmonic spinors on homogeneous spaces*, Represent. Theory **4** (2000), 466–473.

[LY05] J.-L. Li and S.-T. Yau, *Existence of supersymmetric string theory with torsion*, J. Differential Geom. **70** (2005), 143–182 and hep-th/0411136.

[Li63] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris **257** (1963), 7–9.

[Li87] ———, *Spin manifolds, Killing spinors and universality of the Hijazi inequality*, Lett. Math. Phys. **13** (1987), 331–344.

[Lich88] ———, *Les spineurs-twisteurs sur une variété spinorielle compacte*, C. R. Acad. Sci. Paris Sér. I Math. **306** (1988), 381–385.

[LT89] D. Lüst and S. Theisen, *Lectures on String Theory*, Springer-Verlag, 1989.

[MSY06] D. Martelli, J. Sparks, S.-T. Yau, *Sasaki-Einstein Manifolds and Volume Minimisation*, hep-th/0603021.

[MC05] F. Martín Cabrera, *Special almost Hermitian geometry*, J. Geom. Phys. **55** (2005), 450–470.

[CMS96] F. Martín Cabrera, M. D. Monar Hernandez, A. F. Swann, *Classification of* G_2*-structures*, J. London Math. Soc. II. Ser. **53** (1996), 407–416.

[MCS04] F. Martín Cabrera and A. F. Swann, *Almost Hermitian structures and quaternionic geometries*, Differential Geom. Appl. **21** (2004), 199–214.

[McKW89] Y. McKenzie Wang, *Parallel spinors and parallel forms*, Ann. Global Anal. Geom. **7** (1989), 59–68.

[MZ04] S. Mehdi and R. Zierau, *Principal Series Representations and Harmonic Spinors*, to appear in Adv. Math. (preprint at `http://www.math.okstate.edu/~zierau/papers.html`).

[MS00] J. Michelson, A. Strominger, *The geometry of (super) conformal quantum mechanics*, Comm. Math. Phys. **213** (2000), 1–17.

[Miq82] V. Miquel, *The volume of small geodesic balls for a metric connection*, Compositio Math. **46** (1982), 121–132.

[Miq01] ———, *Volumes of geodesic balls and spheres associated to a metric connection with torsion*, Contemp. Math. **288** (2001), 119–128.

[Na95] S. Nagai, *Naturally reductive Riemannian homogeneous structure on a homogeneous real hypersurface in a complex space form*, Boll. Un. Mat. Ital. A (7) **9** (1995), 391–400.

[Na96] ———, *Naturally reductive Riemannian homogeneous structures on some classes of generic submanifolds in complex space forms*, Geom. Dedicata **62** (1996), 253–268.

[Na97] ———, *The classification of naturally reductive homogeneous real hypersurfaces in complex projective space*, Arch. Math. **69** (1997), 523–528.

[Na02a] P.-A. Nagy, *Nearly Kähler geometry and Riemannian foliations*, Asian J. Math. **6** (2002) 481–504.

[Na02b] ———, *On nearly-Kähler geometry*, Ann. Global Anal. Geom. **22** (2002), 167–178.

[NP97] P. Nurowski, M. Przanowski, *A four-dimensional example of Ricci-flat metric admitting almost-Kähler non-Kähler structure*, ESI preprint 477, 1997; Classical Quantum Gravity **16** (1999), L9–L16.

[Nu06] P. Nurowski, *Distinguished dimensions for special Riemannian geometries*, math.DG/0601020.

[Par72] R. Parthasarathy, *Dirac operator and the discrete series*, Ann. of Math. **96** (1972), 1–30.

[Pen83] R. Penrose, *Spinors and torsion in general relativity*, Found. of Phys. 13 (1983), 325-339.

[PS01] Y. S. Poon and A. F. Swann, *Potential functions of HKT spaces*, Classical Quantum Gravity **18** (2001), 4711–4714.

[PS03] ———, *Superconformal symmetry and HyperKähler manifolds with torsion*, Comm. Math. Phys. **241** (2003), 177–189.

[Pu06] Chr. Puhle, *The Killing equation with higher order potentials*, Ph. D. Thesis, Humboldt-Universität zu Berlin, 2006/07.

[Reich07] W. Reichel, *Über trilineare alternierende Formen in sechs und sieben Veränderlichen und die durch sie definierten geometrischen Gebilde*, Druck von B. G. Teubner in Leipzig 1907, Dissertation an der Universität Greifswald.

[Roc92] M. Rocek, *Modified Calabi-Yau manifolds with torsion*, in: Yau, Shing-Tung (ed.), Essays on mirror manifolds. Cambridge, MA: International Press. 1992, 480–488.

[RT03] M. L. Ruggiero and A. Tartaglia, *Einstein–Cartan theory as a theory of defects in space-time*, Amer. J. Phys. **71** (2003), 1303–1313.

[Sal89] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Res. Notes Math. Ser. **201**, Jon Wiley & Sons, 1989.

[Sal01] ———, *Complex structures on nilpotent Lie algebras*, J. Pure Appl. Algebra **157** (2001), 311–333.

[Sal03] ———, *A tour of exceptional geometry*, Milan J. Math. **71** (2003), 59–94.

[Sek87] K. Sekigawa, *On some compact Einstein almost Kähler manifolds*, J. Math. Soc. Japan **39** (1987), 677–684.

[Sch06] N. Schoemann, *Almost hermitian structures with parallel torsion*, PhD thesis, Humboldt-Universität zu Berlin, 2006.

[Sch24] J. A. Schouten, *Der Ricci-Kalkül*, Grundlehren Math. Wiss. **10**, Springer-Verlag Berlin, 1924.

[Sch31] ———, *Klassifizierung der alternierenden Größen dritten Grades in* 7 *Dimensionen*, Rend. Circ. Mat. Palermo **55** (1931), 137–156.

[Schr32] E. Schrödinger, *Diracsches Elektron im Schwerefeld I*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, Phys.-Math. Klasse 1932, Verlag der Akademie der Wissenschaften Berlin, 1932, 436–460.

[Sim62] J. Simons, *On the transitivity of holonomy systems*, Ann. of Math. **76** (1962), 213–234.

[Sle87a] S. Slebarski, *The Dirac operator on homogeneous spaces and representations of reductive Lie groups I*, Amer. J. Math. **109** (1987), 283–301.

[Sle87b] S. Slebarski, *The Dirac operator on homogeneous spaces and representations of reductive Lie groups II*, Amer. J. Math. **109** (1987), 499–520.

[SSTP88] P. Spindel, A. Sevrin, W. Troost, and A. van Proeyen, *Extended supersymmetric σ-models on group manifolds*, Nuclear Phys. B **308** (1988), 662–698.

[Ste99] S. Sternberg, *Lie algebras*, Lecture Notes in Math. 1999.

[Str86] A. Strominger, *Superstrings with torsion*, Nuclear Phys. B **274** (1986), 253–284.

[Str69] K. Strubecker, *Differentialgeometrie. II: Theorie der Flächenmetrik*, Sammlung Göschen, W. de Gruyter, Berlin, 1969.

[Sw89] A. F. Swann, *Aspects symplectiques de la géométrie quaternionique*, C. R. Acad. Sci. Paris, Sér. I **308** (1989), 225–228.

[Sw91] ———, *HyperKähler and quaternionic Kähler geometry*, Math. Ann. **289** (1991), 421–450.

[Sw00] , ———, *Weakening holonomy*, ESI preprint No. 816 (2000); in S. Marchiafava et. al. (eds.), Proc. of the Second Meeting on Quaternionic Structures in Mathematics and Physics, Roma 6-10 September 1999, World Scientific, Singapore 2001, 405–415.

[Taf75] J. Tafel, *A class of cosmological models with torsion and spin*, Acta Phys. Polon. B **6** (1975), 537–554.

[Ta89] S. Tanno, *Variational problems on contact Riemannian manifolds*, Trans. Amer. Math. Soc. **314** (1989), 349–379.

[Thu76] W. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. **55** (1976), 467–468.

[Tra73a] A. Trautman, *On the structure of the Einstein-Cartan equations*, Sympos. Math. **12** (1973), 139–162.

[Tra73b] ———, *Spin and torsion may avert gravitational singularities*, Nature Phys. Sci. **242** (1973) 7.

[Tra99] ———, *Gauge and optical aspects of gravitation*, Classical Quantum Gravity **16** (1999), 157–175.

[TV83] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lecture Notes Series, vol. 83, Cambridge Univ. Press, Cambridge, 1983.

[TV84a] ———, *Geodesic spheres and naturally reductive homogeneous spaces*, Riv. Mat. Univ. Parma **10** (1984), 123–131.

[TV84b] ———, *Naturally reductive homogeneous spaces and generalized Heisenberg groups*, Compositio Math. **52** (1984), 389–408.

[Vai76] I. Vaisman, *On locally conformal almost kähler manifolds*, Israel J. Math. **24** (1976), 338–351.

[Vai79] ———, *Locally conformal Kähler manifolds with parallel Lee form*, Rend. Math. Roma **12** (1979), 263–284.

[VanN81] P. Van Nieuwenhuizen, *Supergravity*, Phys. Rep. **68** (1981), 189–398.

[Van01] J. Vanžura, *One kind of multisymplectic structures on* 6*-manifolds*, Proceedings of the Colloquium on Differential Geometry, Debrecen, 2000, 375–391 (2001).

[Ve02] M. Verbitsky, *HyperKähler manifolds with torsion, supersymmetry and Hodge theory*, Asian J. Math. **6** (2002), 679–712.

[VE88] A. B. Vinberg and A. G. Ehlahvili, *Classification of trivectors of a* 9*-dimensional space*, Sel. Math. Sov. **7** (1988), 63–98. Translated from Tr. Semin. Vektorn. Tensorm. Anal. Prilozh. Geom. Mekh. Fiz. **18** (1978), 197–233.

[McKW89] M. Y. Wang, *Parallel spinors and parallel forms*, Ann. Global Anal. Geom. **7** (1989), 59–68.

[WZ85] M. Y. Wang and W. Ziller, *On normal homogeneous Einstein manifolds*, Ann. Sci. Éc. Norm. Sup., 4^e série 18 (1985), 563–633.

[We81] R. Westwick, *Real trivectors of rank seven*, Linear and Multilinear Algebra **10** (1981), 183–204.

[Wi04] F. Witt, *Generalised* G_2*-manifolds*, Comm. Math. Phys. **265** (2006), 275–303, math.DG/0411642.

[Wi06] ———, *Special metrics and Triality*, math.DG/0602414.

[Wol74] J. A. Wolf, *Partially harmonic spinors and representations of reductive Lie groups*, J. Funct. Anal. **15** (1974), 117–154.

[Yau78] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampr̀e equations. I*, Comm. Pure Appl. Math. **31** (1978), 339–411.

[Zil77] W. Ziller, *The Jacobi equation on naturally reductive compact Riemannian homogeneous spaces*, Comment. Math. Helv. **52** (1977), 573–590.

[Zil82] W. Ziller, *Homogeneous Einstein metrics on spheres and projective spaces*, Math. Ann. **259** (1982), 351–358.

Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
Sitz: John-von-Neumann-Haus, Adlershof
D-10099 Berlin, Germany
E-mail: agricola@mathematik.hu-berlin.de