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CHARACTERIZATION OF $E\mathcal{F}$-SUBCOMPACTIFICATION

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Abstract. For extending the notion of E-algebra, as defined in [2], we present an example of an m-admissible algebra which is not an E - algebra. Then we define E-subcompactification and $E\mathcal{F}$-subcompactification to study the universal E-subcompactification and the universal $E\mathcal{F}$-subcompactification from the function algebras point of view.

1. Introduction and preliminaries

A semigroup S is called *right reductive* if for each $a, b \in S$, the equality $at{=}bt$ for every $t \in S$, implies that $a{=}b$. For example, all right cancellative semigroups and semigroups with a right identity, are right reductive.

For notation and terminology our ground reference is the extensive book of Berglund et al.[1]. From now on S will be a semitopological semigroup. By a *semigroup compactification* of S we mean a pair (ψ, X), where X is a compact Hausdorff right topological semigroup, and $\psi : S \longrightarrow X$ is a continuous homomorphism with dense image such that, for each $s \in S$, the mapping $x \longrightarrow \psi(s)x : X \longrightarrow X$ is continuous. The C^*-algebra of all bounded complex-valued continuous functions on S, will be denoted by $\mathcal{C}(S)$. For $\mathcal{C}(S)$ the left and right translations, L_s and R_t, are defined for each $s, t \in S$ by $(L_s f)(t) = f(st) = (R_t f)(s), f \in \mathcal{C}(S)$. A subset $\mathcal{F}$ of $\mathcal{C}(S)$ is said to be left translation invariant, if for all $s \in S$, $L_s\mathcal{F} \subseteq \mathcal{F}$. A left translation invariant unital C^*-subalgebra $\mathcal{F}$ of $\mathcal{C}(S)$ is called *m-admissible* if the function $s \longrightarrow T_\mu f(s) = \mu(L_s f)$ is in $\mathcal{F}$ for all $f \in \mathcal{F}$ and $\mu \in S^{\mathcal{F}}$ (=the spectrum of $\mathcal{F}$). Then the product of $\mu, \nu \in S^{\mathcal{F}}$ can be defined by $\mu\nu = \mu \circ T_\nu$ and the Gelfand topology on $S^{\mathcal{F}}$ makes $(\epsilon, S^{\mathcal{F}})$ a semigroup compactification (called the $\mathcal{F}$-compactification) of S, where $\epsilon : S \longrightarrow S^{\mathcal{F}}$ is the evaluation mapping.

Some m-admissible subalgebras of $\mathcal{C}(S)$ that we will need in the sequel are: $\mathcal{LMC}$:= left multiplicatively continuous functions, $\mathcal{D}$:=distal functions, $\mathcal{MD}$:=minimal distal functions, and $\mathcal{SD}$:=strongly distal functions. We also write $\mathcal{GP}$ for $\mathcal{MD} \cap \mathcal{SD}$; and we define $\mathcal{LZ} := \{f \in \mathcal{C}(S); f(st) = f(s) \text{ for all } s, t \in S\}$

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and $\mathcal{R}Z := \{f \in \mathcal{C}(S); f(st) = f(t) \text{ for all } s,t \in S\}$.
For a discussion of the universal property of the corresponding compactifications of these function algebras see [1] and also [4].

Let (ψ, X) be a compactification of S, then the mapping $\sigma : S \times X \longrightarrow X$, defined by $\sigma(s,x) = \psi(s)x$, is separately continuous and so (S, X, σ) is a flow. If Σ_X denotes the enveloping semigroup of the flow (S, X, σ) (i.e., the pointwise closure of semigroup $\{\sigma(s,\cdot) : s \in S\}$ in X^X) and the mapping $\sigma_X : S \longrightarrow \Sigma_X$ defined by $\sigma_X(s) = \sigma(s,\cdot)$ for all $s \in S$, then (σ_X, Σ_X) is a compactification of S (see [1; 1.6.5]).

One can easily verify that $\Sigma_X = \{\lambda_x : x \in X\}$, where $\lambda_x(y) = xy$ for each $y \in X$. If we define the mapping $\theta : X \longrightarrow \Sigma_X$ by $\theta(x) = \lambda_x$, then θ is a continuous homomorphism with the property that $\theta \circ \psi = \sigma_X$. So (σ_X, Σ_X) is a factor of (ψ, X), that is $(\psi, X) \geq (\sigma_X, \Sigma_X)$. By definition, θ is one-to-one, if and only if X is right reductive. So we get the next proposition, which is an extension of the Lawson's result [5; 2.4(ii)].

Proposition 1.1. *Let (ψ, X) be a compactification of S. Then $(\sigma_X, \Sigma_X) \cong (\psi, X)$, if and only if X is right reductive.*

A compactification (ψ, X) is called *reductive*, if X is right reductive. For example, the $\mathcal{M}D$, $\mathcal{G}P$ and $\mathcal{L}Z$-compactifications, are reductive.
In [2] an m-admissible subalgebra $\mathcal{F}$ of $\mathcal{C}(S)$ is defined as an *E-algebra* if there is a compactification (ψ, X) such that $(\sigma_X, \Sigma_X) \cong (\epsilon, S^{\mathcal{F}})$. In this setting (ψ, X) is called an *$E\mathcal{F}$-compactification* of S. Clearly every reductive compactification is an E-compactification but the converse is not, in general true; for example see [2; 2.2].

Now we present an example of m-admissible subalgebra of $\mathcal{C}(S)$ which is not an E - algebra. For this purpose we need the following lemma.

Lemma 1.2. *Let S be a factorizable semigroup, i.e. $S = S^2$ (for instance, let S be a regular semigroup, see* [3]*) and (ψ, X) be a compactification of S such that $xyz = yz$ for every $x, y, z \in X$. Then X is a right zero semigroup.*

Proof. We show that $yz = z$ for each $y, z \in X$. First suppose that $z \in \psi(S)$. So $z = \psi(s)$ for some $s \in S$. Hence $yz = y\psi(s) = y\psi(\underline{s_1 s_2}) = y\psi(s_1)\psi(s_2) = \psi(s_1)\psi(s_2) = \psi(s) = z$. Now let $y \in \psi(S)$ and $z \in X = \overline{\psi(S)}$. So $y = \psi(t)$ for some $t \in S$ and there exist a sequence $\{\psi(t_n)\}$ in $\psi(S)$ such that $\psi(t_n) \to z$. Since $\psi(S) \subset \Lambda(X)$, we have $\psi(t_n) = \underline{y\psi(t_n)} \to yz$. Therefore $yz = z$.

Now suppose that $y \in X = \overline{\psi(S)}$ and $z \in X$. Then there exists a sequence $\{\psi(s_n)\}$ in $\psi(S)$ such that $\psi(s_n) \to y$. Since X is right topological, $\psi(s_n)z \to yz$. But $\psi(s_n)z = z$ for all n, and so $yz = z$ for every $y, z \in X$, as claimed. □

Example 1.3. If S is a factorizable semigroup, then $\mathcal{R}Z$ is not an E-algebra. Indeed, let (ψ, X) be a compactification of S such that $(\sigma_X, \Sigma_X) \cong (\epsilon, S^{\mathcal{R}Z})$, then Σ_X must be a right zero semigroup. It is easy to see that Σ_X is a right zero semigroup if and only if $xyz = yz$ for every $x, y, z \in X$. Now by Lemma 1.2 X is a right zero semigroup and so Σ_X is a trivial semigroup.

2. E-SUBCOMPACTIFICATION

In this section we extend the notion of $E\mathcal{F}$-compactification (see [2]), to $E\mathcal{F}$-subcompactification.

Definition 2.1. Let (ψ, X) be a compactification of S. We say that a compactification (ϕ, Y) is an E-subcompactification of (ψ, X) if (σ_Y, Σ_Y) is a factor of (ψ, X), (in symbol, $(\sigma_Y, \Sigma_Y) \leq (\psi, X)$).

Trivially, every compactification of S is an E-subcompactification of itself. Now we are going to construct the universal E-subcompactification of S.

Lemma 2.2. *Let (ϕ, Y) be the subdirect product of the family $\{(\phi_i, Y_i) : i \in I\}$ of compactifications of S. Then (σ_Y, Σ_Y) is isomorphic to the subdirect product of the family $\{(\sigma_{Y_i}, \Sigma_{Y_i}) : i \in I\}$ (i.e., $\vee(\sigma_{Y_i}, \Sigma_{Y_i}) \cong (\sigma_Y, \Sigma_Y)$).*

Proof. By [1; 3.2.5], for each $i \in I$, there exists a homomorphism p_i of (ϕ, Y) onto (ϕ_i, Y_i). So, by [1; 1.6.7], for each $i \in I$, there exists a unique continuous homomorphism π_i of (σ_Y, Σ_Y) onto $(\sigma_{Y_i}, \Sigma_{Y_i})$ such that

$$\pi_i(\zeta)\big(p_i(y)\big) = p_i\big(\zeta(y)\big) \quad y \in Y,\ \zeta \in \Sigma_Y\,.$$

Suppose that $\zeta_1, \zeta_2 \in \Sigma_Y$. If $\pi_i(\zeta_1) = \pi_i(\zeta_2)$ for all $i \in I$, then

$$p_i\big(\zeta_1(y)\big) = \big(\pi_i(\zeta_1)\big)\big(p_i(y)\big) = \big(\pi_i(\zeta_2)\big)\big(p_i(y)\big) = p_i\big(\zeta_2(y)\big)\,,$$

for all $y \in Y$ and $i \in I$. Thus $\zeta_1 = \zeta_2$. Therefore the family $\{\pi_i : i \in I\}$ separates the points of Σ_Y. Now the conclusion follows from [1; 3.2.5]. □

Theorem 2.3. *Every compactification (ψ, X) of S has the universal E-subcompactification.*

Proof. Let (ψ, X) be a compactification of S. Suppose $\{(\phi_i, Y_i) : i \in I\}$ is a family of E-subcompactifications of (ψ, X), and (ϕ, Y) is the subdirect product of this family. We show that (ϕ, Y) is an E-subcompactification of (ψ, X), and so it is the universal E-subcompactification of (ψ, X). To see this, for each $i \in I$, we have $(\sigma_{Y_i}, \Sigma_{Y_i}) \leq (\psi, X)$. So, by the subdirect product property and the previous lemma we have, $(\sigma_Y, \Sigma_Y) \cong \vee(\sigma_{Y_i}, \Sigma_{Y_i}) \leq (\psi, X)$. This means that (ϕ, Y) is an E-subcompactification of (ψ, X). □

Definition 2.4. Let $\mathcal{F}$ be an m-admissible subalgebra of $\mathcal{C}(S)$. The compactification (ψ, X) of S is called an $E\mathcal{F}$-subcompactification of S if $(\sigma_X, \Sigma_X) \leq (\epsilon, S^{\mathcal{F}})$.

Now we are going to prove the next theorem which is an extension of [2; 2.6].

Theorem 2.5. *Every m- admissible subalgebra $\mathcal{F}$ of $\mathcal{C}(S)$ has the universal $E\mathcal{F}$-subcompactification.*

Proof. Set

$$G_{\mathcal{F}} := \{f \in \mathcal{L}MC : T_\nu f \in \mathcal{F} \text{ for all } \nu \in S^{\mathcal{L}MC}\}\,.$$

It is easy to verify that $G_{\mathcal{F}}$ is an m-admissible subalgebra of $\mathcal{C}(S)$ containing $\mathcal{F}$. By definition of $G_{\mathcal{F}}$ we can define the mapping $\theta : S^{\mathcal{F}} \longrightarrow \Sigma_{S^{G_{\mathcal{F}}}}$ by $\theta(\mu) = \lambda_{\tilde{\mu}}$, where $\tilde{\mu}$ is an extension of μ to $S^{G_{\mathcal{F}}}$. Clearly θ is continuous and $\theta \circ \epsilon = \sigma_{S^{G_{\mathcal{F}}}}$.

Thus $(\epsilon, S^{\mathcal{F}}) \geq (\sigma_{S^{G_{\mathcal{F}}}}, \Sigma_{S^{G_{\mathcal{F}}}})$. So $G_{\mathcal{F}}$ is an $E\mathcal{F}$-subcompactification of S. Finally, if (ψ, X) is an $E\mathcal{F}$-subcompactification of S and $f \in \psi^*\big(\mathcal{C}(X)\big)$ (where ψ^* is the adjoint of ψ), then by [2; 2.5.], $T_\mu f \in \sigma_X^*\big(\mathcal{C}(\Sigma_X)\big) \subset \mathcal{F}$ for all $\mu \in S^{\mathcal{L}MC}$. Therefore $\psi^*\big(\mathcal{C}(X)\big) \subset G_{\mathcal{F}}$ and $(\psi, X) \leq (\epsilon, S^{G_{\mathcal{F}}})$. $\square$

References

[1] Berglund, J. F., Junghenn, H. D. and Milnes, P., *Analysis on Semigroups, Function spaces, Compactifications, Representations*, Wiley, New York, 1989.

[2] Fattahi, A., Pourabdollah, M. A. and Sahleh, A., *Reductive Compactifications of Semitopological Semigroups*, Internat. J. Math. Math. Sci. **51** (2003), 3277–3280.

[3] Howie, J. M., *An Introduction to Semigroup Theory*, Academic Press, New York, 1976.

[4] Junghenn, H. D., *Distal compactifications of semigroups*, Trans. Amer. Math. Soc. **274** (1982), 379–397.

[5] Lawson, J. D., *Flows and compactifications*, J. London Math. Soc. **46** (1992), 349–363.

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