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FIBER PRODUCT PRESERVING BUNDLE FUNCTORS ON ALL MORPHISMS OF FIBERED MANIFOLDS

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ABSTRACT. We describe the fiber product preserving bundle functors on the category of all morphisms of fibered manifolds in terms of infinite sequences of Weil algebras and actions of the skeleton of the category of r-jets by algebra homomorphisms. We deduce an explicit formula for the iteration of two such functors. We characterize the functors with values in vector bundles.

Let $\mathcal{M}f$ be the category of all manifolds and all smooth maps, $\mathcal{FM}$ be the category of all smooth fibered manifolds and all fibered morphisms and $\mathcal{FM}_m \subset \mathcal{FM}$ be the subcategory of fibered manifolds with m-dimensional bases and fibered morphisms with local diffeomorphisms as base maps. Starting from the seminal result asserting that all product preserving bundle functors on $\mathcal{M}f$ are the Weil functors T^A, see [6] for a survey, we characterized the fiber product preserving bundle functors on $\mathcal{FM}_m$ by means of Weil algebras in [7]. The latter result already found interesting geometric applications in the theory of functorial prolongations of projectable tangent valued forms, [1], principal and associated bundles, [5], Lie groupoids, [4], and Lie algebroids, [3].

In the present paper we study the fiber product preserving bundle (in short: f.p.p.b.) functors on the whole category $\mathcal{FM}$. The well known example of such functor is the vertical Weil functor V^A. In Section 1 we recall the identification of the f.p.p.b. functors of base order r on $\mathcal{FM}_m$ with the triples (A, H, t), where A is a Weil algebra, H is a group homomorphism of the r-th jet group G^r_m in dimension m into the group Aut A of all algebra isomorphisms of A and t is an equivariant algebra homomorphism from $\mathbb{D}^r_m = J^r_0(\mathbb{R}^m, \mathbb{R})$ into A, [7]. Lemma 1 reads that if this functor is a restriction $F_m = (A_m, H_m, t_m)$ of a f.p.p.b. functor F on the category $\mathcal{FM}$, then t_m is a zero homomorphism. We interpret this condition geometrically and we present further examples of f.p.p.b. functors on $\mathcal{FM}$. We deduce that V^A is characterized by the fiber isomorphism property. In Section 2 we clarify that if F has base order r, it can be identified with the sequence

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$A = (A_0, \dots, A_m, \dots)$ of Weil algebras and an action H of the skeleton L^r of the category of r-jets on A by algebra homomorphisms. Section 3 is devoted to the natural transformations of two such functors. An example in Section 4 shows that F need not to be of finite base order. We prove that F is locally of finite base order. In this general case, the action of L^r should be replaced by an action of locally finite order of the skeleton L^∞ of the category of jets of infinite order. Section 5 contains an auxiliary construction of vector valued bundle functors on the category $\mathcal{M}f$, which is similar to Section 2. In Section 6, this construction is used for an explicit description of the f.p.p.b. functors on $\mathcal{FM}$ with values in the category of vector bundles. In the last section we deduce an explicit formula for the iteration of two f.p.p.b. functors on $\mathcal{FM}$ (the case of $\mathcal{FM}_m$ was studied in [2]).

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notations from [6].

1. The first lemma and examples

Every homomorphism of Weil algebras $\mu : A \to B$ induces a natural transformation $\mu_M : T^A M \to T^B M$ for every manifold M. By definition, these homomorphisms are unital, so that $\mathbb{R} \subset A$ is transformed into $\mathbb{R} \subset B$ by the identity map. The zero algebra homomorphism maps the whole nilpotent part of A into $0 \in B$. We write $M \times N$ for the product fibered manifold $M \times N \to M$ and $i : \mathcal{M}f \to \mathcal{FM}$ for the "identity" injection $iM = (\mathrm{id}_M : M \to M)$ and $if = (f, f) : iM \to iN$. We denote by $\widehat{x}$ the constant map of a given manifold into the point x.

According to [7], we have a bijection between the f.p.p.b. functors F of the base order r on $\mathcal{FM}_m$ and the following triples (A, H, t). The Weil algebra A is $F_0(\mathbb{R}_m \times \mathbb{R})$. For $g = j_0^r \gamma \in G_m^r$, we have

$$H(g) = F_0(\gamma \times \mathrm{id}_{\mathbb{R}}) : A \to A \,. \tag{1}$$

For $j_0^r \varphi \in \mathbb{D}_m$, we consider the base preserving morphism $(\mathrm{id}_{\mathbb{R}^m}, \varphi) : i\mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}$, $x \mapsto \big(x, \varphi(x)\big)$ and define

$$t(j_0^r \varphi) = F(\mathrm{id}_{\mathbb{R}^m}, \varphi)(0) \,. \tag{2}$$

Conversely, given (A, H, t), we construct F as follows. In the product case, $F(M \times N)$ is the associated fiber bundle

$$F(M \times N) = P^r M[T^A N, H_N] \,,$$

where $P^r M$ is the r-th order frame bundle of M and H_N is the action of G_m^r on $T^A N$ by the natural transformations $H(g)_N$, $g \in G_m^r$. For an arbitrary fibered manifold $p : Y \to M$, $FY \subset P^r M[T^A Y, H_Y]$ is the subset of all equivalence classes $\{u, Z\}$,

$$u \in P^r M \,, \quad Z \in T^A Y \quad \text{satisfying} \quad t_M(u) = T^A p(Z) \,. \tag{3}$$

Let F be a f.p.p.b. functor on $\mathcal{FM}$. Write $F_m = (A_m, H_m, t_m)$ for its restriction to $\mathcal{FM}_m \subset \mathcal{FM}$.

Lemma 1. *For every f.p.p.b. functor F on $\mathcal{FM}$, all t_m are zero homomorphisms.*

Proof. If we replace F by F_m in (2), the right hand side can be written as

$$\big(F(\mathrm{id}_{\mathbb{R}^m},\varphi)\circ F(i\widehat{0}_m)\big)(0_m) = F\big(\widehat{0}_m,\widehat{\varphi(0_m)}\big)(0_m)\,.$$

Further, consider $j^r_{0_m}\widehat{\varphi(0_m)} \in \mathbb{D}^r_m$. On one hand, since t_m is an algebra homomorphism, $t_m\big(j^r_{0_m}\widehat{\varphi(0_m)}\big)$ is the number $\varphi(0_m)$ in the real part of A. On the other hand,

$$\begin{aligned} t_m\big(j^r_{0_m}(\widehat{\varphi(0_m)})\big) &= F\big(\,\mathrm{id}_{\mathbb{R}^m},\widehat{\varphi(0_m)}\big)(0_m) = \big(F\big(\,\mathrm{id}_{\mathbb{R}^m},\widehat{\varphi(0_m)}\big)\circ \\ &F(i\widehat{0}_m)\big)(0_m) = F\big(\widehat{0}_m,\widehat{\varphi(0_m)}\big)(0_m)\,. \end{aligned}$$

Hence $t_m(j^r_0\varphi)) = \varphi(0_m)$, so that t_m is the zero homomorphism. $\square$

Thus, if $\{u,Z\}\in FY$, $u\in P^r_xM$, then $T^{A_m}p(Z) = j^{A_m}\widehat{x}$. In other words, Z lies in the fiber of $T^{A_m}Y$ over $j^{A_m}\widehat{x}\in T^{A_m}M$.

Now we add some further examples of f.p.p.b. functors on $\mathcal{FM}$ to the well known case of V^A. We write $\underline{f}$ for the base map of an $\mathcal{FM}$-morphism f.

Example 1. Let V be the vertical tangent functor. We define

$$FY = VY\otimes TM\,,\quad Ff = Vf\otimes T\underline{f}\,.$$

Since $V(Y_1\times Y_2)\otimes TM = VY_1\otimes TM\times_M VY_2\otimes TM$, F is a f.p.p.b. functor.

Example 2. More generally, let G be a bundle functor on $\mathcal{M}f$ with values in the category $\mathcal{VB}$ of vector bundles. We set

$$FY = VY\otimes GM\,,\quad Ff = Vf\otimes G\underline{f}\,.$$

Example 3. Example 2 can be modified to

$$FY = V(V^AY)\otimes GM\,,\quad Ff = V(V^Af)\otimes G\underline{f}\,,$$

the tensor product being over V^AY.

It is remarkable that functor V^A can be characterized by the following additional property.

Definition 1. We say that a bundle functor F on $\mathcal{FM}$ has the fiber isomorphism property, if for every $\mathcal{FM}$-morphism f isomorphic on the fibers, Ff is also isomorphic on the fibers.

Proposition 1. *If F is a f.p.p.b. functor on $\mathcal{FM}$ with the fiber isomorphism property, then $F = V^A$ for a Weil algebra A.*

Proof. Consider the fiber Y_x of $p: Y\to M$ over $x\in M$ as the fibered manifold $Y_x\to pt$, where pt is one-point set. We have $F(Y_x\to pt) = T^{A_0}(Y_x)\to pt$. By the fiber isomorphism property, the injection of $Y_x\to pt$ into Y induces a diffeomorphism between $T^{A_0}Y_x$ and F_xY. $\square$

2. The description of the functors of base order r

The restrictions $F \mid \mathcal{FM}_m$ determine an infinite sequence of Weil algebras

$$A = (A_0, \dots, A_m, \dots)\,. \tag{4}$$

Assume the base order of F is r. Using the notation $L^r_{m,n} = J^r_{0_m}(\mathbb{R}^m, \mathbb{R}^n)_{0_n}$ from [6], $L^r = (L^r_{m,n})$ is a category over $\mathbb{N}$, the skeleton of the category of r-jets. Let $g = j^r_{0_m}\gamma \in L^r_{m,n}$, $\gamma : \mathbb{R}^m \to \mathbb{R}^n$, $\gamma(0_m) = 0_n$. Consider $\gamma \times \mathrm{id}_{\mathbb{R}} : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$ and define

$$H_{m,n}(g) = F_{0_m}(\gamma \times \mathrm{id}_{\mathbb{R}}) : A_m \to A_n\,. \tag{5}$$

Lemma 2. *Each $H_{m,n}(g)$ is an algebra homomorphism. If $\overline{g} \in L^r_{n,p}$, so that $\overline{g} \circ g \in L^r_{m,p}, H_{n,p}(\overline{g}) : A_n \to A_p$, $H_{m,p}(\overline{g} \circ g) : A_m \to A_p$, then*

$$H_{m,p}(\overline{g} \circ g) = H_{n,p}(\overline{g}) \circ H_{m,n}(g)\,. \tag{6}$$

Proof. Write $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for the addition of reals and $a_m = \mathrm{id}_{\mathbb{R}^m} \times a : \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}$. Then the addition in A_m is $F_{0_m}(a_m) : A_m \times A_m \to A_m$. Clearly, the following diagram commutes

$$\begin{tikzcd} \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \arrow[r, "a_m"] \arrow[d, "f \times \mathrm{id}_{\mathbb{R}} \times \mathrm{id}_{\mathbb{R}}"'] & \mathbb{R}^m \times \mathbb{R} \arrow[d, "f \times \mathrm{id}_{\mathbb{R}}"] \\ \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \arrow[r, "a_n"] & \mathbb{R}^n \times \mathbb{R} \end{tikzcd} \tag{7}$$

Applying F, we deduce that $H_{m,n}(g)$ commutes with the additions in A_m and A_n. Replacing the addition of reals by the multiplication of reals, we prove that $H_{m,n}(g)$ is an algebra homomorphism. Formula (6) is a direct consequence of the fact F is a functor. □

Definition 2. We say that (6) is an action of the category L^r on the sequence (4) by algebra homomorphisms.

The following assertion is a direct consequence of the definition of the natural transformation determined by an algebra homomorphism,

Lemma 3. *For every manifold Q, the natural transformation $H_{m,n}(g)_Q : T^{A_m}Q \to T^{A_n}Q$ coincides with $F_{0_m}(\gamma \times \mathrm{id}_Q) : T^{A_m}Q \to T^{A_n}Q$.*

Consider an $\mathcal{FM}$-morphism $f : \mathbb{R}^m \times Q \to \mathbb{R}^n \times Q$ over $\gamma : \mathbb{R}^m \to \mathbb{R}^n$. Write $f(x,y) = \big(\gamma(x), \overline{f}(x,y)\big)$, $x \in \mathbb{R}^m$, $y \in Q$, so that $\overline{f} : \mathbb{R}^m \times Q \to Q$. Define

$$\widetilde{f} : \mathbb{R}^m \times Q \to \mathbb{R}^m \times Q\,, \quad \widetilde{f}(x,y) = \big(x, \overline{f}(x,y)\big)\,.$$

Then we have $f = (\gamma \times \mathrm{id}_Q) \circ \widetilde{f}$. This implies $F_{0_m} f = F_{0_m}(\gamma \times \mathrm{id}_Q) \circ F_{0_m}\widetilde{f}$. But $\widetilde{f}$ is in $\mathcal{FM}_m$. By [7] and Lemma 1,

$$F_{0_m}\widetilde{f}(y) = T^{A_m}\widetilde{f}(0_m, y)\,. \tag{8}$$

Now we can prove

Proposition 2. *The f.p.p.b. functors on $\mathcal{FM}$ of base order r are in bijection with the actions $H = (H_{m,n})$ of L^r on the sequences (4) by algebra homomorphisms.*

Proof. We have to clarify how to reconstruct F from A and H. By Section 1, for $p : Y \to M$ with $\dim M = m$ we define FY as the space of equivalence classes

$$\{u, Z\}, \quad u \in P^r_x M, \quad Z \in T^{A_m} Y \quad \text{satisfying} \quad T^{A_m} p(Z) = j^{A_m} \widehat{x}. \tag{9}$$

For an $\mathcal{FM}$-morphism $f : Y \to \overline{Y}$ over $\underline{f} : M \to \overline{M}$, $n = \dim \overline{M}$, we use the decomposition (not unique) $j^r_x \underline{f} = v \circ g \circ u^{-1}$, $u \in P^r_x M$, $v \in P^r_{\underline{f}(x)} \overline{M}$, $g \in L^r_{m,n}$. Taking into account Lemma 3 and (8), we define

$$Ff\big(\{u, Z\}\big) = \big\{v, H_{m,n}(g)_{\overline{Y}}\big(T^{A_m} f(Z)\big)\big\}, \tag{10}$$

where $T^{A_m} f(Z) \in T^{A_m} \overline{Y}$ is transformed by $H_{m,n}(g)_{\overline{Y}}$ into $T^{A_n} \overline{Y}$. By equivariancy of all quantities, this is a correct definition. One verifies directly that F coincides with the original functor. □

Consider an action H of L^r on A by algebra homomorphisms. The following assertion is geometrically very instructive.

Lemma 4. *If $g \in L^r_{m,n}$ is r-jet of immersion or submersion, then the algebra homomorphism $H_{m,n}(g)$ is injective or surjective, respectively.*

Proof. If $g = j^r_{0_m} \gamma$ is r-jet of an immersion $\gamma : \mathbb{R}^m \to \mathbb{R}^n$, $\gamma(0_m) = 0_n$, then there is a local map $\overline{\gamma} : \mathbb{R}^n \to \mathbb{R}^m$ satisfying $\overline{\gamma} \circ \gamma = \mathrm{id}_{\mathbb{R}^m}$. Hence for $\overline{g} = j^r_{0_n} \overline{\gamma}$ we have $H_{n,m}(\overline{g}) \circ H_{m,n}(g) = \mathrm{id}_{A_m}$, so that $H_{m,n}(g)$ is injective. The second assertion can be proved analogously. □

In particular, the canonical immersions $\mathbb{R}^m \to \mathbb{R}^{m+1}$ induce a sequence of algebra injections

$$A_m \hookrightarrow A_{m+1}. \tag{11}$$

Example 4. We describe the functor F in Example 1 from the viewpoint of Proposition 2. We have $A_m = T\mathbb{R} \otimes T_{0_m} \mathbb{R}^m = \mathbb{R} \times \mathbb{R}^m$, because the tensor product is over $\mathbb{R}$. Write $\nu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for the multiplication of reals. Then the multiplication in A_m is determined by $F_{0_m}(\mathrm{id}_{\mathbb{R}^m} \times \nu)$. Hence we have the zero product in the nilpotent part of A_m, so that $A_m = \mathbb{D}^1_m$. The base order of F is 1. Every $g \in L^1_{m,n}$ is identified with a linear map $l(g) : \mathbb{R}^m \to \mathbb{R}^n$. Evaluating (5), we find

$$H_{m,n}(g) = \mathrm{id}_{\mathbb{R}} \times l(g) : \mathbb{D}^1_m \to \mathbb{D}^1_n. \tag{12}$$

Since we have the zero product in the nilpotent parts of $\mathbb{D}^1_m$ and $\mathbb{D}^1_n$, each map (12) is an algebra homomorphism.

3. Natural transformations

Let $\overline{F} = (\overline{A}, \overline{H})$ be another f.p.p.b. functor on $\mathcal{FM}$ of the base order r and $\tau : F \to \overline{F}$ be a natural transformation. One verifies directly that

$$\mu_m := (\tau_{\mathbb{R}^m \times \mathbb{R}})_{0_m} : F_{0_m}(\mathbb{R}^m \times \mathbb{R}) \to \overline{F}_{0_m}(\mathbb{R}^m \times \mathbb{R}) \tag{13}$$

form an equivariant sequence $\mu = (\mu_0, \dots, \mu_m, \dots)$ of algebra homomorphisms $\mu_m : A_m \to \overline{A}_m$, i.e.

$$\overline{H}_{m,n}(g)\big(\mu_m(a)\big) = \mu_n\big(H_{m,n}(g)(a)\big), \quad g \in L^r_{m,n}, \ a \in A_m. \tag{14}$$

Conversely, given such equivariant sequence μ of algebra homomorphisms $\mu_m : A_m \to \overline{A}_m$, we define a natural transformation $\mu_Y : FY \to \overline{F}Y$ by

$$\mu_Y\big(\{u, Z\}\big) = \big\{u, (\mu_m)_Y(Z)\big\}\,. \tag{15}$$

Then one proves directly

Proposition 3. *The natural transformations of two f.p.p.b. functors $F = (A, H)$ and $\overline{F} = (\overline{A}, \overline{H})$ on $\mathcal{FM}$ of the same base order r are in bijection with the equivariant sequences of algebra homomorphisms $\mu : A \to \overline{A}$.*

4. The case of infinite base order

The second author constructed a bundle functor Q on $\mathcal{M}f$ with values in $\mathcal{VB}$ of infinite order, [6], [8]. If we put $G = Q$ in Example 2, we obtain a f.p.p.b. functor on $\mathcal{FM}$ of infinite base order. We are going to describe all such functors. Let $\mathcal{FM}_{m,n}$ be the category of fibered manifolds with m-dimensional bases and n-dimensional fibers and their local isomorphisms. Write $r(m, n)$ for the order of the restriction $F \mid \mathcal{FM}_{m,n}$. We deduce that every f.p.p.b. functor F on $\mathcal{FM}$ has locally finite order in the following sense.

Consider a fibered manifold $Y \to M$ with $\dim M = m$, $\dim Y = m + n$ and an arbitrary fibered manifold $\overline{Y} \to \overline{M}$. Write $r = \max\big(r(m+1, 1), r(m, 1)\big)$.

Proposition 4. *Let F be a f.p.p.b. functor on $\mathcal{FM}$, $f, g : Y \to \overline{Y}$ be two $\mathcal{FM}$-morphisms and $y \in Y$. Then $j^r_y f = j^r_y g$ implies $Ff \mid F_yY = Fg \mid F_yY$.*

Proof. We may assume $Y = \mathbb{R}^m \times \mathbb{R}^n$, $\overline{Y} = \mathbb{R}^p \times \mathbb{R}^q$, $y = (0_m, 0_n)$, $f(y) = g(y) = (0_p, 0_q)$. Let $j^r_{(0_m,0_n)} f = j^r_{(0_m,0_n)} g$. Analogously to Section 2, we introduce $\overline{f}, \overline{g} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^q$ by

$$f(x, y) = \big(\underline{f}(x), \overline{f}(x, y)\big)\,, \quad g(x, y) = \big(\underline{g}(x), \overline{g}(x, y)\big)\,, \quad x \in \mathbb{R}^m,\ y \in \mathbb{R}^n\,.$$

We define a bundle functor $\overline{F}$ on $\mathcal{M}f$ by $\overline{F}M = F(M \times \mathbb{R})$, $\overline{F}(h) = F(h \times \mathrm{id}_{\mathbb{R}})$. By Corollary 4.2 in [9], we have $\overline{F}(f) = \overline{F}(g)$ over 0_m because the functor $\overline{F} \mid \mathcal{M}f_{m+1}$ is of the order $r(m+1, 1)$ and $\underline{f}$ and $\underline{g}$ have the same $r(m+1, 1)$-jet at 0_m. Then $F(\underline{f} \times \mathrm{id}_{\mathbb{R}^q}) = F(\underline{g} \times \mathrm{id}_{\mathbb{R}^q})$ over $(0_m, 0_q)$, because F is fiber product preserving. Further, consider $F_m = F \mid \mathcal{FM}_m$. Define $\widetilde{f}, \widetilde{g} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^q$,

$$\widetilde{f}(x, y) = \big(x, \overline{f}(x, y)\big)\,, \quad \widetilde{g}(x, y) = \big(x, \overline{g}(x, y)\big)\,.$$

By [6], Theorem 22.3 and [7], we have $F\widetilde{f} = F\widetilde{g}$ over $(0_m, 0_n)$ because $F_m \mid \mathcal{FM}_{m,n+1}$ has order $r(m, 1)$ and $\widetilde{f}$ and $\widetilde{g}$ have the same $r(m, 1)$-jet at $(0_m, 0_n)$. Since $Ff = F(\underline{f} \times \mathrm{id}_{\mathbb{R}^q}) \circ F\widetilde{f}$ and similarly for g, we have proved $Ff = Fg$ over $(0_m, 0_n)$. □

Consider the category L^∞ over $\mathbb{N}$, $L^\infty_{m,n} = J^\infty_{0_m}(\mathbb{R}^m, \mathbb{R}^n)_{0_n}$ and a sequence A of Weil algebras.

Definition 3. An action of L^∞ on A by algebra homomorphisms is said to be of locally finite order, if each $H_{m,n} : L^\infty_{m,n} \to \mathrm{Hom}\,(A_m, A_n)$ factorizes through a jet projection $L^\infty_{m,n} \to L^{r_{m,n}}_{m,n}$.

Let F be an arbitrary f.p.p.b. functor on $\mathcal{FM}$. Using Proposition 4, we deduce in the same way as in Section 2, that the f.p.p.b. functors on $\mathcal{FM}$ are in bijection with the pairs (A, H), where A is a sequence of Weil algebras and H is an action of locally finite order of L^∞ on A by algebra homomorphisms. The natural transformations $(A, H) \to (\overline{A}, \overline{H})$ are in bijection with the equivariant sequences of algebra homomorphisms.

5. $\mathcal{VB}$-valued bundle functors on all manifolds

These functors can be characterized similarly to Sections 2 and 4. First we discuss the r-th order $\mathcal{VB}$-valued functors on $\mathcal{M}f$. Let

$$V = (V_0, \dots, V_m, \dots)$$

be an infinite sequence of vector spaces and $K = (K_{m,n})$,

$$K_{m,n} : L^r_{m,n} \to \mathrm{Lin}\,(V_m, V_n)$$

be an action of L^r on V by linear maps, i.e. the condition analogous to (6)

$$K_{m,p}(\overline{g} \circ g) = K_{n,p}(\overline{g}) \circ K_{m,n}(g) \tag{16}$$

holds. The pair (V, K) defines a $\mathcal{VB}$-valued bundle functor E on $\mathcal{M}f$ as follows.

The restriction of $K_{m,n}$ to $G^r_m \subset L^r_{m,n}$ is a linear action K_m of G^r_m on V_m. Hence the associated bundle

$$EM = P^r M[V_m, K_m] \tag{17}$$

is a vector bundle for every m-dimensional manifold M. For a map $f : M \to \overline{M}$, $\dim \overline{M} = n$, we use the same expression $j^r_x f = v \circ g \circ u^{-1}$, $g \in L^r_{m,n}$, $u \in P^r_x M$, $v \in P^r_{f(x)}\overline{M}$ as in Section 2 and we define $Ef : EM \to E\overline{M}$ by

$$Ef\big(\{u, S\}\big) = \big\{v, K_{m,n}(g)(S)\big\}\,, \quad S \in V_m\,. \tag{18}$$

Analogously to Section 2, one proves

Proposition 5. *The r-th order $\mathcal{VB}$-valued functors on $\mathcal{M}f$ are in bijection with the actions of L^r on V by linear maps.*

For the $\mathcal{VB}$-valued bundle functors on $\mathcal{M}f$ of infinite order we should replace the action of L^r by an action of locally finite order of L^∞ on V by linear maps.

6. F.p.p.b. functors with values in vector bundles

Let F be a f.p.p.b. functor on $\mathcal{FM}$ of base order r with values in $\mathcal{VB}$. (We underline that FY is a vector bundle over Y.) For every $m \in \mathbb{N}$, we construct a $\mathcal{VB}$-valued functor E_m on $\mathcal{M}f$ by

$$E_m N = F_{0_m}(\mathbb{R}^m \times N)\,, \quad E_m f = F_{0_m}(\mathrm{id}_{\mathbb{R}^m} \times f)\,. \tag{19}$$

Clearly, each E_m preserves products. By [6], Lemma 37.2, E_m is the tangent functor T tensorized by a vector space V_m and its identity. Hence $A_m = F_{0_m}(\mathbb{R}^m \times \mathbb{R}) = \mathbb{R} \times V_m$ and the action $H = (H_{m,n})$ of L^r corresponding to F is determined by the restrictions

$$\widetilde{H}_{m,n} : L^r_{m,n} \times V_m \to V_n\,.$$

Since F is $\mathcal{VB}$-valued, for every $g \in L^r_{m,n}$ we have a linear map $V_m \to V_n$. This is the situation of Section 5. Hence F determines a $\mathcal{VB}$-valued functor G of order r on $\mathcal{M}f$. In the same way as in Example 4, we find

$$FY = VY \otimes GM\,, \quad Ff = Vf \otimes G\underline{f}\,. \tag{20}$$

If F is of infinite order, we proceed according to Section 4. Thus, we have proved

Proposition 6. *Every f.p.p.b. functor F on $\mathcal{FM}$ with values in $\mathcal{VB}$ is of the form* (20)*, where G is a $\mathcal{VB}$-valued functor on $\mathcal{M}f$.*

7. The problem of iteration

Consider $F = (A, H)$ of base order r and another f.p.p.b. functor $E = (B, K)$ on $\mathcal{FM}$ of base order s, so that $B = (B_0, \dots, B_m, \dots)$ is a sequence of Weil algebras and $K = (K_{m,n})$, $K_{m,n} : L^s_{m,n} \to \operatorname{Hom}(B_m, B_n)$. The base order of the composition $F \circ E$ is at most $s + r$. Write $F \circ E = (C, D)$, so that $D = (D_{m,n})$ is an action of L^{r+s} on a sequence $C = (C_0, \dots, C_m, \dots)$ of Weil algebras by algebra homomorphisms. We remark that the iteration problem on $\mathcal{FM}_m$ was solved in [2].

We shall use the following basic facts, [6], [10]. If V is a vector space, then $T^A V = V \otimes A$. For a linear map $f : V_1 \to V_2$, $T^A f$ is of the form

$$f \otimes \mathrm{id}_A : V_1 \otimes A \to V_2 \otimes A\,. \tag{21}$$

If $\mu : A \to B$ is an algebra homomorphism, then the natural transformation $\mu_V : T^A V \to T^B V$ is of the form

$$\mathrm{id}_V \otimes \mu : V \otimes A \to V \otimes B\,. \tag{22}$$

By the very definition (5),

$$D_{m,n}(j^{r+s}_{0_m}\gamma) = F_{0_m}\big(E(\gamma \times \mathrm{id}_{\mathbb{R}})\big)\,, \tag{23}$$

$\gamma : \mathbb{R}^m \to \mathbb{R}^n$, $\gamma(0_m) = 0_n$. For $x \in \mathbb{R}^m$, we write t_x for the translation on $\mathbb{R}^m$ transforming 0_m into x. We express $E(\gamma \times \mathrm{id}_{\mathbb{R}}) : \mathbb{R}^m \times B_m \to \mathbb{R}^n \times B_n$ as a pair of γ and a map $\mathbb{R}^m \times B_m \to B_n$. Using the standard identifications, [6], [2], we find

$$E(\gamma \times \mathrm{id}_{\mathbb{R}}) = \big(\gamma, K_{m,n}(j^s_{0_m}(t^{-1}_{\gamma(x)} \circ \gamma \circ t_x))\big)\,.$$

In the same way, by (8) and (10) we obtain for $x = 0$

$$\begin{aligned} &F\big(\gamma, K_{m,n}(j^s_{0_m}(t^{-1}_{\gamma(x)} \circ \gamma \circ t_x))\big) \\ &\qquad = \big(\gamma, H_{m,n}\big(j^r_{0_m}(t^{-1}_{\gamma(x)} \circ \gamma \circ t^{-1}_x)\big) \circ T^{A_m}\big(K_{m,n}(j^s_{0_m}(t^{-1}_{\gamma(x)} \circ \gamma \circ t_x))\big)\big)\,. \end{aligned}$$

By (21) or (22), the first or the second term of the composition on the right hand side is $H_{m,n}(j^r_{0_m}\gamma) \otimes \mathrm{id}_{B_n}$ or $\mathrm{id}_{A_m} \otimes K_{m,n}(j^s_{0_m}\gamma)$, respectively. Thus, we have proved

Proposition 7. *If we express $F\circ E$ in the form (C, D), where $C = (C_0, \dots, C_m, \dots)$ and $D = (D_{m,n})$ with $D_{m,n} : L^{r+s}_{m,n} \to \operatorname{Hom}(C_m, C_n)$, then $C_m = A_m \otimes B_m$ and*

$$D_{m,n}(j^{r+s}_{0_m}\gamma) = H_{m,n}(j^r_{0_m}\gamma) \otimes K_{m,n}(j^s_{0_m}\gamma)\,. \tag{24}$$

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