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NON–EXISTENCE OF NATURAL OPERATORS TRANSFORMING CONNECTIONS ON $Y \to M$ INTO CONNECTIONS ON $FY \to Y$

W. M. MIKULSKI

Abstract. Under some weak assumptions on a bundle functor $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ we prove that there is no $\mathcal{FM}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on $FY \to Y$.

Introduction

Let $\mathcal{M}f$ be the category of manifolds and their maps. Let $\mathcal{M}f_n$ be the category of n-dimensional manifolds and their local diffeomorphisms. Let $\mathcal{FM}$ be the category of fibered manifolds and their fibered maps. Let $\mathcal{FM}_{m,n}$ be the category of fibered manifolds with m-dimensional bases and n-dimensional fibers and their local fibered diffeomorphisms.

We recall that a (general) connection on a fibered manifold $p : Y \to M$ is a smooth section $\Gamma : Y \to J^1Y$ of the first prolongation of Y, which can be also interpreted as the lifting map $\Gamma : Y \times_M TM \to TY$, see [1].

There are known the following facts, see e.g. [1]:

Fact 1. *There is no first order $\mathcal{FM}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on the vertical bundle $VY \to Y$.*

Fact 2. *There is no first order $\mathcal{FM}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on the tangent bundle $TY \to Y$.*

Fact 3. *There is no first order $\mathcal{FM}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on $J^1Y \to Y$.*

The purpose of this short note is the following general result:

Theorem 1. *Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor such that the corresponding natural bundle $\tilde{F} : \mathcal{M}f_n \to \mathcal{FM}$, $\tilde{F}N = F(\mathbf{R}^m \times N)$, $\tilde{F}\varphi = F(\mathrm{id}_{\mathbf{R}^m} \times \varphi)$,*

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$N \in \mathrm{Obj}(\mathcal{M}f_n)$, $\varphi \in \mathrm{Morph}(\mathcal{M}f_n)$ is not of order 0. Then there is no $\mathcal{FM}_{m,n}$--natural operator transforming connections on $Y \to M$ into connections on $FY \to Y$.

That the assumption on F is essential is remarked. Applications of Theorem 1 are given. A generalization of Theorem 1 is proved.

All manifolds are assumed to be finite dimensional and of class $\mathcal{C}^\infty$. All maps between manifolds are assumed to be of class $\mathcal{C}^\infty$.

1. Proof of Theorem 1

Lemma 1. *Let $G : \mathcal{M}f_n \to \mathcal{FM}$ be a natural bundle of order $r \geq 1$. Then any natural operator $\mathcal{V} : T_{\mathcal{M}f_n} \rightsquigarrow TG$ of vertical type is of order $r-1$.*

Proof. Let $X_1, X_2 \in \mathcal{X}(N)$ be two vector fields with $j^{r-1}_x(X_1) = j^{r-1}_x(X_2)$, $x \in N$. Let $w \in G_x N$. Because of the regularity of $\mathcal{V}$ we can assume that $X_1(x) \neq 0$. There is an x-preserving local diffeomorphism $\varphi : N \to N$ such that $j^r_x \varphi = \mathrm{id}$ and $\varphi_* X_1 = X_2$ near x, see [1]. Then $\mathcal{V}(X_2)(w) = \mathcal{V}(\varphi_* X_1)(w) = TG_x(\varphi) \circ \mathcal{V}(X_1) \circ G_x(\varphi^{-1})(w) = \mathcal{V}(X_1)(w)$ because of $G_x(\varphi) = \mathrm{id}$ as G is of order r and $j^r_x\varphi = \mathrm{id}$. □

Proof of Theorem 1. Suppose D is such an operator. Then for any n-manifold N we have the connection

$$\Gamma_N = D(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}) : F(\mathbf{R}^m \times N) \times_{\mathbf{R}^m \times N} T(\mathbf{R}^m \times N) \to TF(\mathbf{R}^m \times N)$$

on $F(\mathbf{R}^m \times N) \to \mathbf{R}^m \times N$, where $x^1, \dots, x^m$ are the usual coordinates on $\mathbf{R}^m$. Define a 0-order natural operator $A : T_{\mathcal{M}f_n} \rightsquigarrow T\tilde{F}$ by

$$A(X)_w = \Gamma_N(w, X(y)),$$

$X \in \mathcal{X}(N)$, $w \in F_{(x,y)}(\mathbf{R}^m \times N) \subset \tilde{F}N$, $(x,y) \in \mathbf{R}^m \times N$, $X(y) \in T_y N = \{0_x\} \times T_y N \subset T_{(x,y)}(\mathbf{R}^m \times N)$. Then $A(X)$ is a projectable vector field covering X. Then

$$A = \tilde{\mathcal{F}} + \mathcal{V},$$

where $\tilde{\mathcal{F}} : T_{\mathcal{M}f_n} \rightsquigarrow T\tilde{F}$ is the flow operator and $\mathcal{V} : T_{\mathcal{M}f_n} \rightsquigarrow T\tilde{F}$ is a natural operator of vertical type. Let $r \geq 1$ be the (minimal) order of $\tilde{F}$. Any vertical type natural operator $\mathcal{V} : T_{\mathcal{M}f_n} \rightsquigarrow T\tilde{F}$ is of order $r-1$ and $\tilde{\mathcal{F}}$ is of (minimal) order r. Then A is not of order $r-1$. Contradiction. □

2. Essentiality of the assumption of Theorem 1

Example 1. Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be the trivial bundle functor with fiber $\mathbf{R}$. For any $\mathcal{FM}_{m,n}$-object Y we have the trivial connection D on $FY = Y \times \mathbf{R} \to Y$. D is an (absolute) natural operator transforming connections on $Y \to M$ into connections on $FY \to Y$.

3. Applications of Theorem 1

Corollary 1. *Let $F : \mathcal{M}f \to \mathcal{FM}$ be a non-trivial bundle functor with the point property, e.g. the tangent functor T, the r-tangent functor T^r, the Weil functor T^A corresponding to an r-order Weil algebra A, the vector r-tangent functor $T^{(r)} = (J^r(.,\mathbf{R})_0)^*$ for $r \geq 1$, e.t.c. Write $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ for the composition of F with the forgetfull functor $\mathcal{FM}_{m,n} \to \mathcal{M}f$. Then there is no $\mathcal{FM}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on $FY \to Y$. In particular, for $F = T$ we reobtain Fact 2 without any assumption on the order of operators.*

Corollary 2. *Let $F : \mathcal{M}f_n \to \mathcal{FM}$ be a bundle functor of non-zero order, e.g. the tangent functor T, the r-tangent functor, the Weil functor T^A corresponding to a Weil algebra A, the vector r-tangent functor $T^{(r)} = (J^r(.,\mathbf{R})_0)^*$, the r-cotangent functor $T^{r*} = J^r(.,\mathbf{R})_0$, e.t.c. Let $V^F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be the vertical modification on F, $V^F Y = \bigcup_{x\in M} F(Y_x)$, $V^F\varphi = \bigcup_{x\in M} F(\varphi_x)$. Then there is no $\mathcal{FM}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on $FY \to Y$. In particular, for $F = T$ we have $V^F = V$ and we reobtain Fact 1 without any assumption on the order of operators.*

Corollary 3. *Let $F = J^r : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be the functor of r-jet prolongation, $r \geq 1$. Then there is no $\mathcal{FM}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on $J^rY \to Y$. In particular, for $r = 1$ we reobtain Fact 3 without any assumption on the order of operators.*

Clearly, the list of applications of Theorem 1 is not complete.

4. A generalization of Theorem 1

Theorem 2. *Let $H : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor such that there is $u_0 \in H_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n)$ with $H(\mathrm{id}_{\mathbf{R}^m} \times \varphi)(u_0) = u_0$ for any 0-preserving embedding $\varphi : \mathbf{R}^n \to \mathbf{R}^n$. Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor such that the corresponding natural bundle $\tilde{F} : \mathcal{M}f_n \to \mathcal{FM}$, $\tilde{F}N = F(\mathbf{R}^m \times N)$, $\tilde{F}\varphi = F(\mathrm{id}_{\mathbf{R}^m} \times \varphi)$, $N \in \mathrm{Obj}(\mathcal{M}f_n)$, $\varphi \in \mathrm{Morph}(\mathcal{M}f_n)$ is not of order 0. Then there is no $\mathcal{FM}_{m,n}$-natural operator transforming sections of $HY \to Y$ into connections on $FY \to Y$.*

Proof. Using u_0 we produce a section σ_0 of $H(\mathbf{R}^m \times N) \to \mathbf{R}^m \times N$ which is invariant with respect to $\mathcal{FM}_{m,n}$-maps of the form $\mathrm{id}_{\mathbf{R}^m} \times \varphi$. Next we modify the proof of Theorem 1 replacing connections $\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}$ by σ_0. $\square$

5. Applications of Theorem 2

Corollary 4. *Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor such that the corresponding natural bundle $\tilde{F} : \mathcal{M}f_n \to \mathcal{FM}$, $\tilde{F}N = F(\mathbf{R}^m \times N)$, $\tilde{F}\varphi = F(\mathrm{id}_{\mathbf{R}^m} \times \varphi)$, $N \in \mathrm{Obj}(\mathcal{M}f_n)$, $\varphi \in \mathrm{Morph}(\mathcal{M}f_n)$ is not of order 0. Then there is no $\mathcal{FM}_{m,n}$-natural operator transforming pairs of an s-order connection on M and a connection on $Y \to M$ into connections on $FY \to Y$.*

Corollary 5. *Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor such that the corresponding natural bundle $\tilde{F} : \mathcal{M}f_n \to \mathcal{FM}$, $\tilde{F}N = F(\mathbf{R}^m \times N)$, $\tilde{F}\varphi = F(\mathrm{id}_{\mathbf{R}^m} \times \varphi)$, $N \in \mathrm{Obj}(\mathcal{M}f_n)$, $\varphi \in \mathrm{Morph}(\mathcal{M}f_n)$ is not of order 0. Then there is no $\mathcal{FM}_{m,n}$-natural operator transforming a finite number of connections on $Y \to M$ into connections on $FY \to Y$.*

Corollary 6. *Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor such that the corresponding natural bundle $\tilde{F} : \mathcal{M}f_n \to \mathcal{FM}$, $\tilde{F}N = F(\mathbf{R}^m \times N)$, $\tilde{F}\varphi = F(\mathrm{id}_{\mathbf{R}^m} \times \varphi)$, $N \in \mathrm{Obj}(\mathcal{M}f_n)$, $\varphi \in \mathrm{Morph}(\mathcal{M}f_n)$ is not of order 0. Then there is no $\mathcal{FM}_{m,n}$-natural operator transforming a finite number of (projectable) $(1, k)$-tensor fields on Y into connections on $FY \to Y$.*

Corollary 7. *Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor such that the corresponding natural bundle $\tilde{F} : \mathcal{M}f_n \to \mathcal{FM}$, $\tilde{F}N = F(\mathbf{R}^m \times N)$, $\tilde{F}\varphi = F(\mathrm{id}_{\mathbf{R}^m} \times \varphi)$, $N \in \mathrm{Obj}(\mathcal{M}f_n)$, $\varphi \in \mathrm{Morph}(\mathcal{M}f_n)$ is not of order 0. Then there is no $\mathcal{FM}_{m,n}$-invariant connection on $FY \to Y$. In particular for $m = 0$ we obtain that if $F : \mathcal{M}f_n \to \mathcal{FM}$ is a natural bundle which is not of order 0 then there is no canonical connection on $FN \to N$.*

References

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Institute of Mathematics, Jagiellonian University
Reymonta 4, Kraków, Poland
E-mail: mikulski@im.uj.edu.pl