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IDEAL TUBULAR HYPERSURFACES IN REAL SPACE FORMS

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Abstract. In this article we give a classification of tubular hypersurfaces in real space forms which are $\delta(2,2,\dots,2)$-ideal.

1. Ideal immersions

Let M be a Riemannian n-manifold. Denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $e_1,\dots,e_n$ of the tangent space T_pM, the scalar curvature τ at p is defined to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j)\,. \tag{1}$$

When L is a 1-dimensional subspace of T_pM, we put $\tau(L) = 0$. If L is a subspace of T_pM of dimension $r \geq 2$, we define the scalar curvature $\tau(L)$ of L by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \qquad 1 \leq \alpha, \beta \leq r\,, \tag{2}$$

where $\{e_1,\dots,e_r\}$ is an orthonormal basis of L.

For an integer $k \geq 0$, denote by $\mathcal{S}(n,k)$ the finite set consisting of unordered k-tuples $(n_1,\dots,n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. Let $\mathcal{S}(n)$ be the union $\cup_{k\geq 0}\,\mathcal{S}(n,k)$. If $n = 2$, we have $k = 0$ and $\mathcal{S}(2) = \{\emptyset\}$.

For each $(n_1,\dots,n_k) \in \mathcal{S}(n)$, the invariant $\delta(n_1,\dots,n_k)$ is defined in [3] by:

$$\delta(n_1,\dots,n_k)(p) = \tau(p) - S(n_1,\dots,n_k)(p)\,, \tag{3}$$

where

$$S(n_1,\dots,n_k)(p) = \inf\big\{\tau(L_1) + \cdots + \tau(L_k)\big\}$$

and $L_1,\dots,L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1,\dots,k$. Clearly, the invariant $\delta(\emptyset)$ is nothing but the scalar curvature τ of M.

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For a given partition $(n_1, \dots, n_k) \in \mathcal{S}(n)$, we put

$$b(n_1, \dots, n_k) = \frac{1}{2}\Big(n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\Big)\,, \tag{4}$$

$$c(n_1, \dots, n_k) = \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)}\,. \tag{5}$$

For each real number c and each $(n_1, \dots, n_k) \in \mathcal{S}(n)$, the associated normalized invariant $\Delta_c(n_1, \dots, n_k)$ is defined by

$$\Delta_c(n_1, \dots, n_k) = \frac{\delta(n_1, \dots, n_k) - b(n_1, \dots, n_k)c}{c(n_1, \dots, n_k)}\,. \tag{6}$$

We recall the following general result from [3].

Theorem 1. *Let M be an n-dimensional submanifold of a real space form $R^m(c)$ of constant sectional curvature c. Then for each $(n_1, \dots, n_k) \in \mathcal{S}(n)$ we have*

$$H^2 \geq \Delta_c(n_1, \dots, n_k)\,, \tag{7}$$

where H^2 is the squared norm of the mean curvature vector.

The equality case of inequality (7) holds at a point $p \in M$ if and only if, with respect to a suitable orthonormal basis $e_1, \dots, e_n, e_{n+1}, \dots, e_m$ at p, the shape operators $A_r = A_{e_r}$, $r = n+1, \dots, m$ of M in $R^m(c)$ at p take the following forms:

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}, \tag{8}$$

$$A_r = \begin{pmatrix} A_1^r & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & A_k^r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad r = n+2, \dots, m, \tag{9}$$

where $a_1, \dots, a_n$ satisfy

$$\begin{aligned} a_1 + \cdots + a_{n_1} &= \cdots = a_{n_1+\dots n_{k-1}+1} + \cdots + a_{n_1+\cdots+n_k} \\ &= a_{n_1+\dots n_k+1} = \cdots = a_n \end{aligned} \tag{10}$$

and each A_j^r is an $n_j \times n_j$ submatrix such that

$$\operatorname{trace}(A_j^r) = 0\,, \quad (A_j^r)^t = A_j^r\,, \quad r = n+2, \dots, m\,; \quad j = 1, \dots, k\,. \tag{11}$$

For an isometric immersion $x : M \to R^m(c)$ of a Riemannian n-manifold into $R^m(c)$, this theorem implies that

$$H^2(p) \geq \hat{\Delta}_c(p)\,, \tag{12}$$

where $\hat{\Delta}_c$ denotes the invariant on M defined by

$$\hat{\Delta}_c = \max\left\{\Delta_c(n_1,\ldots,n_k) \mid (n_1,\ldots,n_k) \in \mathcal{S}(n)\right\}. \tag{13}$$

In general, there do not exist direct relations between these new invariants.

Applying inequality (12) B. Y. Chen introduced in [4] the notion of ideal immersions as follows.

Definition 1. *An isometric immersion $x : M \to R^m(c)$ is called an ideal immersion if the equality case of* (12) *holds at every point $p \in M$. An isometric immersion is called $(n_1,\ldots,n_k)$-ideal if it satisfies $H^2 = \Delta_c(n_1,\ldots,n_k)$ identically for $(n_1,\ldots,n_k) \in \mathcal{S}(n)$.*

Physical Interpretation of Ideal Immersions. An isometric immersion $x : M \to R^m(c)$ is ideal means that M receives the least possible amount of tension (given by $\hat{\Delta}_c(p)$) at each point $p \in M$ from the ambient space. This is due to (12) and the well-known fact that the mean curvature vector field is exactly the tension field for isometric immersions. Therefore, the squared mean curvature $H^2(p)$ at a point $p \in M$ simply measures the amount of tension M is receiving from the ambient space $R^m(c)$ at that point.

2. Tubular hypersurfaces

Recall the definition of the exponential mapping exp of a Riemannian manifold M. Denote by γ_v, $v \in T_pM$, the geodesic of M through p such that $\gamma'(p) = v$. Then we have that

$$\exp : TM \to M : (p,v) \mapsto \exp_p(v) = \gamma_v(1)$$

for every $v \in T_pM$ for which γ_v is defined on $[0,1]$.

Let B^ℓ be a topologically imbedded ℓ-dimensional ($\ell < n$) submanifold in an $n+1$-dimensional real space form $R^{n+1}(c)$. Denote by $\nu_1(B^\ell)$ the unit normal subbundle of the normal bundle $T^\perp(B^\ell)$ of B^ℓ in $R^{n+1}(c)$. Then, for a sufficiently small $r > 0$, the mapping

$$\psi : \nu_1(B^\ell) \to R^{n+1}(c) : (p,e) \mapsto \exp_\nu(re)$$

is an immersion which is called the *tubular hypersurface* with radius r about B^ℓ. We denote it by $T_r(B^\ell)$.

In this article, we consider $r > 0$ such that the map is an immersion only. Thus, the shape operator of the tubular hypersurface $T_r(B^\ell)$ is a well defined self-adjoint linear operator at each point.

Now take an arbitrary point p in B^ℓ and a vector u in $\nu_1(B^\ell)$. Denote with $\kappa_1(u),\dots,\kappa_\ell(u)$ the eigenvalues of the shape operator of B^ℓ in $R^{n+1}(c)$ with respect to u at the point p. Then we can give an expression for the principal curvatures $\bar\kappa_1,\dots,\bar\kappa_m$ of the tubular hypersurface in the point $\exp(p,u)$. We consider three cases.

(i) $c=0$. In the Euclidean case, we find

$$\bar\kappa_i = \frac{\kappa_i(u)}{1-r\kappa_i(u)}\,, \qquad i=1,\dots,\ell\,, \tag{14}$$

$$\bar\kappa_\alpha(r) = -\frac{1}{r}\,, \qquad \alpha=\ell+1,\dots,n\,. \tag{15}$$

(ii) $c=1$. For the unit sphere, we can simplify the expressions by denoting $\kappa_1(u)=\tan(\theta_1),\dots,\kappa_\ell(u)=\tan(\theta_\ell)$ with $-\frac{\pi}{2}<\theta_i<\frac{\pi}{2}$. Then we have

$$\bar\kappa_i = \tan(\theta_i+r)\,, \qquad i=1,\dots,\ell\,, \tag{16}$$

$$\bar\kappa_\alpha(r) = -\cot(r)\,, \qquad \alpha=\ell+1,\dots,n\,. \tag{17}$$

(iii) $c=-1$. In the hyperbolic space we have

$$\bar\kappa_i = \frac{\kappa_i(u)\coth(r)-1}{\coth(r)-\kappa_i(u)}\,, \qquad i=1,\dots,\ell\,, \tag{18}$$

$$\bar\kappa_\alpha(r) = -\coth(r)\,, \qquad \alpha=\ell+1,\dots,n\,. \tag{19}$$

More details can be found in [2].

3. $\delta_{(2,2\dots,2)}$-ideal tubular hypersurfaces

In this section we will give a complete classification of tubular hypersurfaces in real space forms for which the immersion defined in the previous section is a $\delta_{(2,2\dots,2)}$-ideal immersion. We again consider three cases.

In the Euclidean space $\mathbb{E}^{n+1}$.

Theorem 2. *A tubular hypersurface $T_r(B^\ell)$ in $\mathbb{E}^{n+1}$ $(n>2)$ satisfies equality in (7) for k-tuple $(n_1,\dots,n_k)=(2,\dots,2)$ if and only if one of the following three cases occurs:*

(1) *$\ell=0$ and the tubular hypersurface is a hypersphere.*
(2) *$\ell=k\in\{1,\dots,[\frac{n}{2}]\}$ and the tubular hypersurface is an open part of a spherical hypercylinder: $\mathbb{E}^\ell\times S^{n-\ell}(r)$.*
(3) *n is even, $\ell=k=\frac{n}{2}$ and B^ℓ is totally umbilical.*

Proof. Let $\kappa_1(u),\dots,\kappa_\ell(u)$ be the eigenvalues of the shape operator of B^ℓ in $\mathbb{E}^{n+1}$ with respect to a unit normal vector u at p. Then we find, according to the previous section, that the principal curvatures of the tubular hypersurface $T_r(B)$

at $p + ru$ are given by

$$\bar{\kappa}_i = \frac{\kappa_i(u)}{1 - r\kappa_i(u)}\,, \qquad i = 1, \dots, \ell\,, \tag{20}$$

$$\bar{\kappa}_\alpha(r) = -\frac{1}{r}\,, \qquad \alpha = \ell + 1, \dots, n\,. \tag{21}$$

Suppose now that $T_r(B)$ satisfies equality in (7) for a k-tuple $(n_1, \dots, n_k) = (2, \dots, 2)$.

If $\ell = 0$, the tubular hypersurface is an open part of an hypersphere. This gives us the first case in the theorem.

If $\ell = 1$, the multiplicity of $-\frac{1}{r}$ is $n-1$. From (8) and (10) we find the following three cases:

- $\bar{\kappa}_1 + \left(-\frac{1}{r}\right) = -\frac{1}{r}$, which implies that $\bar{\kappa}_1 = 0$.
- $\frac{\kappa_1}{1 - r\kappa_1} = -\frac{1}{r} - \frac{1}{r} = -\frac{2}{r}$, so we have that $r\kappa_1 = 2$. This gives a contradiction with the fact that $\kappa_1(-u) = -\kappa_1(u)$.
- $\frac{\kappa_1}{1 - r\kappa_1} + \left(-\frac{1}{r}\right) = -\frac{2}{r}$, from which we also get a contradiction.

So we see that $\bar{\kappa}_1 = 0$ and that $k = 1$. Thus B^1 is an open part of a line segment and the tubular hypersurface is an open part of $\mathbb{E}^1 \times S^{n-1}(r)$. This gives a special case of case (2) of the theorem.

Suppose now that $\ell \geq 2$, then (8) and (10) imply that we have one of the following five cases:

(a) for all unit normal vectors u of B^ℓ, we have

$$\kappa_1(u) = \cdots = \kappa_\ell(u) = 0 \tag{22}$$

and $\ell = k \leq \frac{n}{2}$;

(b) for all unit normal vectors u of B^ℓ, we have

$$\bar{\kappa}_1(u) = \cdots = \bar{\kappa}_\ell(u) \neq 0\,, \tag{23}$$

n is even and $k = \ell = \frac{n}{2}$;

(c) for all $i \in \{1, \dots, \ell\}$ there exists a $j \in \{1, \dots, \ell\}$ such that $i \neq j$ and such that:

$$\frac{\kappa_i(u)}{1 - r\kappa_i(u)} + \frac{\kappa_j(u)}{1 - r\kappa_j(u)} = -\frac{1}{r}\,; \tag{24}$$

(d) for all $i \in \{1, \dots, \ell\}$ there exists a $j \in \{1, \dots, \ell\}$ such that $i \neq j$ and such that:

$$\frac{\kappa_i(u)}{1 - r\kappa_i(u)} - \frac{1}{r} = \frac{\kappa_j(u)}{1 - r\kappa_j(u)}\,; \tag{25}$$

(e) $\ell = k = 2$, $n = 4$ and

$$\frac{\kappa_1(u)}{1 - r\kappa_1(u)} + \frac{\kappa_2(u)}{1 - r\kappa_2(u)} = -\frac{2}{r}\,. \tag{26}$$

Case (a) implies that B^ℓ is totally geodesic. Thus the tubular hypersurface is an open part of a spherical hypercylinder $\mathbb{E}^\ell \times S^{n-\ell}(r)$, which gives case (2) of the theorem.

Case (b) gives us case (3) of the theorem because $\bar{\kappa}_i = \bar{\kappa}_j$ if and only if $\kappa_i = \kappa_j$.

Next we want to proof that cases (c), (d) and (e) cannot occur.

From (24), we find that

$$1 = r^2 \kappa_i(u)\kappa_j(u) \tag{27}$$

for every u. This is impossible since the codimension of B^ℓ in $\mathbb{E}^{n+1}$ is at least 2. We can see this in the following way. Because the codimension is at least 2, we can take a plane in the normal space which contains u. If $\kappa_i(u) = 0$, then we have a contradiction at once. Otherwise $\kappa_i(u)$ is strict positive or strict negative. Then we have that $\kappa_i(-u)$ is strict negative or strict positive respectively. Now we rotate u in the chosen plane to $-u$. Because the principal curvature is a continuous function, there exists a normal vector ξ for which $\kappa_i(\xi) = 0$. Putting ξ in equation (27) gives a contradiction.

From (25) we find analogously that

$$1 - 2r\kappa_i(u) - r^2\kappa_i(u)\kappa_j(u) = 0\,. \tag{28}$$

Because $\kappa_i(-u) = -\kappa_i(u)$ we have also that

$$1 + 2r\kappa_i(u) - r^2\kappa_i(u)\kappa_j(u) = 0\,. \tag{29}$$

Combining (28) and (29) then gives

$$4r\kappa_i(u) = 0\,,$$

which gives a contradiction unless all the principal curvatures of B^ℓ are zero. But then we are again in case (a).

Similarly case (e) gives a contradiction since we find from (26) that $\kappa_1 + \kappa_2 = \frac{2}{r}$.

The converse is trivial. □

In the sphere $S^{n+1}(1)$. First we recall the definition of an austere submanifold in the sense of Harvey and Lawson [5].

Definition 2. *We call a submanifold M of a Riemannian manifold $\widetilde{M}$ austere if for every normal $\xi \in T^\perp M$ the set of all eigenvalues of the shape operator counted with multiplicities is invariant under multiplication with -1.*

Theorem 3. *A tubular hypersurface $T_r(B^\ell)$ in $S^{n+1}(1)$ $(n > 2)$ satisfies equality in (7) for a k-tuple $(n_1, \ldots, n_k) = (2, \ldots, 2)$ if and only if one of the following four cases occur:*

(1) *$\ell = 0$ and the tubular hypersurface is a geodesic sphere with radius $r \in]0, \pi[$.*

(2) *$n > \ell \geq \frac{n}{2}$, $k = n - \ell$, $r = \frac{\pi}{2}$ and B^ℓ is a totally umbilical submanifold in $S^{n+1}(1)$.*

(3) *$\ell = 2k < n$, $r = \frac{\pi}{2}$ and B^ℓ is an austere submanifold in $S^{n+1}(1)$.*
(4) *n is even, $\ell = k = \frac{n}{2}$ and B^ℓ is totally umbilical.*

Proof. Let B^ℓ be an ℓ-dimensional submanifold inbedded in $S^{n+1}(1)$. For every unit normal vector u of B^ℓ at a point p we denote by $\kappa_1(u), \dots, \kappa_\ell(u)$ the eigenvalues of the shape operator of B^ℓ in $S^{n+1}(1)$ with respect to u. Suppose now that

$$\kappa_i(u) = \tan(\theta_i)\,, \quad -\frac{\pi}{2} < \theta_i < \frac{\pi}{2}\,, \qquad 1 \leq i \leq \ell\,. \tag{30}$$

Then we know from the previous section that the principal curvatures of the tubular hypersurface $T_r(B^\ell)$ in $S^{n+1}(1)$ at $\cos(r)p + \sin(r)u$ are given by

$$\bar{\kappa}_i = \tan(\theta_i + r)\,, \qquad i = 1, \dots, \ell\,, \tag{31}$$

$$\bar{\kappa}_\alpha(r) = -\cot(r)\,, \qquad \alpha = \ell + 1, \dots, n\,.$$

Suppose that $T_r(B^\ell)$ satisfies (7) for a k-tupple $(n_1, \dots, n_k) = (2, \dots, 2)$.

If $\ell = 0$, the tubular hypersurface is totally umbilical in $S^{n+1}(1)$. Then theorem 1 implies that $T_r(B^\ell)$ with radius $r \in\ ]0, \pi[$ satisfies (7) for a k-tuple $(n_1, \dots, n_k) = (2, \dots, 2)$ if and only if $k = 0$ or $k = \frac{n}{2}$. So we find that $T_r(B^\ell)$ is a geodesic sphere. This gives us case (1).

If $\ell = 1$, then (8) and (10) imply that we are in one of the following cases:

- $\frac{\kappa_1 + \tan r}{1 - \kappa_1 \tan r} + (-\cot(r)) = -\cot(r)$, which implies that $\kappa_1(u) = -\tan(r)$ for every unit normal vector u of B^1 in $S^{n+1}(1)$. This gives a contradiction with the fact that $\kappa_1(-u) = -\kappa_1(u)$.
- $\frac{\kappa_1 + \tan r}{1 - \kappa_1 \tan r} = -2\cot r$, so we find $\kappa_1 \tan r = 2 + \tan^2 r$. Because $\kappa_1(-u) = -\kappa_1(u)$ we have $2 + \tan^2 r = 0$ which also gives a contradiction.
- $\frac{\kappa_1 + \tan r}{1 - \kappa_1 \tan r} + (-\cot r) = -2\cot r$, which becomes $\tan^2 r = -1$. This clearly also gives a contradiction.

In each case we get a contradiction, so $\ell = 1$ cannot occur.

Suppose now that $\ell \geq 2$, then theorem 1 implies that we are in one of the following cases:

(a) for all unit normal vectors u of B^ℓ we have that

$$\tan(\theta_j + r) = 0\,, \qquad j = 1, \dots, \ell \tag{32}$$

and $\ell = k \leq \frac{n}{2}$;

(b) for any unit normal vector u of B^ℓ we have that

$$\tan(\theta_1 + r) = \dots = \tan(\theta_\ell + r) \neq 0\,, \tag{33}$$

n is even and $k = \ell = \frac{n}{2}$;

(c) for all $i \in \{1, \dots, \ell\}$ there exists a $j \in \{1, \dots, \ell\}$ such that $i \neq j$ and such that:

$$\tan(\theta_i + r) - \cot(r) = \tan(\theta_j + r)\,; \tag{34}$$

(d) for all $i \in \{1,\dots,\ell\}$ there exists a $j \in \{1,\dots,\ell\}$ such that $i \neq j$ and such that:

$$\tan(\theta_i + r) + \tan(\theta_j + r) = -\cot(r)\,; \tag{35}$$

(e) $\ell = k = 2$, $n = 4$ and

$$\tan(\theta_1 + r) + \tan(\theta_2 + r) = -2\cot(r)\,. \tag{36}$$

Suppose now that we are in case (a) and thus (32) holds. Then we see that $\kappa_j(u)\cot(r) + 1 = 0$ for any unit normal vector u of B^ℓ in $S^{n+1}(1)$. This is impossible since $\kappa_j(-u) = -\kappa_j(u)$.

If case (b) holds, then we get case (4) of the theorem, since

$$\frac{\kappa_i + \tan r}{1 - \kappa_i \tan r} = \frac{\kappa_j + \tan r}{1 - \kappa_j \tan r}$$

implies that

$$(\kappa_i - \kappa_j)(1 + \tan^2 r) = 0\,.$$

Suppose now that we are in case (c). Then we have from (34) that:

$$\cot^3(r) - 2\kappa_i \cot^2(r) + \kappa_i\kappa_j \cot(r) + (\kappa_j - \kappa_i) = 0\,. \tag{37}$$

We use again the fact that $\kappa_i(-u) = -\kappa_i(u)$ and therefore we find

$$\cot(r)(\cot^2(r) + \kappa_i(u)\kappa_j(u)) = 0 \tag{38}$$

and

$$2\kappa_i(u)\cot^2(r) + \kappa_i(u) - \kappa_j(u) = 0\,. \tag{39}$$

If $\cot(r) \neq 0$, then (38) implies that $\cot^2(r) = -\kappa_i(u)\kappa_j(u)$. Because $\ell < n$ we get a contradiction with the same argument as in the preceding proof.

Thus we have $\cot(r) = 0$, and thus $r = \frac{\pi}{2}$. From (39) we also see that $\kappa_i(u) = \kappa_j(u)$. Without loss of generality, we may assume

$$a_1 = \mu,\ a_2 = 0,\ a_3 = \mu,\ a_4 = 0,\ \dots,\ a_{2k-1} = \mu,\ a_{2k} = 0,\ a_{2k+1} = \mu, \dots,\ a_n = \mu$$

where $\mu = -\frac{1}{\kappa_1}$ and $a_1, \dots, a_n$ are given by theorem (1).

Furthermore we see that $\tan(\theta_i + r) \neq 0$ since $-\frac{\pi}{2} < \theta_i < \frac{\pi}{2}$ and from (31) we find that $\cot(r)$ has multiplicity $n - \ell$. So theorem (1) implies that $\ell \geq \frac{n}{2}$ and $\tan(\theta_1 + r) = \dots = \tan(\theta_\ell + r)$. This implies also that $\tan(\theta_1) = \dots = \tan(\theta_\ell)$ and thus that B^ℓ is totally umbilical. Moreover we see that theorem (1) implies that $k = n - \ell$. This gives rise to case (2).

Suppose now that we are in case (d) and thus that (35) holds. Then we have

$$\cot^3(r) + 2\cot(r) - \kappa_i\kappa_j \cot(r) - (\kappa_i + \kappa_j) = 0\,. \tag{40}$$

If we use that $\kappa_i(-u) = -\kappa_i(u)$ we find

$$\cot(r)(\cot^2(r) + 2 - \kappa_i\kappa_j) = 0 \tag{41}$$

and

$$\kappa_i + \kappa_j = 0\,. \tag{42}$$

Like in case (c) we get a contradiction if $\cot(r) \neq 0$. So we find $\cot(r) = 0$ and thus $r = \frac{\pi}{2}$. Moreover we have $\kappa_i = -\kappa_j$. Without loss of generality, we may assume

$$a_1 = \tan(\theta_1 + r) = -\frac{1}{\kappa_1}\,, \quad a_2 = \tan(\theta_2 + r) = -\frac{1}{\kappa_2}\,, \ldots, \, a_n = -\cot(r) = 0$$

We also know that $\tan(\theta_j + r) \neq 0$ (since $-\frac{\pi}{2} < \theta_j < \frac{\pi}{2}$). Thus (31) and theorem 1 imply that B^ℓ is an austere submanifold in $S^{n+1}(1)$; in particular ℓ is even. This gives case (3).

A similar computation as in case (d) shows that case (e) gives a contradiction.

The converse can be verified easily. □

In the hyperbolic space $H^{n+1}(-1)$.

Theorem 4. *A tubular hypersurface $T_r(B^\ell)$ in $H^{n+1}(-1)$ $(n > 2)$ satisfies equality in (7) for a k-tuple $(n_1, \ldots, n_k) = (2, \ldots, 2)$ if and only if we are in one of the following three cases:*

(1) *$\ell = 0$ and the tubular hypersurface is a geodesic sphere with radius $r > 0$.*

(2) *$\ell = 2k$, B^ℓ is totally geodesic and $r = \coth^{-1}(\sqrt{2})$.*

(3) *n is even, $\ell = k = \frac{n}{2}$ and B^ℓ is totally umbilical.*

Proof. Let B^ℓ be an ℓ-dimensional submanifold in the hyperbolic space $H^{n+1}(-1)$ and $T_r(B^\ell)$ be the tubular hypersurface of B^ℓ in $H^{n+1}(-1)$. Suppose that $T_r(B^\ell)$ satisfies (7) for a k-tuple $(n_1, \ldots, n_k) = (2, \ldots, 2)$. For any unit normal vector u of B^ℓ at a point p of B^ℓ denote with $\kappa_1(u), \ldots, \kappa_\ell(u)$ the principal curvatures of B^ℓ in $H^{n+1}(-1)$ at p with respect to u. Then it follows from section 2 that the principal curvatures $\bar{\kappa}_1, \ldots, \bar{\kappa}_n$ of the shape operator of $T_r(B^\ell)$ are given by:

$$\bar{\kappa}_i = \frac{\kappa_i(u)\coth(r) - 1}{\coth(r) - \kappa_i(u)}\,, \qquad i = 1, \ldots, \ell\,, \tag{43}$$

$$\bar{\kappa}_\alpha(r) = -\coth(r)\,, \qquad \alpha = \ell + 1, \ldots, n\,. \tag{44}$$

If $\ell = 0$, then the tubular hypersurface is totally umbilical. So we find from theorem (1) that $k = 0$ or $k = \frac{n}{2}$ and $T_r(B^\ell)$ is a geodesic sphere. Thus we are in case (1).

If $\ell = 1$, then from theorem 1 and (43) it follows that we are in one of the following cases:

- $\bar{\kappa}_1 - \cot r = -\cot r$, which implies immediately that $\bar{\kappa}_1(u) = 0$ for any unit normal vector u of B^1 in $S^{n+1}(1)$. Then (43) would imply that $\kappa_1(u) = -\tanh(r)$ which gives a contradiction with the fact that $\kappa_1(-u) = -\kappa_1(u)$ since $r \in \mathbb{R}_0^+$.
- $\frac{\kappa_1 \coth r - 1}{\coth r - \kappa_1} = -2\coth r$, so we find $\kappa_1 \coth r = 2\coth^2 r - 1$. Because $\kappa_1(-u) = -\kappa_1(u)$ this implies that $\coth^2 r = \frac{1}{2}$ which gives a contradiction since $\coth^2 r$ is always greater than 1.
- $\frac{\kappa_1 \coth r - 1}{\coth r - \kappa_1} + (-\cot r) = -2\cot r$, this implies $\coth^2 r = 1$ which gives a contradiction as above.

Thus we see that the case $\ell = 1$ cannot occur.

Suppose now that $\ell \geq 2$, then theorem (1) implies that one of the following cases occur:

(a) for all unit normal vectors u of B^ℓ we have

$$\kappa_i(u)\coth(r) = 1\,, \qquad \text{for all } i \in \{1, \dots \ell\} \tag{45}$$

and $\ell = k \leq \frac{n}{2}$;

(b) for alle unit normal vectors u of B^ℓ we have

$$\bar{\kappa}_1(u) = \cdots = \bar{\kappa}_\ell(u) \neq 0\,, \tag{46}$$

n is even and $k = \ell = \frac{n}{2}$;

(c) for all $i \in \{1, \dots, \ell\}$ there exists a $j \in \{1, \dots, \ell\}$ such that $i \neq j$ and such that:

$$\frac{\kappa_i(u)\coth(r) - 1}{\coth(r) - \kappa_i(u)} - \coth(r) = \frac{\kappa_j(u)\coth(r) - 1}{\coth(r) - \kappa_j(u)}\,; \tag{47}$$

(d) for all $i \in \{1, \dots, \ell\}$ there exists a $j \in \{1, \dots, \ell\}$ such that $i \neq j$ and such that:

$$\frac{\kappa_i(u)\coth(r) - 1}{\coth(r) - \kappa_i(u)} + \frac{\kappa_j(u)\coth(r) - 1}{\coth(r) - \kappa_j(u)} = -\coth(r)\,; \tag{48}$$

(e) $\ell = k = 2$, $n = 4$ and

$$\frac{\kappa_1(u)\coth(r) - 1}{\coth(r) - \kappa_1(u)} + \frac{\kappa_2(u)\coth(r) - 1}{\coth(r) - \kappa_2(u)} = -2\coth(r)\,. \tag{49}$$

We see at once that (45) and thus case (a) cannot occur since $\kappa_i(-u) = -\kappa_i(u)$.

Suppose now that we are in case (b). The condition $\bar{\kappa}_i = \bar{\kappa}_j$ gives us

$$(\kappa_i - \kappa_j)(\coth^2 r - 1) = 0\,.$$

Because $\coth^2 r > 1$ this implies $\bar{\kappa}_i = \bar{\kappa}_j$ if and only if $\kappa_i = \kappa_j$. This is case (3) of the theorem.

Suppose that we are in case (c). Then from (47), we find

$$\coth^3(r) - 2\kappa_i \coth^2(r) + \kappa_i\kappa_j \coth(r) + \kappa_i - \kappa_j = 0\,. \tag{50}$$

Because $\kappa_i(-u) = -\kappa_i(u)$ we have

$$\coth^3(r) + \kappa_i\kappa_j \coth(r) = 0\,, \tag{51}$$

$$-2\kappa_i \coth^2(r) + \kappa_i - \kappa_j = 0\,. \tag{52}$$

From (51), it follows that $\kappa_i(u)\kappa_j(u) = -\coth^2(r)$ since $\coth(r) \neq 0$. But this gives a contradiction with the same argument as in the Euclidean case because the codimension is at least 2.

Analogously from (48) we find:

$$(\kappa_i + \kappa_j)\tanh^3(r) - (2 + \kappa_i\kappa_j)\tanh^2(r) + 1 = 0\,. \tag{53}$$

By switching to $-u$ we get:

$$-(\kappa_i + \kappa_j)\tanh^3(r) - (2 + \kappa_i\kappa_j)\tanh^2(r) + 1 = 0\,. \tag{54}$$

This implies that $\kappa_i(u) + \kappa_j(u) = 0$. Substituting this in (53) gives $\kappa_i(u)^2 = 2 - \coth^2(r)$. We can also substitute the other way round, then we find $\kappa_j(u)^2 = 2-\coth^2(r)$. Thus κ_i must be zero for every $i \in \{1, \dots, \ell\}$. We see that B^ℓ is totally geodesic. We see also that in this case $r = \coth^{-1}(\sqrt{2})$. Thus we get as principal curvatures for $T_r(B^\ell)$ $\bar{\kappa}_i = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$, $i = 1, \dots, \ell$ and $\bar{\kappa}_\alpha = -\sqrt{2}$, $\alpha = \ell + 1, \dots, n$. From theorem (1) it follows that $\ell = 2k$. So we get case (2).

Case (e) cannot occur since similar computations as in case (d) give a contradiction.

The converse can be verified easily. □

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