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A NOTE ON THE COUNTABLE EXTENSIONS OF SEPARABLE $p^{\omega+n}$–PROJECTIVE ABELIAN p–GROUPS

Peter Danchev

Abstract. It is proved that if G is a pure $p^{\omega+n}$-projective subgroup of the separable abelian p-group A for $n \in \mathbb{N} \cup \{0\}$ such that $|A/G| \leq \aleph_0$, then A is $p^{\omega+n}$-projective as well. This generalizes results due to Irwin-Snabb-Cutler (Comment. Math. Univ. St. Pauli, 1986) and the author (Arch. Math. (Brno), 2005).

Throughout this brief note all groups are assumed to be abelian p-primary, written additively as is customary when regarding the group structure. Since we shall deal exclusively only with p-torsion abelian groups, for some arbitrary but a fixed prime p, there should be no confusion in future removing the phrase *"is an abelian p-group"*. Concerning the terminology, under the term a *separable* group, we mean a reduced group without elements of infinite height (as computed in the full group). All other notation and notions are standard. For example, for any group A, the letter $A^1 = p^\omega A = \cap_{k<\omega} p^k A$ traditionally denotes the first *Ulm* subgroup of A.

Before stating and proving our main attainment, to make the article more nearly self contained, we give a systematic introducing in the basic concepts of the principle best known results in this theme.

In [5], Wallace has proven the following important assertion, which he used in the classification of rank one mixed abelian groups having totally projective primary components.

Theorem. *If the reduced group A possesses a totally projective subgroup G such that A/G is countable, then A is totally projective.*

As an immediate valuable consequence, we have the following.

Corollary. *If the separable group A has a subgroup G which is a direct sum of cyclic groups so that A/G is countable, then A is a direct sum of cyclic groups.*

In [2], Irwin-Snabb-Cutler established the following strengthening of the foregoing corollary for a more large class of groups, called by Nunke $p^{\omega+n}$-projective groups, where $n \in \mathbb{N} \cup \{0\}$ (see e.g. [3] and [4]).

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Theorem. *Let A be a separable group whose subgroup G satisfies the following conditions:*

i) *G is pure and dense in A;*

ii) *A/G is countable.*

Then A is $p^{\omega+1}$-projective if and only if G is $p^{\omega+1}$-projective.

In [1] we obtained an affirmation independent of the preceding one by dropping off the limitation on A to be separable but incorporating the additional restriction on G to be nice in A; thereby G is not dense in A if G is a proper subgroup different from A.

Theorem ([1])**.** *Suppose A is a reduced group of length not exceeding $\omega + n$ for some non-negative integer n with a pure and nice subgroup G such that A/G is countable. Then A is $p^{\omega+n}$-projective if and only if G is $p^{\omega+n}$-projective.*

Although the properties of $p^{\omega+1}$-projective groups are not always preserved by the $p^{\omega+n}$-projective ones over $n \geq 2$, the purpose of this exploration is to enlarge and improve the foregoing quoted second theorem on p. 51 of [2] to $p^{\omega+n}$-projective groups $\forall n \geq 0$ by simplifying the idea for its proof and by deleting the restriction on density.

The main statement of the present short paper is the following. As we have already seen it extends the alluded to above two corresponding assertions; we omit the limitation from the third Theorem on G to be nice in A but, however, only when the whole group A is separable.

Theorem. *Suppose that A is a separable group and G is a subgroup of A such that it satisfies the following conditions:*

j) *G is pure in A;*

jj) *A/G is countable.*

Then, for each non-negative integer n, A is $p^{\omega+n}$-projective if and only if G is $p^{\omega+n}$-projective.

Proof. The necessity is straightforward since each subgroup of a $p^{\omega+n}$-projective group is again $p^{\omega+n}$-projective (see cf. [3]).

We now concentrate on the more difficult converse implication. Consulting with the Nunke's criterion for $p^{\omega+n}$-projectivity ([3]), given $C \leq G[p^n]$ such that G/C is a direct sum of cyclic groups for an arbitrary fixed natural number n. Thus C is nice in G and $G^1 \subseteq C$; actually within the current case $G^1 = 0$ since A is separable.

Letting $C^- = \cap_{k<\omega}(C + p^k A)$ be the closure of C in A with respect to the relative p-adic topology of A, we clearly observe that $p^n C^- \subseteq A^1 = 0$ hence $C^- \subseteq A[p^n]$. Moreover, appealing to the modular law, we have $G \cap C^- = \cap_{k<\omega}[C + (G \cap p^k A)] = \cap_{k<\omega}(C + p^k G) = C + G^1 = C$. Consequently, $(G + C^-)/C^- \cong G/(G \cap C^-) = G/C$ is a direct sum of cycles. On the other hand, $A/C^- \cong A/C/C^-/C = A/C/(A/C)^1$ is separable, and $A/C^-/(G+C^-)/C^- \cong A/(G+C^-)$ is countable as an epimorphic image of the countable quotient A/G. Finally, the Corollary of the Wallace theorem enables us to infer that A/C^- is a direct sum of cyclic groups, whence the early used necessary and sufficient condition due to

Nunke is a guarantor that A is $p^{\omega+n}$-projective, as asserted. This completes the proof. $\square$

Remark. It is still unknown at this stage whether or not under the required circumstances j) and jj) the claim remains true for $p^{\omega+n}$-projective groups of length $\in (\omega, \omega+n]$.

So, we may state the following more concrete and yet unanswered question.

Problem. Can the assumptions on G to be pure or nice in A as well as on A to be separable be ignored?

Incidentally, we proceed by proving the following particular solution. In order to do this, we foremost need the following well-known technical tool.

Lemma. *If the subgroup G is pure in A, then the factor-group $(G+A^1)/A^1$ is pure in A/A^1.*

Proof. By definition, $G \cap p^n A = p^n G$, $\forall n \geq 1$. Therefore, owing to the modular law, we calculate that $[(G+A^1)/A^1] \cap p^n(A/A^1) = [(G+A^1)/A^1] \cap (p^n A/A^1) = [(G+A^1) \cap p^n A]/A^1 = [A^1 + (G \cap p^n A)]/A^1 = (A^1 + p^n G)/A^1 = p^n((G+A^1)/A^1)$, and thus the wanted purity follows. The proof is finished. $\square$

And so, we are now ready with the promised particular answer.

Corollary. *Let A be a group whose subgroup G satisfies conditions* j) *and* jj)*. Then*

1) *A/A^1 is $p^{\omega+n}$-projective $\iff$ G/G^1 is $p^{\omega+n}$-projective.*
2) *G being $p^{\omega+n}$-projective $\Rightarrow$ A/A^1 is $p^{\omega+n}$-projective.*
3) *G being $p^{\omega+n}$-projective $\Rightarrow$ A is $p^{\omega+2n}$-projective, provided length$(A) \leq \omega+n$.*

Proof. 1) First of all, notice that $A/A^1 \supseteq (G+A^1)/A^1 \cong G/G^1$ since $G \cap A^1 = G^1$. Next, we routinely observe that $A/A^1/(G+A^1)/A^1 \cong A/(G+A^1)$ is at most countable because so is A/G. Furthermore, the utilization of the Lemma along with the previous Theorem, both applied to the group $(G+A^1)/A^1$, substantiates the equivalence.

2) If G is $p^{\omega+n}$-projective, then so does G/G^1 (see, for instance, [4, p. 194, Corollary 2.4]). Indeed, there is $C \leq G$ with $p^n C = 0$ and G/C a direct sum of cycles. Consequently, $G^1 \subseteq C$ and $p^n(C/G^1) = 0$ with $G/G^1/C/G^1 \cong G/C$ is a direct sum of cyclic groups. Knowing this, the aforementioned Nunke's criterion works.

Another confirmation that A/A^1 must be $p^{\omega+n}$-projective is like this. For C^- taken as in the proof of the Theorem, we see that $A/A^1/C^-/A^1 \cong A/C^-$ is a direct sum of cycles and besides $p^n(C^-/A^1) = 0$. That is why, with the aid of Nunke's criterion, we are done.

So, in both variants presented, the first point can be employed to derive the claim.

3) For C^- as in the Theorem, we easily find that $p^n C^- \subseteq A^1 \subseteq A[p^n]$ whence $C^- \subseteq A[p^{2n}]$. Further, with this in hand, we ascertain at once that the same arguments as in the just cited Theorem are applicable to get the desired implication. This concludes the proof. $\square$

Corrections. In ([1], p. 265), the year "1981" should be replaced by "1971". Moreover, in ([1], p. 270, Theorem 4), the group A should be "reduced".

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13, General Kutuzov Street, block 7, floor 2, flat 4
4003 Plovdiv, Bulgaria
E-mail: pvdanchev@yahoo.com