ARCHIVUM MATHEMATICUM (BRNO)
Tomus 42 (2006), 1 – 10

PERIODIC SOLUTIONS FOR A NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATION WITH MULTIPLE VARIABLE LAGS

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Abstract. By means of the Krasnoselskii fixed piont theorem, periodic solutions are found for a neutral type delay differential system of the form

$$x'(t) + cx'(t-\tau) = A(t, x(t))\, x(t) + f(t, x(t-r_1(t)), \dots, x(t-r_k(t)))\,.$$

1. Introduction

Periodic solutions of delay differential equations are important in ecological models and design of electronic devices, and appear in many investigations (see, e.g. [1-10]). In particular, periodic solutions of linear differential system of the form

$$x'(t) = A(t)x(t)\,, \tag{1}$$

where $A(t)$ is a continuous n by n real matrix function such that $A(t+T) = A(t)$ for some $T > 0$ and all $t \in R$, have been studied to a great extent. When periodic perturbations exist, and when lags in the model are present, the above system should be modified so as to reflect these additional factors.

In this paper, we consider the existence of periodic solutions of one such system

$$x'(t) + cx'(t-\tau) = A(t, x(t))\, x(t) + f(t, x(t-r_1(t)), \dots, x(t-r_k(t)))\,, \tag{2}$$

where τ and c are constants, $|c| < 1$, $r_i(t)$, $i = 1, 2, ..., k$, are real continuous functions on R with period $T > 0$. $A(t, x)$ is a $n \times n$ real continuous matrix function defined on $R \times R^n$ such that

$$A(t+T, x) = A(t, x),\ (t, x) \in R \times R^n$$

and $f(t, u_1, ..., u_k)$ is a real continuous vector function defined on $R \times R^n \times \cdots \times R^n$ such that

$$f(t+T, u_1, \dots, u_k) = f(t, u_1, \dots, u_k),\ (t, u_1, \dots, u_k) \in R \times R^n \times \cdots \times R^n\,.$$

2000 *Mathematics Subject Classification.* 39A11.
Key words and phrases. Neutral differential system, periodic solutions, fixed point theorem.
Received April 28, 2004, revised November 2005.

We will invoke the Krasnoselskii fixed point theorem for finding T-periodic solutions of (2): Suppose B is a Banach space and G is a nonempty bounded convex and closed subset of B. Let $S, P : G \to B$ satisfy the following conditions: (i) $Sx + Py \in G$ for any $x, y \in G$, (ii) S is a contraction mapping, and (iii) P is completely continuous. Then $S + P$ has a fixed point in G.

2. Preliminaries

First, we recall some basic facts about linear periodic differential system and the matrix measures. Consider the system (1) where $A(t)$ is a $n \times n$ continuous matrix function defined on R such that $A(t+T) = A(t)$. Let $\Phi(t, t_0)$ be the fundamental matrix of (1) which satisfy $\Phi(t_0, t_0) = I$. Recall that

$$\Phi(t, w)\, \Phi(w, s) = \Phi(t, s), \quad t, s, w \in R,$$

and

$$\Phi^{-1}(t, s) = \Phi(s, t), \quad t, s \in R.$$

Let A be a $n \times n$ real matrix. Let $|\cdot|_p$ be the standard p norm for the linear Euclidean space R^n and $\|A\|_p$ the induced matrix norm of A corresponding to the vector norm $|\cdot|_p$. The corresponding matrix measure of the matrix A is the function (see e.g. [12])

$$\mu_p(A) = \lim_{\varepsilon \to 0^+} \frac{\|I + \varepsilon A\|_p - 1}{\varepsilon}.$$

For instance (see e.g. [12]), let $x = (x_1, \ldots, x_n)^T$, $A = (a_{ij})_{n\times n} \in R^{n\times n}$ then $|x|_1 = \sum_{i=1}^n |x_i|$, $\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|$ and $\mu_1(A) = \max_{1\le j\le n} \left\{a_{jj} + \sum_{i\ne j} |a_{ij}|\right\}$.

Lemma 1 ([12])**.** *Let $x(t)$ be a solution of system* (1)*. Then*

$$|x(t_0)|_1 \exp\left\{\int_{t_0}^t (-\mu_1(-A(s)))\, ds\right\} \le |x(t)|_1 \le |x(t_0)|_1 \exp\left\{\int_{t_0}^t (\mu_1(A(s)))\, ds\right\}$$

for $t \ge t_0$.

Lemma 2. *The fundamental matrix of* (1) *satisfies*

$$\|\Phi(t, s)\|_1 \le \exp\left(\int_s^t \mu_1(A(\zeta))\, d\zeta\right), \quad t \geqslant s.$$

Indeed, let $\Phi = (\Phi_{ij})$ and let $\Phi^{(j)}$ be the j-th column of the matrix Φ. Then by Lemma 1,

$$\begin{aligned}\|\Phi(t, s)\|_1 &= \max_{1\le j\le n} \sum_{i=1}^n |\Phi_{ij}(t, s)| = \max_{1\le j\le n} \left|\Phi^{(j)}(t, s)\right|_1 \\ &\le \left|\Phi^{(j)}(s, s)\right|_1 \exp\left(\int_s^t \mu_1(A(\zeta))\, d\zeta\right) = \exp\left(\int_s^t \mu_1(A(\zeta))\, d\zeta\right).\end{aligned}$$

Lemma 3. *If*

$$\exp\left\{\int_0^T (\mu_1(A(s)))\, ds\right\} < 1\,, \tag{3}$$

then the linear system (1) *does not have any nontrivial T-periodic solution.*

Proof. Let $x(t)$ be a T-periodic solution of (1) which does not vanish at t_0. From Lemma 1, we have

$$|x(t_0)|_1 = |x(t_0+T)|_1 \le |x(t_0)|_1 \exp\left\{\int_{t_0}^{t_0+T} (\mu_1(A(s)))\, ds\right\} < |x(t_0)|_1\,, \tag{4}$$

which implies $x(t_0) = 0$. This is a contradiction. □

Lemma 4 ([11])**.** *If the linear system* (1) *does not have any nontrivial T-periodic solution, then for any T-periodic continuous function $f(t)$, the nonhomogeneous system*

$$x'(t) = A(t)\, x(t) + f(t) \tag{5}$$

has a unique T-periodic solution $x(t)$ determined by

$$x(t) = \Phi(t,t_0)\, x(t_0) + \int_{t_0}^{t} \Phi(t,s)\, f(s)\, ds\,, \quad t \in R\,. \tag{6}$$

Under the condition

$$\exp\left\{\int_0^T (\mu_1(A(s)))\, ds\right\} < 1\,, \tag{7}$$

we see from Lemma 3 that the linear system (1) does not have any nontrivial T-periodic solution. Hence by Lemma 4, the unique T-periodic solution $x(t)$ of the nonhomogeneous system (5) can be expressed by (6). We assert further that

$$x(t) = (I - \Phi(t+T,t))^{-1} \int_t^{t+T} \Phi(t+T,s)\, f(s)\, ds\,. \tag{8}$$

To see this, note that by (6), we have

$$x(t) = x(t+T) = \Phi(t+T,t_0)\, x(t_0) + \int_{t_0}^{t+T} \Phi(t+T,s)\, f(s)\, ds\,, \tag{9}$$

and

$$\begin{aligned} \Phi(t+T,t)\, x(t) &= \Phi(t+T,t)\, \Phi(t,t_0)\, x(t_0) + \Phi(t+T,t) \int_{t_0}^{t} \Phi(t,s)\, f(s)\, ds \\ &= \Phi(t+T,t_0)\, x(t_0) + \int_{t_0}^{t} \Phi(t+T,s)\, f(s)\, ds\,. \qquad (10) \end{aligned}$$

Thus,

$$(I - \Phi(t+T,t))\, x(t) = \int_t^{t+T} \Phi(t+T,s)\, f(s)\, ds\,. \tag{11}$$

Furthermore, since

$$\|\Phi(t+T,t)\|_1 \leq \exp\Big(\int_t^{t+T} \mu_1(A(s))\,ds\Big) = \exp\Big(\int_0^T \mu_1(A(s))\,ds\Big) < 1\,, \tag{12}$$

we see that thus, $(I-\Phi(t+T,t))^{-1}$ exists for every $t \in R$. Therefore, we may infer from (11) that (8) holds.

We summarize these as follows.

Lemma 5. *Suppose* (7) *holds. Then equation* (5) *is equivalent to* (8).

3. Main Results

Let X be the Banach space of all real T-periodic continuously differentiable functions of the form $x = x(t)$ which is defined on R and endowed with the usual linear structure as well as the norm $\|x\|^{(2)} = \|x\|^{(0)} + \|x\|^{(1)}$ where $\|x\|^{(0)} = \max_{0\leq t\leq\omega} |x(t)|_1$ and $\|x\|^{(1)} = \max_{0\leq t\leq\omega} |x'(t)|_1$.

For the sake of simplicity, in the sequel, we will write $|x|$, $\|A\|$ and $\mu(A)$ instead of $|x|_1$, $\|A\|_1$ and $\mu_1(A)$.

Theorem 1. *Suppose there exists a T-periodic continuous function $\alpha(t)$ such that*

$$\mu(A(t,x)) \leq \alpha(t)\,, \quad (t,x) \in [0,T]\times R^n \tag{13}$$

and

$$\kappa = \exp\Big\{\int_0^T \alpha(s)\,ds\Big\} < 1\,. \tag{14}$$

Suppose further that there is $M > 0$ such that

$$\frac{1}{M}\int_0^T \sup_{|u_1|\leq M,\ldots,|u_k|\leq M} |f(t,u_1,\ldots,u_k)|\,dt < \frac{1-\kappa}{M_0}(1-|c|) - |c|\,LT\,, \tag{15}$$

where

$$L = \sup_{|x|\leq M,\ 0\leq t\leq T} \|A(t,x)\|$$

and

$$M_0 = \sup_{0\leq s\leq t\leq T} \exp\Big\{\int_s^t \alpha(\theta)\,d\theta\Big\}\,. \tag{16}$$

Then (2) *has a T-periodic solution.*

Proof. For any $u \in X$, consider the linear periodic system

$$x'(t) = A(t,u(t))\,x(t)\,, \tag{17}$$

and

$$x'(t) = A(t,u(t))\,x(t) + f(t,u(t-r_1(t)),\ldots,u(t-r_k(t))) - cu'(t-\tau)\,. \tag{18}$$

From condition (13) and (14), we have

$$\exp\Big\{\int_0^T \mu(A(t,u(t)))\,dt\Big\} \leq \exp\Big\{\int_0^T \alpha(s)\,ds\Big\} < 1\,. \tag{19}$$

By Lemma 3, (17) does not have any nontrivial T-periodic solution. Furthermore, by Lemma 5, (18) is equivalent to the integral equation

$$\begin{aligned} x(t) = {} & \left(I - \Phi_u(t+T,t)\right)^{-1} \int_t^{t+T} \Phi_u(t+T,s) \\ & \times \left[f\left(s, u\left(s - r_1(s)\right), \ldots, u\left(s - r_k(s)\right)\right) - cu'(s-\tau)\right] ds \end{aligned} \tag{20}$$

where $\Phi_u(t,t_0)$ is a fundamental matrix of (17) which satisfies $\Phi_u(t_0,t_0) = I$. Define the mappings $S : X \to X$ and $P : X \to X$ by

$$(Su)(t) = -cu(t-\tau) \tag{21}$$

and

$$\begin{aligned} (Pu)(t) = {} & \left(I - \Phi_u(t+T,t)\right)^{-1} \int_t^{t+T} \Phi_u(t+T,s) \\ & \times \left[f\left(s, u\left(s - r_1(s)\right), \ldots, u\left(s - r_k(s)\right)\right) - cu'(s-\tau)\right] ds + cu(t-\tau) \end{aligned} \tag{22}$$

for $u \in X$. Clearly, if $P + S$ has a fixed point, then this fixed point is periodic solution of (2). To find such a fixed point, we show that the assumptions in the Krasnoselskii theorem are satisfied. Since

$$\Phi_u(s,t_0)\,\Phi_u^{-1}(s,t_0) = I$$

and

$$\left(\Phi_u(t_0,s)\right)^{-1} = \Phi_u(s,t_0)\ ,$$

we see that

$$\begin{aligned} \frac{d}{ds}\Phi_u(t,s) &= \frac{d}{ds}\left(\Phi_u(t,t_0)\,\Phi_u(t_0,s)\right) = \Phi_u(t,t_0)\,\frac{d}{ds}\left(\Phi_u^{-1}(s,t_0)\right) \\ &= -\Phi_u(t,s)\,A(s,u(s)) \end{aligned} \tag{23}$$

and

$$\begin{aligned} & (I - \Phi_u(t+T,t))^{-1} \int_t^{t+T} \Phi_u(t+T,s)\,u'(s-\tau)\,ds \\ &\quad = \left(I - \Phi_u(t+T,t)\right)^{-1} \int_t^{t+T} \Phi_u(t+T,s)\,d(u(s-\tau)) \\ &\quad = \left(I - \Phi_u(t+T,t)\right)^{-1} \Phi_u(t+T,s)\,u(s-\tau)\,\Big|_{s=t}^{s=t+T} \\ &\qquad - \left(I - \Phi_u(t+T,t)\right)^{-1} \int_t^{t+T} \left(\frac{d}{ds}\Phi_u(t+T,s)\right) u(s-\tau)\,ds \\ &\quad = u(t-\tau) + \left(I - \Phi_u(t+T,t)\right)^{-1} \\ &\qquad \times \int_t^{t+T} \Phi_u(t+T,s)\,A(s,u(s))\,u(s-\tau)\ ds\,. \end{aligned} \tag{24}$$

In view of (22) and (24),

$$
\begin{aligned}
(Pu)(t) = {} & (I - \Phi_u(t+T,t))^{-1} \int_t^{t+T} \Phi_u(t+T,s) \\
& \times [f(s, u(s - r_1(s)), \ldots, u(s - r_k(s)))]\, ds \\
& - (I - \Phi_u(t+T,t))^{-1} \int_t^{t+T} \Phi_u(t+T,s)\, A(s,u(s))\, cu(s-\tau)\, ds \qquad (25)
\end{aligned}
$$

Next we will prove that for any $u, v \in X$ which satisfy $|u(t)|, |v(t)| \le M$ for $t \in R$, then

$$|(Pv)(t) + (Su)(t)| \le M,\ t \in R.$$

Indeed, by Lemma 2, (12) -(14) we see that

$$\|\Phi_u(t+T,t)\| \le \kappa < 1,$$

so,

$$
\begin{aligned}
\left\| (I - \Phi_u(t+T,t))^{-1} \right\| &= \left\| \sum_{n=0}^{\infty} (\Phi_u(t+T,t))^n \right\| \\
&\le \sum_{n=0}^{\infty} \|(\Phi_u(t+T,t))\|^n \le \sum_{n=0}^{\infty} \kappa^n = \frac{1}{1-\kappa}. \qquad (26)
\end{aligned}
$$

Furthermore, by Lemma 2 and (16), we get

$$
\begin{aligned}
\|\Phi_u(t+T,s)\| &\le \exp\left\{ \int_s^{t+T} \mu(A(\theta, u(\theta)))\, d\theta \right\} \\
&\le \exp\left\{ \int_s^{t+T} \alpha(\theta)\, d\theta \right\} \le M_0 \qquad (27)
\end{aligned}
$$

for $t \le s \le t+T$. Thus from (15), (21), (25), (26) and (27), we have

$$
\begin{aligned}
|(Pv)(t) + (Su)(t)| &\le |(Pv)(t)| + |(Su)(t)| \le |c|M + \left\| (I - \Phi_v(t+T,t))^{-1} \right\| \\
&\quad \times \int_t^{t+T} \|\Phi_v(t+T,s)\|\, |f(s, v(s - r_1(s)), \ldots, v(s - r_k(s)))|\, ds \\
&\quad + |c| \left\| (I - \Phi_v(t+T,t))^{-1} \right\| \int_t^{t+T} \|\Phi_v(t+T,s)\|\, \|A(s, v(s))\|\, |v(s-\tau)|\, ds \\
&\le |c|M + \frac{M_0}{1-\kappa} \int_t^{t+T} \big| f(s, v(s-r_1(s)), \ldots, v(s-r_k(s))) \big|\, ds + |c|M \frac{M_0 LT}{1-\kappa} \\
&\le M \left\{ |c| + |c| \frac{M_0 LT}{1-\kappa} + \frac{M_0}{1-\kappa} \left[\frac{1-\kappa}{M_0}(1-|c|) - |c|LT \right] \right\} \\
&= M.
\end{aligned}
$$

Let

$$N = \frac{ML + b_0}{1 - |c|} \qquad (28)$$

where

$$b_0 = \sup_{\substack{0\le t\le T\\ |u_1|\le M,\dots,|u_k|\le M}} |f(t,u_1,\dots,u_k)| ,$$

and

$$G = \{u \in X : \ |u(t)| \le M, |u'(t)| \le N,\ 0 \le t \le T\} .$$

It is easily seen that G is a nonempty bounded, convex and closed subset of X. Now we show that for any $u, v \in G$,

$$\left|\frac{d}{dt}[(Pv)(t) + (Su)(t)]\right| \le N, \qquad t \in R.$$

Indeed, since

$$\frac{d}{dt}(Su)(t) = -cu'(t-\tau) , \tag{29}$$

by (22) and (29), we know that $(Pv)(t) + (Sv)(t)$ is a periodic solution of the system of the form

$$x'(t) = A(t, v(t))\, x(t) + f(t, v(t - r_1(t)), \dots, v(t - r_k(t))) - cv'(t-\tau) .$$

Hence,

$$\begin{aligned}\frac{d}{dt}(Pv)(t) &= A(t, v(t))\,[(Pv)(t) + (Sv)(t)]\\ &\quad + f(t, v(t - r_1(t)), \dots, v(t - r_k(t))) ,\end{aligned} \tag{30}$$

we have

$$\begin{aligned}\left|\frac{d}{dt}[(Pv)(t) + (Su)(t)]\right| &\le \|A(t, v(t))\|\,[|(Pv)(t) + (Sv)(t)|]\\ &\quad + |f(t, v(t - r_1(t)), \dots, v(t - r_k(t)))| + |c|\,N\\ &\le LM + b_0 + |c|\,N\\ &= N ,\end{aligned}$$

so that

$$\begin{aligned}\|(Pv) + (Su)\|^{(2)} &= \max_{0\le t\le T} |(Pv)(t) + (Su)(t)|\\ &\quad + \max_{0\le t\le T}\left|\frac{d}{dt}((Pv)(t) + (Su)(t))\right| \le M + N .\end{aligned}$$

Now we have proved that for any $u, v \in G$, $Su + Pv \in G$. Note that for any $u, v \in G$ are T-periodic, thus we have

$$\begin{aligned}\|S(u-v)\|^{(2)} &= \max_{0\le t\le T} |c(u-v)(t-\tau)| + \max_{0\le t\le T}\left|c(u-v)'(t-\tau)\right|\\ &= |c|\left(\max_{0\le t\le T} |(u-v)(t)| + \max_{0\le t\le T}\left|(u-v)'(t)\right|\right)\\ &= |c|\,\|u-v\|^{(2)} .\end{aligned}$$

In view of the condition $|c| < 1$, we now know that S is a contraction mapping.

Next we will prove that P is a completely continuous operator from G into G. To see this, for any $u, v \in G$, let $H = Pu - Pv$. By (30),

$$H'(t) = \{A(t, u(t))[(Pu)(t)+(Su)(t)]+f(t, u(t-r_1(t)), \ldots, u(t-r_k(t)))\}$$

$$-\{A(t, v(t))[(Pv)(t)+(Sv)(t)]+f(t, v(t-r_1(t)), \ldots, v(t-r_k(t)))\}. \tag{31}$$

Let

$$\begin{aligned} h^*(t, u(t), v(t)) = {} & A(t, u(t))\, c\,[v(t-\tau) - u(t-\tau)] \\ & + [A(t, u(t)) - A(t, v(t))][(Pv)(t) + (Sv)(t)] \\ & + \big\{ f(t, u(t-r_1(t)), \ldots, u(t-r_k(t))) \\ & - f(t, v(t-r_1(t)), \ldots, v(t-r_k(t))) \big\}. \end{aligned} \tag{32}$$

Let $G_1 = \{x \in R^n : |x| \le M\}$. Since $A(t,x)$ and $f(t, u_1, \ldots, u_k)$ for $0 \le t \le T$ are uniform continuous on G_1 and $|(Sv)(t) + (Pv)(t)|$ is bounded, we see that as $\|u - v\|^{(2)} \to 0$, $|h^*(t, u, v)| \to 0$ uniformly holds for $0 \le t \le T$. By (31), we have

$$H'(t) = A(t, u(t))\, H(t) + h^*(t, u(t), v(t)), \tag{33}$$

that is, $H(t)$ is a T-periodic solution of (33). By Lemma 4, we have

$$\begin{aligned} |H(t)| &\le \left\|(I - \Phi_u(t+T, t))^{-1}\right\| \int_t^{t+T} \|\Phi_u(t+T, s)\| |h^*(s, u(s), v(s))|\, ds \\ &\le \frac{M_0}{1-\kappa} \int_t^{t+T} |h^*(s, u(s), v(s))|\, ds. \end{aligned}$$

Thus, when $\|u - v\|^{(0)} \to 0$, $\|Pu - Pv\|^{(0)} = \|H\|^{(0)} \to 0$. On the other hand, in view of (31), we see that as $\|u - v\|^{(0)} \to 0$, $\|Pu - Pv\|^{(1)} = \|H\|^{(1)} = \|H'\|^{(0)} \to 0$. Hence if $\|u - v\|^{(2)} \to 0$, then $\|u - v\|^{(0)} \to 0$, and so $\|Pu - Pv\|^{(2)} = \|Pu - Pv\|^{(0)} + \|Pu - Pv\|^{(1)} \to 0$. That is, P is a continuous mapping on G. Next, we will prove that PG is relatively compact. Note that $PG \subset G$. In view of the definition of G, we know that G is uniformly bounded and equicontinuous. Thus PG is uniformly bounded and equicontinuous. For any $\{Pu_n\} \subset G$, there is a convergent subsequence of $\{Pu_n\}$. We may assume without loss of generality that $\{Pu_n\}$ converges in the norm $\|\cdot\|_0$. Next we will prove that $\{Pu_n\}$ has a subsequence which converges in the norm $\|\cdot\|^{(2)}$. Indeed, since $\|Pu\|^{(1)} \le N$ for $u \in G$, we know that $\left\|\frac{d}{dt}Pu\right\|^{(0)} \le N$ for $u \in G$. That is, $\left\{\frac{d}{dt}(Pu) : u \in G\right\}$ is uniformly bounded. Furthermore, for any $u \in G$, we have

$$\frac{d}{dt}(Pu)(t) = A(t, u(t))[(Pu)(t) + (Su)(t)]+f(t, u(t-r_1(t)), \ldots, u(t-r_k(t))).$$

Since $A(t,x)$ and $f(t, u_1, \ldots, u_k)$ are uniformly continuous on $[0, T] \times G_1$, and G and PG are equicontinuous, so $\left\{\frac{d}{dt}(Pu) : u \in G\right\}$ is equicontinuous. Since $\left\{\frac{d}{dt}(Pu_n)\right\} \subset \left\{\frac{d}{dt}(Pu) : u \in G\right\}$, we see that $\left\{\frac{d}{dt}(Pu_n)\right\}$ has a subsequence $\left\{\frac{d}{dt}(Pu_{n_k})\right\}$ which converges in the norm $\|\cdot\|^{(0)}$, that is, $\{Pu_{n_k}\}$ converges in the norm $\|\cdot\|^{(1)}$. Thus, P is a completely continuous mapping from G into G.

By means of Krasnoselskii's theorem, we know that $P+S$ has a fixed point in G which is a T-periodic solution of (2). The proof is complete. □

As an example, consider the two dimensional nonlinear neutral differential system of the form

$$x'(t)-\frac{1}{16}x'(t-\tau)=A(t,x(t))\,x(t)+f(t,x(t-\sin 2\pi t),x(t-\cos 2\pi t)), \tag{34}$$

where

$$A(t,x)=\begin{pmatrix} \frac{-1}{4} & \frac{\sin 2\pi t}{8}\exp\left(-x_1^2-x_2^2\right) \\ \frac{\sin 2\pi t}{8}\exp\left(-x_1^4-x_2^4\right) & \frac{-1}{4} \end{pmatrix},$$

and

$$f(t,v,w)=\begin{pmatrix} \frac{\sin 2\pi t}{4}\exp\left(-v_1^2-v_2^2\right) \\ \frac{\sin 2\pi t}{8}\exp\left(-w_1^8-w_2^8\right) \end{pmatrix}.$$

If we take $p=1$ in $|\cdot|, \|\cdot\|$ and $\mu(\cdot)$, then it is easy to see that $|a_{ii}(t,x)|=1/4<1$ for $i=1$ and 2, and $\mu(A(t,x))\le -1/8$. If we let $\alpha(t)=-1/8$ and $M=16$, then $\kappa=\exp\left(\int_0^1\alpha(\theta)\,d\theta\right)=e^{-1/8}$, $M_0=1$, $L=\sup_{|x|<16,0\le t\le 1}\|A(t,x)\|=3/8$ and $\sup_{|v|\le 16,|w|\le 16}|f(t,u,v)|=3|\sin 2\pi t|/8$.

In view of these calculations, we may see that the conditions of Theorem 1 are satisfied. Hence (34) has a 1-periodic solution. This solution is also nontrivial, since $f(t,0,0)$ is not identically zero.

Acknowledgment. We appreciate the suggestions of the reviewer which are helpful in preparing the final version of our paper.

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