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ON THE FIRST EIGENVALUE OF SPACELIKE HYPERSURFACES IN LORENTZIAN SPACE

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Abstract. In this paper we obtain a lower bound for the first Dirichlet eigenvalue of complete spacelike hypersurfaces in Lorentzian space in terms of mean curvature and the square length of the second fundamental form. This estimate is sharp for totally umbilical hyperbolic spaces in Lorentzian space. We also get a sufficient condition for spacelike hypersurface to have zero first eigenvalue.

1. Introduction

Let M^n be a complete noncompact Riemannian manifold and $\Omega \subset M^n$ a domain with compact closure and nonempty boundary $\partial\Omega$. The Dirichlet eigenvalue $\lambda_1(\Omega)$ of Ω is defined by

$$\lambda_1(\Omega) = \inf \Big(\frac{\int_\Omega |\nabla f|^2 dM}{\int_\Omega f^2 dM} : f \in L^2_{1,0}(\Omega)\backslash\{0\} \Big) ,$$

where dM is the volume element on M^n and $L^2_{1,0}(\Omega)$ the completion of C_0^∞ with respect to the norm

$$\parallel \varphi \parallel^2_\Omega = \int_\Omega \varphi^2 dM + \int_\Omega |\varphi|^2 dM \,.$$

If $\Omega_1 \subset \Omega_2$ are bounded domains, then $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2) \geq 0$. Thus one may define the first Dirichlet eigenvalue of M^n as the following limit

$$\lambda_1(M) = \lim_{r\to\infty} \lambda_1(B(p,r)) \geq 0 \,,$$

where $B(p,r)$ is the geodesic ball of M^n with radius r centered at p. It is clear that the definition of $\lambda_1(M)$ does not depend on the center point p. According to Schoen and Yau [6] it is an important question to find conditions which will imply $\lambda_1(M) > 0$. The best well-known result toward the question is due to Mckean [4]. He showed that if M^n is an n-dimensional, complete noncompact,

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simply connected Riemannian manifold with sectional curvature $K_M \le -a^2 < 0$, then

$$\lambda_1(M) \ge \frac{(n-1)^2 a^2}{4}.$$

Mckean's estimate is sharp in the sense that for hyperbolic space $H^n(-a^2)$ with constant curvature $-a^2$, we have $\lambda_1(H^n(-a^2)) = \frac{1}{4}(n-1)^2 a^2$.

There is a version of Mckean's theorem for submanifolds of the hyperbolic space with bounded mean curvature due to Cheung and Leung [1]. They proved that for an n-dimensional complete noncompact immersed submanifold M^n in $H^{n+p}(-1)$ with bounded mean curvature $H \le c < n-1$, one has

$$\lambda_1(M) \ge \frac{(n-1-c)^2}{4},$$

and the estimate is sharp for totally geodesic $H^n(-1)$ in $H^{n+p}(-1)$.

Besides the above mentioned results, recently Pacellibessa and Montenegro[5] discussed similar problem in a little more general situations. Our motivation comes from the fact that the hyperbolic space, as the totally umbilical spacelike hypersurface in Lorentzian space, has positive first eigenvalue. This suggests that we look at spacelike hypersurfaces in Lorentzian space. First let us recall the fact that complete spacelike hypersurfaces in Lorentzian $(n+1)$-space L^{n+1} are all noncompact, as one can verify that the projection $\Pi : M^n \to R^n_a$ defined by any unit timelike vector a is a diffeomorphism(for details see §3). We shall prove the following

Theorem 1. *Let M^n be a complete spacelike hypersurface in Lorentzian $(n+1)$-space L^{n+1}. Suppose that both*

$$A = \inf\left(\frac{1}{n}S + (n-2)H^2 - 2H\sqrt{\frac{n-1}{n}(S-nH^2)} - n|\nabla H|\right) \tag{1.1}$$

and

$$B = \sup\left(H + \sqrt{\frac{n-1}{n}(S-nH^2)}\right) \tag{1.2}$$

are positive constants, then

$$\lambda_1(M) \ge \frac{A^2}{4B^2}, \tag{1.3}$$

where H and S denote the mean curvature and square length of second fundamental form of M^n in L^{n+1},respectively.

Remark. (1.3) is sharp in the sense that for the hyperbolic space in L^{n+1}, it becomes an equality.

On the other hand, it is also interesting to ask that for what geometries a noncompact manifold M^n has zero first eigenvalue. Cheng and Yau [2] showed that $\lambda_1(M) = 0$ if M^n has polynomial volume growth. In this paper we shall prove the following

Theorem 2. *Let M^n be a complete spacelike hypersurface in L^{n+1} whose Gauss map is bounded. Then $\lambda_1(M) = 0$.*

2. Preliminaries

Let L^{n+1} be the Lorentzian $(n+1)$-space and M^n be a complete spacelike hypersurface in L^{n+1}. We choose a local Lorentzian frames $e_1, e_2, \cdots, e_{n+1}$ in L^{n+1} such that, restricted to M^n, $e_1, e_2, \cdots, e_n$ are tangent to M^n, and e_{n+1} is future-directed. We use the following convention on the ranges of indices:

$$1 \leq A, B, \cdots \leq n+1, \quad 1 \leq i, j, \cdots \leq n$$

Let $\omega_1, \omega_2, \cdots, \omega_{n+1}$ be the dual frames of $e_1, e_2, \cdots, e_{n+1}$ so that the Lorentzian metric on L^{n+1} is given by $d\widetilde{s}^2 = \omega_1^2 + \cdots + \omega_n^2 - \omega_{n+1}^2 = \sum_A \varepsilon_A \omega_A^2$, where $\varepsilon_1 = \cdots = \varepsilon_n = 1$ and $\varepsilon_{n+1} = -1$. The structure equations of L^{n+1} are given by

$$de_A = -\sum_B \varepsilon_A \omega_{AB} e_B \,, \quad \omega_{AB} + \omega_{BA} = 0 \,, \tag{2.1}$$

$$d\omega_A = -\sum_B \varepsilon_B \omega_{AB} \wedge \omega_B \,, \tag{2.2}$$

$$d\omega_{AB} = -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} \,. \tag{2.3}$$

When restricted on M^n, we have $\omega_{n+1} = 0$, and the induced Riemannian metric of M^n is written as $ds^2 = \sum_i (\omega_i)^2$. Since $0 = d\omega_{n+1} = -\sum_i \omega_{n+1,i} \wedge \omega_i$, by Cartan's lemma we may write

$$\omega_{n+1,i} = \sum_j h_{ij} \omega_j \,, \quad h_{ij} = h_{ji} \,. \tag{2.4}$$

We call $h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j$ the second fundamental form of M^n. The mean curvature and the square length of the second fundamental form of M^n is defined by $H = (1/n) \sum_i h_{ii}$ and $S = \sum_{i,j} h_{ij}^2$, respectively. The covariant differentiation of h is defined by $\nabla h = \sum_{i,j,k} h_{ijk} \omega_i \otimes \omega_j \otimes \omega_k$, where

$$\sum_k h_{ijk} \omega_k = dh_{ij} - \sum_k (h_{kj} \omega_{ki} + h_{ik} \omega_{kj}) \,, \tag{2.5}$$

and it satisfies the following Codazzi equation:

$$h_{ijk} = h_{ikj} \,. \tag{2.6}$$

In order to prove Theorem 1 we need a lemma due to Pacellibessa and Montenegro [5]. Let $\Omega \subset M^n$ be a domain with compact closure in M^n, and $\chi(\Omega)$ be the set of all smooth vector fields X on Ω with $\| X \|_\infty = \sup_\Omega |X| < \infty$ and inf div$X > 0$. Define $c(\Omega)$ by

$$c(\Omega) = \sup \left(\frac{\inf \operatorname{div} X}{\| X \|_\infty} : X \in \chi(\Omega) \right) .$$

Remark. To show that $\chi(\Omega) \neq \emptyset$, consider the boundary value problem $\Delta f = 1$ in Ω, and $f = 0$ on $\partial\Omega$ and set $X = \nabla f$, then div$X = 1$ and $\| X \|_\infty < \infty$.

Lemma 3. [5] *Let $\Omega \subset M^n$ be a domain with compact closure ($\partial\Omega \neq \emptyset$) in M^n. Then*

$$\lambda_1(\Omega) \geq \frac{c(\Omega)^2}{4} > 0\,.$$

3. The proof of theorems

In this section we shall complete the proof of Theorems 1 and 2. For complete spacelike hypersurface M^n in L^{n+1}, the Gauss map is defined by $e_{n+1} : M^n \to H^n(-1) \subset L^{n+1}$. Let us fix a future-directed unit timelike vector $a \in H^n(-1)$ and define the projection $\Pi : M^n \to R^n_a$ by

$$\Pi(x) = x + \langle x, a\rangle a\,, \tag{3.1}$$

where $\langle , \rangle$ is the standard Lorentzian inner product on L^{n+1} and R^n_a the totally geodesic Euclidean n-space determined by a which is defined by

$$R^n_a = \{x \in L^{n+1} : \langle x, a\rangle = 0\}\,. \tag{3.2}$$

It is clear from (3.1) that

$$d\Pi(X) = X + \langle X, a\rangle a \tag{3.3}$$

for any tangent vector field on M^n, and consequently,

$$|d\Pi(X)|^2 = |X|^2 + \langle X, a\rangle^2\,. \tag{3.4}$$

(3.4) means that the map $\Pi : M^n \to R^n_a$ increases the distance. If a map, from a complete Riemannian manifold M_1 into another Riemannian manifold M_2 of the same dimension, increases the distance, then it is a covering map and M_2 is complete [3,VIII, Lemma 8.1]. Hence Π is a covering map,but R^n_a being simply connected this means that Π is in fact a diffeomorphism between M^n and R^n_a, and thus M^n is noncompact.

Let us first prove Theorem 2.Assume that the Gauss map $e_{n+1} : M^n \to H^n(-1)$ is bounded, then there exists $\rho > 0$ such that

$$1 \leq -\langle a, e_{n+1}\rangle \leq \rho\,. \tag{3.5}$$

Write

$$a = a^T - \langle a, e_{n+1}\rangle e_{n+1}\,, \tag{3.6}$$

where a^T denotes the component of a which is tangent to M^n. Since $a \in H^n(-1)$ we have

$$-1 = |a^T|^2 - \langle a, e_{n+1}\rangle^2\,. \tag{3.7}$$

It follows from (3.4)–(3.7) that

$$|X|^2 \leq |d\Pi(X)|^2 \leq \rho^2 |X|^2 \tag{3.8}$$

for any tangent vector field on M^n. Let $B(p, r)$ is the geodesic ball of M^n with radius r centered at $p \in M^n$. We claim that $\Pi\big(B(p, r)\big) \subset \widetilde{B}(\widetilde{p}, \rho r)$, where $\widetilde{B}(\widetilde{p}, \rho r)$ denotes the geodesic ball of R^n_a with radius ρr centered at $\widetilde{p} = \Pi(p)$. In fact, for any $\widetilde{q} \in \Pi(B(p,r))$ let $q \in B(p,r)$ be the unique point such that $\Pi(q) = \widetilde{q}$, and $\gamma : [a,b] \to M^n$ is the minimal geodesic joining p and q, then from (3.8) we have

$$\begin{aligned}\widetilde{d}(\widetilde{p},\widetilde{q}) \leq L(\Pi \circ \gamma) &= \int_a^b |d\Pi\big(\gamma'(t)\big)|\, dt \\ &\leq \rho \int_a^b |\gamma'(t)|\, dt = \rho L(\gamma) = \rho d(p,q) < \rho r\,,\end{aligned}$$

where $\widetilde{d}$ and d denote the distance in R_a^n and M^n, respectively. This prove our claim.

Let dV denotes the n-dimensional volume element on R_a^n. Using (3.3) and (3.6) it follows that

$$\begin{aligned}\Pi^*(dV)(X_1, \cdots, X_n) &= \det(d\Pi(X_1), \cdots, d\Pi(X_n), a) = \det(X_1, \cdots, X_n, a) \\ &= -\langle a, e_{n+1}\rangle \det(X_1, \cdots, X_n, e_{n+1}) \\ &= -\langle a, e_{n+1}\rangle dM(X_1, \cdots, X_n)\end{aligned}$$

for any tangent vector fields $X_1, \cdots, X_n$ of M^n. In other words,

$$\Pi^*(dV) = -\langle a, e_{n+1}\rangle dM\,. \tag{3.9}$$

Since $\Pi(B(p,r)) \subset \widetilde{B}(\widetilde{p}, \rho r)$ and $\Pi : M^n \to R_a^n$ is a diffeomorphism, it follows from (3.5), (3.8) and (3.9) that

$$\begin{aligned}\rho^n r^n \omega_n &= \mathrm{vol}(\widetilde{B}(\widetilde{p}, \rho r)) \geq \mathrm{vol}\big(\Pi(B(p,r))\big) = \int_{\Pi(B(p,r))} dV \\ &= \int_{B(p,r)} \Pi^*(dV) = \int_{B(p,r)} -\langle a, e_{n+1}\rangle dM \geq \int_{B(p,r)} dM = \mathrm{vol}\big(B(p,r)\big)\,,\end{aligned} \tag{3.10}$$

where ω_n denotes the volume of unit ball in Euclidean n-space. (3.10) means that the order of the volume growth of M^n is not larger than n, thus by [2] we see that $\lambda_1(M) = 0$, and Theorem 2 is proved.

Next we want to prove Theorem 1. Using (2.1)–(2.6), a standard computation shows that

$$\triangle\langle a, e_{n+1}\rangle = \langle a, n\nabla H\rangle + S\langle a, e_{n+1}\rangle\,, \tag{3.11}$$

where $\triangle$ is the Laplace-Beltrami operator on M^n, and ∇H the gradient of mean curvature. Let $f = \log(-\langle a, e_{n+1}\rangle)$, then from (3.11) we have

$$\triangle f = S + \frac{\langle a, n\nabla H\rangle}{\langle a, e_{n+1}\rangle} - \frac{|\nabla\langle a, e_{n+1}\rangle|^2}{\langle a, e_{n+1}\rangle^2}\,. \tag{3.12}$$

It is clear from (3.7) that

$$|\langle a, n\nabla H\rangle| \leq n|\nabla H| \cdot |a^T| \leq n|\nabla H| \cdot |\langle a, e_{n+1}\rangle|\,. \tag{3.13}$$

In order to estimate the quantity $|\nabla\langle a, e_{n+1}\rangle|^2$, we need the following lemma which can be easily verified by the method of Lagrange multipliers.

Lemma 4. *Let $\lambda_1, \cdots, \lambda_n$ satisfy $\sum_i \lambda_i = nH$ and $\sum_i \lambda_i^2 = S$, then*

$$\max_i\{\lambda_i\} \le H + \sqrt{\frac{n-1}{n}(S - nH^2)}\,.$$

We can assume that the local frames $e_1, \cdots, e_n$ diagonalize the second fundamental form, i.e., we have $h_{ij} = \lambda_i \delta_{ij}$, thus by (3.7) and Lemma 4 we get

$$|\nabla\langle a, e_{n+1}\rangle|^2 = \sum_i (e_i\langle a, e_{n+1}\rangle)^2 = \sum_i (\lambda_i\langle a^T, e_i\rangle)^2 \le \max_i\{\lambda_i^2\}|a^T|^2$$

$$\le \left(\frac{n-1}{n}S - (n-2)H^2 + 2H\sqrt{\frac{n-1}{n}(S - nH^2)}\right)\langle a, e_{n+1}\rangle^2\,. \tag{3.14}$$

Combining (3.12)–(3.14) we get

$$\Delta f \ge \frac{1}{n}S + (n-2)H^2 - 2H\sqrt{\frac{n-1}{n}(S - nH^2)} - n|\nabla H|\,. \tag{3.15}$$

Now suppose that the numbers A, B defined by (1.1) and (1.2) are both positive constants. For any domain $\Omega \subset M^n$ with compact closure and nonempty boundary, let $X = \nabla f$, then (3.14) and (3.15) implies that $\mathrm{div}X = \triangle f \ge A, \| X \|_\infty \le B$ and consequently, $X \in \chi(\Omega)$, and $c(\Omega) \ge A/B$. By Lemma 3, we have

$$\lambda_1(\Omega) \ge \frac{c(\Omega)^2}{4} \ge \frac{A^2}{4B^2}\,. \tag{3.16}$$

Since $\Omega \subset M^n$ is arbitrary, (3.16) implies (1.3), and so we are done.

References

[1] Cheung, L. F. and Leung, P. F., *Eigenvalue estimates for submanifolds with bounded mean curvature in the hyperbolic space*, Math. Z. **236** (2001), 525–530.

[2] Cheng, S. Y. and Yau, S. T., *Differential equations on Riemannian manifolds and geometric applications*, Comm. Pure Appl. Math. **28** (1975), 333–354.

[3] Kobayashi, S. and Nomizu, K., *Foundations of Differential Geometry*, vol II, Interscience, New York, 1969.

[4] Mckean, H. P., *An upper bound for the spectrum of Δ on a manifold of negative curvature*, J. Differential Geometry **4** (1970), 359–366.

[5] Pacellibessa, G. and Montenegro, J. F., *Eigenvalue estimates for submanifolds with locally bounded mean curvature*, Ann. Glob. Anal. Geom. **24** (2003), 279–290.

[6] Schoen, R. and Yau, S. T., *Lectures on differential geometry*, Lecture Notes in Geom. Topo. **1** (1994).

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