ARCHIVUM MATHEMATICUM (BRNO)
Tomus 42 (2006), 205 – 213

A NONLINEAR PERIODIC SYSTEM WITH NONSMOOTH POTENTIAL OF INDEFINITE SIGN

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Abstract. In this paper we consider a nonlinear periodic system driven by the vector ordinary p-Laplacian and having a nonsmooth locally Lipschitz potential, which is positively homogeneous. Using a variational approach which exploits the homogeneity of the potential, we establish the existence of a nonconstant solution.

1. Introduction

In this paper we study the following nonlinear periodic system with nonsmooth potential

$$\left\{ \begin{array}{l} -\big(\|x'(t)\|^{p-2}x'(t)\big)' \in \partial j\big(t,x(t)\big) \quad \text{a.e. on} \quad T=[0,b] \\ x(0)=x(b),\ x'(0)=x'(b), \quad 1<p<\infty\,. \end{array} \right\} \tag{1.1}$$

Here the potential function $x \to j(t,x)$ is only locally Lipschitz not necessarily C^1 and by $\partial j(t,x)$ we denote the generalized (Clarke) subdifferential of $j(t,\cdot)$ (see Section 2). The purpose of this work is to establish the existence of nontrivial solutions, when the potential is indefinite in sign. In the past this problem has been addressed only in the context of semilinear (i.e. $p=2$), smooth (i.e. $j(t,\cdot) \in C^1\big(\mathbb{R}^{\mathbb{N}},\mathbb{R}\big)$) systems. We refer to the works of Lassoued [10],[11], Ben Naoum-Troestler-Willem [5], Girardi-Matzeu [9], Antonacci [4], Xu-Guo [14] and Tang-Wu [13]. In Lassoued [10],[11] the potential has the form $j(t,x)=b(t)V(x)$ where $b \in L^1(T)$ with changing sign and $V \in C^2(\mathbb{R}^{\mathbb{N}},\mathbb{R})$ is strictly convex and nonnegative. In Lassoued [10] V is subquadratic, while in Lassoued [11] V is positively homogeneous of degree $\theta>2$ (hence V is superquadratic). Her approach is based on the dual action principle of Clarke and on the Lyapunov-Schmidt reduction method. Girardi-Matzeu [9] also assume that $j(t,x)=b(t)V(x)$ and

2000 *Mathematics Subject Classification*: 34C25.
Key words and phrases: locally Lipschitz function, generalized subdifferential, p-Laplacian, homogeneous function, variational method, Poincare-Wirtinger inequality, potential indefinite in sign.
The first author was supported by a grant of the National Scholarship Foundation of Greece (I.K.Y.).
Received September 15, 2004, revised April 2006.

impose on V a kind of generalized Ambrosetti-Rabinowitz condition of the form $|(V'(x),x)_{\mathbb{R}^{\mathbb{N}}} - \beta V(x)| \le c\|x\|^2$ for all $x \in \mathbb{R}^{\mathbb{N}}$ with $\beta > 2$ and $c > 0$. Antonacci [4] and Xu-Guo [14] assume that $j(t,x) = A(t)x + b(t)V(x)$ with $A \in C(T, \mathbb{R}^{\mathbb{N}\times\mathbb{N}})$ indefinite in sign and $V \in C^2(\mathbb{R}^{\mathbb{N}}, \mathbb{R})$ superquadratic. The approach in both papers is similar, variational based on the generalized mountain pass theorem. Finally in Ben Naoum-Troestler-Willem [5] and Tang-Wu [13] the authors do not assume the decomposition $j(t,x) = b(t)V(x)$. Instead, Ben Naoum-Troestler-Willem [5] require that $j(t,\cdot)$ is positively homogeneous of order $\theta \neq 2$, while in Tang-Wu [13] $j(t,x) = b(t)|x|^\theta + W(t,x)$ with $\theta > 2$ and $W(t,\cdot)$ is sublinear. The approach in both papers is variational. In Ben Naoum-Troestler-Willem [5] the authors exploit the homogeneity of the potential, while Tang-Wu [13] employ the generalized mountain pass theorem. Our work here is closer to that of Ben Naoum-Troestler-Willem [5], which we extend to systems driven by the vector ordinary p-Laplacian and having a nonsmooth potential. In the past periodic systems with a nonsmooth potential were studied by Adly-Goeleven [1], Adly-Goeleven-Motreanu [2], Adly-Motreanu [3] (semilinear systems) and E. H. Papageorgiou-N. S. Papageorgiou [12] (nonlinear systems). However, their conditions on the potential function imply that it has definite sign near zero or for large $x \in \mathbb{R}^{\mathbb{N}}$. So our work here appears to have two novel features with respect to the existing relevant literature. On the one hand is the first work on nonlinear systems monitored by the ordinary p-Laplacian and with a potential indefinite in sign and on the other hand we do not assume that the varying sign potential is smooth.

Our approach is variational and uses tools from nonsmooth analysis.

2. Mathematical preliminaries

Let X be a Banach space, X^* its topological dual and let $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair. Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, the *generalized directional derivative* of φ at $x \in X$ in the direction $h \in X$, is given by

$$\varphi^0(x;h) \stackrel{df}{=} \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to check that $\varphi^0(x;\cdot)$ is sublinear, continuous and so by the Hahn-Banach Theorem it is the support function of a nonempty, convex and weakly compact convex set $\partial\varphi(x) \subseteq X^*$. So

$$\partial\varphi(x) \stackrel{df}{=} \left\{x^* \in X^* : \left\langle x^*, h\right\rangle \le \varphi^0(x;h) \text{ for all } h \in X\right\}.$$

The multifunction $x \to \partial\varphi(x)$ is called the *generalized* (or *Clarke*) *subdifferential* of φ. If φ is in addition convex, then the generalized subdifferential coincides with the subdifferential in the sense of convex analysis, which is defined by

$$\partial_c\varphi(x) \stackrel{df}{=} \left\{x^* \in X^* : \left\langle x^*, y - x\right\rangle \le \varphi(y) - \varphi(x) \text{ for all } y \in X\right\}.$$

If $\varphi \in C^1(X, \mathbb{R})$, then $\partial\varphi(x) = \{\varphi'(x)\}$.

Our hypotheses on the nonsmooth potential function $j(t,x)$ are the following:

$H(j)_1$: $j : T \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}^{\mathbb{N}}$, $t \to j(t,x)$ is measurable;
(ii) for $r > 0$ there exists $k_r \in L^1(T)_+$ such that for almost all $t \in T$ and all $x, y \in \mathbb{R}^{\mathbb{N}}$ with $\|x\|, \|y\| \leq r$ we have $|j(t,x) - j(t,y)| \leq k_r(t)\|x - y\|$ and also for almost all $t \in T$, $j(t,\cdot)$ is homogeneous of order $\theta > 1$, $\theta \neq p$;
(iii) there exists $c_1 \in L^1(T)_+$ such that for almost all $t \in T$, all $\|x\| = 1$ and all $u \in \partial j(t,x)$ we have $\|u\| \leq c_1(t)$;
(iv) for all $x \in \mathbb{R}^{\mathbb{N}}$, $x \neq 0$, we have $\int_0^b j(t,x)\,dt < 0$;
(v) there exists $x_0 \in \mathbb{R}^{\mathbb{N}}$ such that for all $t \in C$, $|C|_1 > 0$ (by $|\cdot|_1$ we denote the Lebesgue measure on $\mathbb{R}$), we have $j(t,x_0) > 0$.

Remark 2.1. By virtue of the positive homogeneity of $j(t,\cdot)$ for almost all $t \in T$ (see hypothesis $H(j)$(ii)), we have $j(t,0) = 0$ a.e. on T. Let $\alpha_1, \alpha_2 \in L^1(T)$ such that $\int_0^b \alpha_1(t)\,dt \leq 0$, $\int_0^b \alpha_2(t)\,dt \leq 0$, one of the inequalities is strict and there exist $C \subseteq T$ with $|C|_1 > 0$ such that for almost all $t \in C$ we have $\alpha_1(t) + \alpha_2(t) > 0$. Then the function $j : T \times \mathbb{R}^2 \to \mathbb{R}$ defined by $j(t,x,y) = \alpha_1(t)|x|^3 + \alpha_2(t)|x|y^2$ satisfies hypothesis $H(j)$.

Proposition 2.2. *If hypoheses $H(j)$(i) and (ii) hold, then for almost all $t \in T$ and all $x \in \mathbb{R}^{\mathbb{N}}$, we have $j^0(t,x;x) = \partial j(t,x)$ and $j^0(t,x;-x) = -\partial j(t,x)$.*

Proof. For almost all $t \in T$ and all $x \in \mathbb{R}^{\mathbb{N}}$, by definition we have

$$\begin{aligned} j^0(t,x;x) &= \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{j(t,x'+\lambda x) - j(t,x')}{\lambda} \\ &= \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \left[\frac{j(t,x'+\lambda x) - j(t,x'+\lambda x')}{\lambda} + \frac{j(t,x'+\lambda x') - j(t,x')}{\lambda} \right] \\ &\leq \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \left[k_1(t)\|x - x'\| + \frac{(1+\lambda)^\theta - 1}{\lambda} j(t,x') \right] \end{aligned}$$

for some $k_1(t) \in L^1(T)_+$ (see hypothesis $H(j)_1$(iii))

$$= \theta j(t,x)\,. \tag{2.1}$$

On the other hand, note that

$$\begin{aligned} j^0(t,x;x) &\geq \limsup_{\lambda \downarrow 0} \frac{j(t,x+\lambda x) - j(t,x)}{\lambda} \\ &= \limsup_{\lambda \downarrow 0} \frac{[(1+\lambda)^\theta - 1]}{\lambda} j(t,x) = \theta j(t,x)\,. \end{aligned}$$

From (2.1) and (2.2) we conclude that for almost all $t \in T$ and all $x \in \mathbb{R}^{\mathbb{N}}$, we have

$$j^0(t,x;x) = \theta j(t,x)\,.$$

In a similar fashion we show that for almost all $t \in T$ and all $x \in \mathbb{R}^{\mathbb{N}}$, we have

$$j^0(t,x,-x) = -\theta j(t,x)\,. \qquad \square$$

Next we consider the following minimization problem:

$$\left\{\begin{array}{ll} & \frac{1}{p}\|x'\|_p^p \to \inf = m \\ \text{subject to} & \int_0^b j\big(t, x(t)\big)\, dt = 1\,. \end{array}\right. \tag{2.2}$$

Proposition 2.3. *If hypotheses $H(j)$ hold then the feasible set of problem* (2.2) *is nonempty.*

Proof. Let $E = \{t \in T : j(x, t_0) > 0\}$. By hypothesis $H(j)(v)$ we know that $|E|_1 > 0$. Let $\chi_E(t) = \begin{cases} 1 & \text{if } t \in T \\ 0 & \text{if } t \in \mathbb{R} \setminus E \end{cases}$ (the characteristic function of the set E). Given $\varepsilon > 0$, consider a mollifier function $\varphi_\varepsilon \in C_c^\infty(-\frac{b}{2}, \frac{b}{2})$, $\varphi_\varepsilon \geq 0$ with supp$\varphi_\varepsilon \subseteq [-\varepsilon, \varepsilon]$ and $\int_{\mathbb{R}^{\mathbb{N}}} \varphi_\varepsilon(t)\, dt = 1$. Extend φ_ε by b-periodicity on $\mathbb{R}$. We know (see for example Denkowski-Migorski-Papagorgiou [7], p.342) that $\chi_{\varepsilon_n} = (\varphi_{\varepsilon_n} * \chi_E) \to \chi_E$ in $L^1(T)$ as $\varepsilon_n \downarrow 0$ (here $*$ denotes the operation of convolution). By passing to a suitable subsequence if necessary, we may assume that $\chi_{\varepsilon_n}(t) \to \chi_E(t)$ a.e. on T as $\varepsilon_n \downarrow 0$. Hence, because $\chi_{\varepsilon_n} \geq 0$, we have

$$\chi_{\varepsilon_n}(t)^\theta j(t, x_0) = j(t, \chi_{\varepsilon_n}(t)x_0) \to \chi_E(t) j(t, x_0) \quad \text{a.e. on } T \text{ as } \varepsilon_n \downarrow 0\,.$$

Note that

$$\begin{aligned} \chi_{\varepsilon_n}(0) &= \int_{\mathbb{R}} \varphi_{\varepsilon_n}(0 - s)\chi_E(s) ds = \int_E \varphi_{\varepsilon_n}(-s) ds \\ &= \int_{E+b} \varphi_{\varepsilon_n}(b - s) ds = \int_E \varphi_{\varepsilon_n}(b - s) ds = (\varphi_{\varepsilon_n} * \chi_E)(b) = \chi_{\varepsilon_n}(b) \\ &\Rightarrow \chi_{\varepsilon_n}(\cdot) x_0 = y_n(\cdot) \in C^1_{\text{per}}(T, \mathbb{R}^{\mathbb{N}})\,. \end{aligned}$$

We have

$$\int_0^b j\big(t, y_n(t)\big)\, dt \to \int_0^b j\big(t, \chi_E(t)x_0\big)\, dt = \int_E j(t, x_0)\, dt > 0$$

(see hypotheses $H(j)$(ii) and (v)).

Therefore we can find $n_0 \geq 1$ large enough such that

$$\int_0^b j\big(t, y_{n_0}(t)\big)\, dt > 0\,.$$

Then for some $\lambda > 0$, we have

$$\begin{aligned} \lambda^\theta \int_0^b j\big(t, y_{n_0}(t)\big)\, dt &= 1\,, \\ \Rightarrow \int_0^b j\big(t, \lambda y_{n_0}(t)\big)\, dt &= 1 \end{aligned}$$

(see hypothesis $H(j)$(ii)).

Therefore $\lambda y_{n_0} \in C^1_{\text{per}}(T, \mathbb{R}^{\mathbb{N}})$ is a feasible function for the minimization problem (2.2). $\square$

Now that we have established the feasibility of problem (2.2), we proceed to solve it. In what follows $W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}}) = \{x \in W^{1,p}((0,b),\mathbb{R}^{\mathbb{N}}) : x(0) = x(b)\}$. Since $W^{1,p}((0,b),\mathbb{R}^{\mathbb{N}}) \subseteq C(T,\mathbb{R}^{\mathbb{N}})$, the pointwise evaluations at $t = 0$ and $t = b$ make sense.

Proposition 2.4. *If hypotheses $H(j)$ hold, then problem* (2.2) *has a nonconstant solution $x \in W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}})$.*

Proof. Evidently $m \geq 0$. Let $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}})$ be a minimizing sequence for problem (2.2).

We have

$$\frac{1}{p}\|x_n'\|_p^p \downarrow m \ \text{ as } \ n \to \infty \ \text{ and } \ \int_0^b j\big(t, x_n(t)\big)\, dt = 1 \ \text{ for all } \ n \geq 1\,.$$

Consider the direct sum decomposition

$$W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}}) = \mathbb{R}^{\mathbb{N}} \oplus V \ \text{ with } \ V = \{v \in W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}}) : \int_0^b v(t)\, dt = 0\}.$$

For every $n \geq 1$ we have $x_n = \bar{x}_n + \widehat{x}_n$ with $\bar{x}_n \in \mathbb{R}^{\mathbb{N}}$ and $\widehat{x}_n \in V$. Since $\{x_n' = \widehat{x}_n'\}_{n\geq 1} \subseteq L^p(T,\mathbb{R}^{\mathbb{N}})$ is bounded, from the Poincare-Wirtinger inequality (see for example Denkowski-Migorski-Papageorgiou [7], p.357), we deduce that $\{\widehat{x}_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}})$ is bounded. Suppose that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}})$ is unbounded. By passing to a suitable subsequence if necessary, we may assume that $\|x_n\| \to +\infty$. Set $y_n = \dfrac{x_n}{\|x_n\|}$, $n \geq 1$. Since $\|y_n\| = 1$ for all $n \geq 1$, we may assume that

$$y_n \xrightarrow{w} y \ \text{ in } \ W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}}) \ \text{ and } \ y_n \to y \ \text{ in } \ C(T,\mathbb{R}^{\mathbb{N}}).$$

Because $\{\widehat{x}_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}})$ is bounded, we have $y \in \mathbb{R}^{\mathbb{N}}$. For all $n \geq 1$, we have

$$\begin{aligned}
&\int_0^b j\big(t, x_n(t)\big)\, dt = 1\,,\\
&\qquad \Rightarrow \frac{1}{\|x_n\|^{\theta}} \int_0^b j\big(t, x_n(t)\big)\, dt = \frac{1}{\|x_n\|^{\theta}}\,,\\
&\qquad \Rightarrow \int_0^b j\big(t, y_n(t)\big)\, dt = \frac{1}{\|x_n\|^{\theta}}\,,\\
&\qquad \Rightarrow \int_0^b j(t, y)\, dt = 0\,.
\end{aligned}$$

Since $y \in \mathbb{R}^{\mathbb{N}}$, from hypothesis $H(j)$(iv), we infer that $y = 0$. But then $y_n \to 0$ in $W^{1,p}_{\text{per}}\big((0,b),\mathbb{R}^{\mathbb{N}}\big)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \geq 1$. This proves that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}\big((0,b)\mathbb{R}^{\mathbb{N}}\big)$ is bounded. Thus we may assume that

$x_n \xrightarrow{w} x$ in $W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}})$ and $x_n \to x$ in $C(T,\mathbb{R}^{\mathbb{N}})$. So we have

$$\frac{1}{p}\|x'\|_p^p \leq \frac{1}{p}\liminf_{n\to\infty}\|x_n'\|_p^p = m \ \text{ and } \ \lim\int_0^b j\big(t,x_n(t)\big)\,dt = \int_0^b j\big(t,x(t)\big)\,dt = 1\,,$$

$$\Rightarrow x \in W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}}) \quad \text{is a solution of (2.2).}$$

Because of hypothesis $H(j)$(iv), x is nonconstant. □

3. Existence theorem

In this section we prove the existence of a nonconstant solution for problem (1.1).

Theorem 3.1. *If hypotheses $H(j)$ hold, then problem* (1.1) *has a nonconstant solution $y \in C^1_{\text{per}}(T,\mathbb{R}^{\mathbb{N}})$ such that $\|y'\|^{p-2}y' \in W^{1,1}((0,b),\mathbb{R}^{\mathbb{N}})$.*

Proof. Let $x \in W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}})$ be a nonconstant solution of (2.2) (see Proposition 2.4). Consider the integral functional $I_j : W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}}) \to \mathbb{R}$ defined by $I_j(y) = \int_0^b j\big(t,y(t)\big)\,dt$. Clearly I_j is locally Lipschitz (see hypothesis $H(j)$(ii)) and for every $u \in \partial I_j(y)$, we have that $u \in L^1(T,\mathbb{R}^{\mathbb{N}})$ and $u(t) \in \partial j\big(t,y(t)\big)$ a.e. on T (see Denkowski-Migorski-Papageorgiou [7], p.617). Also let $A : W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}}) \to W^{1,p}_{\text{per}}\big((0,b),\mathbb{R}^{\mathbb{N}}\big)^*$ be the nonlinear operator defined by

$$\langle A(v),y\rangle = \int_0^b \|v'(t)\|^{p-2}\big(v'(t),y'(t)\big)_{\mathbb{R}^{\mathbb{N}}}\,dt \ \text{ for all } \ v,y \in W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}})\,.$$

It is easy to see that A is monotone, demicontinuous, hence it is maximal monotone (see Denkowski-Migorski-Papageorgiou [8], p.37). Since $x \in W^{1,p}_{\text{per}}\big((0,b),\mathbb{R}^{\mathbb{N}}\big)$ is a solution of (2.2), from the nonsmooth multiplier rule of Clarke [6], we can find $\beta,\mu \in \mathbb{R}$, $\beta \geq 0$, not both equal to zero such that

$$\beta A(x) + \mu u = 0 \ \text{ with } \ u \in L^1(T,\mathbb{R}^{\mathbb{N}}) \ \ u(t) \in \partial j\big(t,x(t)\big) \ \text{ a.e. on } T\,.$$

If $\beta = 0$, then $\mu u = 0$, hence $u \equiv 0$ and so $j^0\big(t,x(t);-x(t)\big) \geq 0$ a.e. on T. But from Proposition 2.2 we know that $j_0\big(t,x(t);-x(t)\big) = -\theta j\big(t,x(t)\big)$ a.e. on T. So $\theta\int_0^b j\big(t,x(t)\big)\,dt \leq 0$, a contradiction to the fact that $\int_0^b j\big(t,x(t)\big)\,dt = 1$. So $\beta \neq 0$ and without any loss of generality, we may assume that $\beta = 1$. So we have

$$A(x) + \mu u = 0\,, \tag{3.1}$$

$$\Rightarrow \|x'\|_p^p + \mu\int_0^b \big(u(t),x(t)\big)_{\mathbb{R}^{\mathbb{N}}}\,dt = 0 \ \text{ (acting with the test function } x)\,.$$

Suppose that $\mu \geq 0$. Then we have

$$\|x'\|_p^p + \mu\int_0^b j^0\big(t,x(t);x(t)\big)\,dt \geq 0\,,$$

$$\Rightarrow \|x'\|_p^p + \mu\theta \geq 0 \tag{3.2}$$

(see Proposition (2.2) and recall that $\int_0^b j\big(t,x(t)\big)\,dt = 1$).

On the other hand using as a test function $-x$, from (3.1) we have

$$
\begin{aligned}
& -\|x'\|_p^p + \mu \int_0^b j^0\big(t, x(t); -x(t)\big)\, dt \geq 0 \\
& \qquad \text{(since we have assumed that } \mu \geq 0\text{)}, \\
& \Rightarrow -\|x'\|_p^p - \mu\theta \geq 0 \\
& \qquad \text{(again by Proposition 2.2 and since } \textstyle\int_0^b j\big(t, x(t)\big)\, dt = 1\text{)}, \\
& \Rightarrow \|x'\|_p^p + \mu\theta \leq 0\,. \qquad (3.3)
\end{aligned}
$$

From (3.2) and (3.3) it follows that

$$\|x'\|_p^p + \mu\theta = 0\,. \tag{3.4}$$

Since x is nonconstant, $\|x'\|_p > 0$. Also $\mu \geq 0$ and $\theta > 0$. All these facts contradict equality (3.4). Therefore $\mu < 0$. Let $x = \lambda y$, $\lambda > 0$. We have

$$A(\lambda y) + \mu u = 0\,, \quad u \in L^1(T, \mathbb{R}^{\mathbb{N}})\,, \quad u(t) \in \partial j\big(t, \lambda y(t)\big) \quad \text{a.e. on } T.$$

Note that for all $v, h \in \mathbb{R}^{\mathbb{N}}$, because of hypothesis $H(j)$(ii), we have

$$
\begin{aligned}
\lambda^{\theta-1} j^0(t, v; h) = \limsup_{\substack{v' \to v \\ \lambda \downarrow 0}} \frac{j(t, \lambda v' + r\lambda h) - j(t, \lambda v')}{\lambda r} = j^0(t, \lambda v; h)\,, \\
\Rightarrow \partial j\big(t, \lambda y(t)\big) = \lambda^{\theta-1} \partial j\big(t, y(t)\big) \quad \text{for a.a. } t \in T \\
\text{and so } \ u(t) = \lambda^{\theta-1} v(t)\,, \quad v(t) \in \partial j\big(t, y(t)\big) \quad \text{a.e. on } T.
\end{aligned}
$$

Therefore $\lambda^{p-1} A(y) + \mu\lambda^{\theta-1} v = 0$. If $\lambda > 0$ is such that $\mu\lambda^{\theta-1} = -\lambda^{p-1}$, then $A(y) - v = 0$. Let $\psi \in C_c^1((0,b), \mathbb{R}^{\mathbb{N}})$. Since $(\|y'\|^{p-2} y')' \in W^{-1,q}((0,b), \mathbb{R}^{\mathbb{N}}) = W_0^{1,p}((0,b), \mathbb{R}^{\mathbb{N}})^*$ $\dfrac{1}{p} + \dfrac{1}{q} = 1$ (see Denkowski-Migorski-Papageorgiou [7], p.362), we have

$$\langle -(\|y'\|^{p-2} y')', \psi \rangle_0 = \int_0^b \big(v(t), \psi(t)\big)_{\mathbb{R}^{\mathbb{N}}}\, dt$$

(by $\langle \cdot, \cdot \rangle_0$ we denote the duality brackets for the pair $(W_0^{1,p}((0,b), \mathbb{R}^{\mathbb{N}}),$ $W^{-1,q}((0,b), \mathbb{R}^{\mathbb{N}}))$. Because $C_c^1((0,b), \mathbb{R}^{\mathbb{N}})$ is dense in $W_0^{1,p}((0,b), \mathbb{R}^{\mathbb{N}})$, it follows that

$$
\begin{aligned}
& -\big(\|y'(t)\|^{p-2} y'(t)\big)' = v(t) \quad \text{a.e. on } T, \quad y(0) = y(b) \qquad (3.5) \\
& \Rightarrow \|y'\|^{p-2} y' \in W^{1,1}((0,b), \mathbb{R}^{\mathbb{N}})\,, \\
& \text{hence } \ y' \in C(T, \mathbb{R}^{\mathbb{N}}), \quad \text{i.e.} \quad y \in C^1(T, \mathbb{R}^{\mathbb{N}})\,.
\end{aligned}
$$

Also if $w \in W^{1,p}_{\text{per}}((0,b),\mathbb{R}^{\mathbb{N}})$, we have

$$\begin{aligned}
\langle A(y), w\rangle &= \int_0^b \big(v(t), w(t)\big)_{\mathbb{R}^{\mathbb{N}}}\, dt \\
&\Rightarrow \langle -(\|y'\|^{p-2}y')', w\rangle + \|y'(b)\|^{p-2}\big(y'(b), w(b)\big)_{\mathbb{R}^{\mathbb{N}}} \\
&- \|y'(0)\|^{p-2}\big(y'(0), w(0)\big)_{\mathbb{R}^{\mathbb{N}}} \\
&= \int_0^b \big(v(t), w(t)\big)_{\mathbb{R}^{\mathbb{N}}}\, dt \quad \text{(by Green's identity)} \\
&\Rightarrow \|y'(0)\|^{p-2}\big(y'(0), w'(0)\big)_{\mathbb{R}^{\mathbb{N}}} = \|y'(b)\|^{p-2}\big(y'(b), w'(b)\big)_{\mathbb{R}^{\mathbb{N}}} \\
\text{for all} \quad w \in W^{1,p}_{\text{per}}&\big((0,b),\mathbb{R}^{\mathbb{N}}\big) \quad \text{(see (3.5))} \\
&\Rightarrow y'(0) = y'(b)\,.
\end{aligned}$$

So $y \in C^1(T,\mathbb{R}^{\mathbb{N}})$ is a nonconstant solution of (1.1) with $\|y'\|^{p-2}y' \in W^{1,1}\big((0,b),\mathbb{R}^{\mathbb{N}}\big)$. $\square$

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