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MULTIPLICATION MODULES AND RELATED RESULTS

SHAHABADDIN EBRAHIMI ATANI

Abstract. Let R be a commutative ring with non-zero identity. Various properties of multiplication modules are considered. We generalize Ohm's properties for submodules of a finitely generated faithful multiplication R-module (see [8], [12] and [3]).

1. Introduction

Throughout this paper all rings will be commutative with identity and all modules will be unitary. If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ will be denoted by $[N : M]$. Then $[0 : M]$ is the annihilator of M, $\mathrm{Ann}(M)$. An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = [N : M]$. Clearly, M is a multiplication module if and only if for each $m \in M$, $Rm = [Rm : M]M$ (see [6]). For an R-module M, we define the ideal $\theta(M) = \sum_{m\in M}[Rm : M]$. If M is multiplication then $M = \sum_{m\in M} Rm = \sum_{m\in M}[Rm : M]M = (\sum_{m\in M}[Rm : M])M = \theta(M)M$. Moreover, if N is a submodule of M, then $N = [N : M]M = [N : M]\theta(M)M = \theta(M)[N : M]M = \theta(M)N$ (see [1]).

An R-module M is secondary if $0 \neq M$ and, for each $r \in R$, the R-endomorphism of M produced by multiplication by r is either surjective or nilpotent. This implies that $\mathrm{nilrad}(M) = P$ is a prime ideal of R, and M is said to be P-secondary. A secondary ideal of R is just a secondary submodule of the R-module R. A secondary representation for an R-module M is an expression for M as a finite sum of secondary modules (see [11]). If such a representation exists, we will say that M is representable. So whenever an R-module M has secondary representation, then the set of attached primes of M, which is uniquely determined, is denoted by $\mathrm{Att}_R(M)$.

A proper submodule N of a module M over a ring R is said to be prime submodule (primary submodule) if for each $r \in R$ the R-endomorphism of M/N produced by multiplication by r is either injective or zero (either injective or

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nilpotent), so $[0 : M/N] = P$ ($\mathrm{nilrad}(M/N) = P'$) is a prime ideal of R, and N is said to be P-prime submodule (P'-primary submodule). So N is prime in M if and only if whenever $rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $rM \subseteq N$. We say that M is a prime module (primary module) if zero submodule of M is prime (primary) submodule of M. The set of all prime submodule of M is called the spectrum of M and denoted by $\mathrm{Spec}(M)$.

Let M be an R-module and N be a submodule of M such that $N = IM$ for some ideal I of R. Then we say that I is a presentation ideal of N. It possible that for a submodule N no such presentation exist. For example, if V is a vector space over an arbitrary field with a proper subspace W ($\neq 0$ and V), then W has not any presentation. Clearly, every submodule of M has a presentation ideal if and only if M is a multiplication module. Let N and K be submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product N and K denoted by NK is defined by $NK = I_1I_2M$. Let $N = I_1M = I_2M = N'$ and $K = J_1M = J_2M = K'$ for some ideals I_1, I_2, J_1 and J_2 of R. It is easy to show that $NK = N'K'$, that is, NK is independent of presentation ideals of N and K ([4]). Clearly, NK is a submodule of M and $NK \subseteq N \cap K$.

2. Secondary modules

Let R be a domain which is not a field. Then R is a multiplication R-module, but it is not secondary and also if p is a fixed prime integer then $E(Z/pZ)$, the injective hull of the Z-module Z/pZ, is not multiplication, but it is representable. Now, we shall prove the following results:

Lemma 2.1. *Let R be a commutative ring, M a multiplication R-module, and N a P-secondary R-submodule of M. Then there exists $r \in R$ such that $r \notin P$ and $r \in \theta(M)$. In particular, rM is a finitely generated R-submodule of M.*

Proof. Otherwise $\theta(M) \subseteq P$. Assume that $a \in N$. Then

$$Ra = \theta(M)Ra \subseteq PRa = Pa \subseteq Ra,$$

so $a = pa$ for some $p \in P$. There exists a positive integer m such that $p^mN = 0$. It follows that $p^ma = a = 0$, and hence $N = 0$, a contradiction. Finally, if $r \in \theta(M)$, then rM is finitely generated by [1, Lemma 2.1]. □

Theorem 2.2. *Let R be a commutative ring, and let M be a representable multiplication R-module. Then M is finitely generated.*

Proof. Let $M = \sum_{i=1}^{k} M_i$ be a minimal secondary representation of M with $\mathrm{Att}_R(M) = \{P_1, P_2, \dots, P_k\}$. By Lemma 2.1, for each i, $i = 1, \dots, k$, there exists $r_i \in R$ such that $r_i \notin P_i$ and $r_i \in \theta(M)$. Then for each i, $i = 1, \dots, k$, we have

$$r_iM = r_iM_1 + \cdots + r_iM_{i-1} + M_i + r_iM_{i+1} + \cdots + r_iM_k\,.$$

It follows that $r = \sum_{i=1}^{k} r_i \in \theta(M)$ and $rM = M$. Now the assertion follows from Lemma 2.1. □

The proof of the next result should be compared with [6, Corollary 2.9].

Corollary 2.3. *Let R be a commutative ring. Then every artinian multiplication R-module is cyclic.*

Proof. Since every artinian module is representable by [11, 2.4], we have from Theorem 2.2 that M is finitely generated and hence M is cyclic by [5, Proposition 8]. □

Lemma 2.4. *Let I be an ideal of a commutative ring R. If M is a representable R-module, then IM is a representable R-module.*

Proof. Let $M = \sum_{i=1}^{n} M_i$ be a minimal secondary representation of M with $\mathrm{Att}_R(M) = \{P_1, \dots, P_n\}$. Then we have $IM = \sum_{i=1}^{n} IM_i$. It is enough to show that for each i, $i = 1, \dots, n$, IM_i is P_i-secondary. Suppose that $r \in R$. If $r \in P_i$, then $r^m IM_i = I(r^m M_i) = 0$ for some m. If $r \notin P_i$, then $r(IM_i) = I(rM_i) = IM_i$, as required. □

Theorem 2.5. *Let R be a commutative ring, and let M be a representable multiplication R-module. Then every submodule of M is representable.*

Proof. This follows from Lemma 2.4. □

Theorem 2.6. *Let R be a commutative ring, and let M be a multiplication representable R-module with* $\mathrm{Att}_R(M) = \{P_1, \dots, P_n\}$. *Then* $\mathrm{Spec}(M) = \{P_1M, \dots, P_nM\}$.

Proof. Let $M = \sum_{i=1}^{n} M_i$ be a minimal secondary representation of M with $\mathrm{Att}_R(M) = \{P_1, \dots, P_n\}$. Then by [11, Theorem 2.3], we have

$$\mathrm{Ann}(M) = \bigcap_{i=1}^{n} \mathrm{Ann} M_i \subseteq \bigcap_{i=1}^{n} P_i \subseteq P_k$$

for all k $(1 \leq k \leq n)$. Note that $P_iM \neq M$ for all i. Otherwise, since from Theorem 2.2 M is a finitely generated R-module, there is an element $p_i \in P_i$ such that $(1 - p_i)M = 0$ and so $1 - p_i \in \mathrm{Ann}(M) \subseteq P_i$. Thus $1 \in P_i$, a contradiction. It follows from [6, Corollary 2.11] that $P_iM \in \mathrm{spec}(M)$ for all i, $i = 1, \dots, n$.

Let N be a prime submodule of M with $[N : M] = P$, where P is a prime ideal of R. Since from [7, Theorem 2.10] M/N is P_i-secondary for some i, we get $P = P_i$. Thus $N = [N : M]M = P_iM$, as required. □

Corollary 2.7. *Let R be a commutative ring, and let M be a multiplication representable R-module with* $\mathrm{Att}_R(M) = \{P_1, \dots, P_n\}$. *Then* $\mathrm{Spec}(R/\mathrm{Ann}(M)) = \{P_1/\mathrm{Ann}(M), \dots, P_n/\mathrm{Ann}(M)\}$.

Proof. Since from Theorem 2.2 M is finitely generated, we have the mapping $\phi : \mathrm{Spec}(M) \longrightarrow \mathrm{Spec}(R/\mathrm{Ann}(M)$ by $P_iM \longmapsto P_i/\mathrm{Ann}(M)$ is surjective by [9, Theorem 2]. As M is multiplication, we have ϕ is one to one, as required. □

Theorem 2.8. *Let R be a commutative ring, and let M be a primary multiplication R-module. Then M is a finitely generated R-module.*

Proof. Let $0 \neq a \in M$. Then $Ra = \theta(M)Ra$, so there exists an element $r \in \theta(M)$ with $ra = a$, and hence $(1-r)a = 0$. Thus $(1-r)^m M = 0$ for some m since M is primary. Therefore we have $(1-r)^m \in \mathrm{Ann}(M) \subseteq \theta(M)$. Note that $(1-r)^m = 1-s$ where $s \in \theta(M)$. Thus $1 \in \theta(M)$, so $\theta(M) = R$, as required. □

Theorem 2.9. *Let R be a commutative ring and M a finitely generated faithful multiplication R-module. A submodule N of M is secondary if and only if there exists a secondary ideal J of R such that $N = JM$.*

Proof. Suppose first that N is a P-secondary submodule of M. There exists an ideal J of R such that $N = JM$. Let $r \in R$. If $r \in P$ then $r^n N = r^n JM = 0$ for some n. It follows that $r^n J = 0$ since M is faithful. If $r \notin P$ then $rN = N$, so $JM = rJM$, and hence $J = rJ$ since M is cancellation.

Conversely, let J be a P-secondary ideal of R and $s \in R$. If $s \in P$ then $s^m N = s^m JM = 0$. If $s \notin R$ then $sN = sJM = JM = N$, as required. □

Proposition 2.10. *Let E be an injective module over a commutative noetherian ring R. If M is a multiplication R-module then* $\mathrm{Hom}_R(M, E)$ *is representable.*

Proof. This follows from [14, Theorem 1] since over R, every multiplication R-module is noetherian. □

Proposition 2.11. *Let R be a commutative ring. Then every multiplication secondary module is a finitely generated primary R-module.*

Proof. This follows from Theorem 2.2 and the fact that, every R-epimorphism $\varphi : M \to M$ is an isomorphism. □

3. The Ohm type properties for multiplication modules

The purpose of this section is to generalize the results of M. M. Ali (see [3]) to the case of submodules of a finitely generated faithful multiplication module.

Throughout this section we shall assume unless otherwise stated, that M is a finitely generated faithful multiplication R-module. Our starting point is the following lemma.

Lemma 3.1. *Let $N = I_1 M$ and $K = I_2 M$ be submodules of M for some ideals I_1 and I_2 of R. Then $[N : K]M = [I_1 : I_2]M$.*

Proof. The proof is completely straightforward. □

Proposition 3.2. *Let N_i $(i \in \Lambda)$ be a collection of submodules of M such that $\sum_{i\in\Lambda} N_i$ is a multiplication module. Then for each $a \in \sum_{i\in\Lambda} N_i$ we have*

$$\Big(\sum_{i\in\Lambda} \big[N_i : \sum_{i\in\Lambda} N_i\big]\Big)M + \mathrm{Ann}(a)M = M\,.$$

Proof. There exist ideals I_i $(i \in \Lambda)$ of R such that $N_i = I_iM$ $(i \in \Lambda)$. Since $\sum_{i\in\Lambda} N_i = (\sum_{i\in\Lambda} I_i)M$, we get from [13, Theorem 10] that $\sum_{i\in\Lambda} I_i$ is a multiplication ideal. Therefore, from Lemma 3.1 and [3, Proposition 1.1], we have

$$\begin{aligned}\Big(\sum_{i\in\Lambda}\big[N_i : \sum_{i\in\Lambda} N_i\big]\Big)M + \mathrm{Ann}(a)M &= \Big(\sum_{i\in\Lambda}\big[I_iM : \sum_{i\in\Lambda} I_iM\big]\Big)M + \mathrm{Ann}(a)M\\ &= \Big(\sum_{i\in\Lambda}\big[I_i : \sum_{i\in\Lambda} I_i\big]\Big)M + \mathrm{Ann}(a)M\\ &= \Big(\sum_{i\in\Lambda}\big[I_i : \sum_{i\in\Lambda} I_i\big] + \mathrm{Ann}(a)\Big)M = RM = M\,.\end{aligned}$$

□

Proposition 3.3. *Let N_i $(1 \le i \le n)$ be a finite collection of submodules of M such that $\sum_{i=1}^n N_i$ is a multiplication module. Then for each $a \in \sum_{i=1}^n N_i$ we have*

$$\Big(\sum_{i=1}^{n}\big[\bigcap_{k=1}^{n} N_k) : \check{N}_\imath\big]\Big)M + \mathrm{Ann}(a)M = M$$

where $\check{N}_\imath$ denotes the intersection of all N_i except $N_\imath$.

Proof. By a similar argument to that in the proposition 3.2, this follows from Lemma 3.1, [6, Theorem] and [3, Proposition 1.2]. □

Lemma 3.4. *Let N and K be submodules of M such that $N+K$ is a multiplication module. Then for every maximal ideal P of R we have $[N_P : K_P]M_P + [K_P : N_P]M_P = M_P$.*

Proof. Let $N = I_1M$ and $K = I_2M$ be submodules of M for some ideals I_1 and I_2 of R. Clearly, $I_1 + I_2$ is multiplication, and it then follows from Lemma 3.1 and [3, Lemma 1.3] that

$$\begin{aligned}[N_P : K_P]M_P + [K_P : N_P]M_P &= [I_PM_P : J_PM_P]M_P + [J_PM_P : I_PM_P]M_P\\ &= ([I_P : J_P] + [J_P : I_P])M_P = R_PM_P = M_P\,.\end{aligned}$$

□

Lemma 3.5. *Let $N = IM$ and $K = JM$ be submodules of M such that $[N : K]M + [K : N]M = M$. Then $[I : J] + [J : I] = R$.*

Proof. By Lemma 3.1, we have

$$\begin{aligned}[N : K]M + [K : N]M &= [IM : JM]M + [JM : IM]M\\ &= ([I : J] + [J : I])M = M = RM\,.\end{aligned}$$

It follows that $[I : J] + [J : I] = R$ since M is a cancellation module. □

Lemma 3.6. *Let N and K be submodules of M such that $(N : K)M + (K : N)M = M$. Then the following statements are true:*

(i) *$NK = (N + K)(N \cap K)$.*

(ii) *$(N \cap K)T = NT \cap KT$ for every submodule T of M.*

Proof. (i) We can write $N = IM$ and $K = JM$ for some ideals I and J of R. Now, by Lemma 3.5 and [3, Lemma 1.4], we have

$$\begin{aligned} NK = IJM &= (I+J)(I\cap J)M = (I+J)M(I\cap J)M \\ &= (IM+JM)(IM\cap JM) = (N+K)(N\cap K)\,. \end{aligned}$$

(ii) This proof is similar to that of case (i) and we omit it. □

Proposition 3.7. *Let N and K be submodules of M such that $[N : K]M + [K : N]M = M$. Then for each positive integer s we have $(N + K)^s = N^s + K^s$. In particular, the claim holds if $N + K$ is a multiplication module.*

Proof. There exist ideals I and J of R such that $N = IM$ and $K = JM$. By Lemma 3.5 and [3, Proposition 2.1], we have

$$(N+K)^n = ((I+J)M)^s = (I+J)^sM = (I^s + J^s)M = N^s + K^s.$$

□

The following theorem is a generalization of Proposition 3.7.

Theorem 3.8. *Let N_i $(i \in \Lambda)$ be a collection of submodules of M such that $\sum_{i\in\Lambda} N_i$ is a multiplication module. Then for each positive integer n we have $(\sum_{i\in\Lambda} N_i)^n = \sum_{i\in\Lambda} N_i^n$.*

Proof. There exist ideals I_i $(i \in \Lambda)$ of R such that $N_i = I_iM$ $(i \in \Lambda)$. Clearly, $\sum_{i\in\Lambda} I_i$ is a multiplication ideal. By [3, Theorem 2.2], we have $(\sum_{i\in\Lambda} N_i)^n = (\sum_{i\in\Lambda} I_iM)^n = \big((\sum_{i\in\Lambda} I_i)M\big)^n = (\sum_{i\in\Lambda} I_i)^nM = (\sum_{i\in\Lambda} I_i^n)M = \sum_{i\in\Lambda} N_i^n$. □

Proposition 3.9. *Let N and K be submodules of M such that $[N : K]M + [K : N]M = M$. Then the following statements are true:*
(i) *$[N^s : K^s]M + [K^s : N^s]M = M$ for each positive integer s.*
(ii) *$(N \cap K)^s = N^s \cap K^s$ for each positive integer s.*

Proof. There exist ideals I and J of R such that $N = IM$ and $K = JM$.
(i) From Lemma 3.5, Lemma 3.1 and [3, Lemma 3.5], we have

$$\begin{aligned} [N^s : K^s]M + [K^s : N^s]M &= [I^sM : J^sM]M + [J^sM : I^sM]M \\ &= ([I^s : J^s] + [J^s : I^s])M = RM = M\,. \end{aligned}$$

(ii) From Lemma 3.5, [6, Theorem 1.6] and [3, Proposition 3.1], we have $(N \cap K)^s = (IM \cap JM)^s = \big((I \cap J)M\big)^s = (I \cap J)^sM = I^sM \cap J^sM = N^s \cap K^s$. □

Theorem 3.10. *Let N_i $(1 \le i \le n)$ be a finite collection of submodules of M such that $\sum_{i=1}^n N_i$ is a multiplication module. Then for each positive integer s we have $(\cap_{i=1}^n N_i)^s = \cap_{i=1}^n N_i^s$.*

Proof. There exist ideals I_i $(1 \le i \le n)$ of R such that $N_i = I_iM$ $(1 \le i \le n)$. Clearly, $\sum_{i=1}^n I_i$ is a multiplication ideal. Therefore, from [6, Theorem 1.6] and [3, Theorem 3.6], we get that $(\cap_{i=1}^n N_i)^s = (\cap_{i=1}^n I_iM)^s = \big((\cap_{i=1}^n I_i)M\big)^s = (\cap_{i=1}^n I_i)^sM = \cap_{i=1}^n I_i^sM = \cap_{i=1}^n N_i^s$. □

Lemma 3.11. *Let I be an ideal of R. Then* $\mathrm{Ann}(IM) = \mathrm{Ann}I$.

Proof. The proof is completely straightforward. □

Lemma 3.12. *Let P be a maximal ideal of R. If $N = IM$ is a multiplication submodule of M, and if I contains no non-zero nilpotent element, then the following statements are true:*

(i) $\mathrm{Ann}N = \mathrm{Ann}N^k$ *for each positive integer k.*

(ii) $\mathrm{Ann}(N_P^k) \subseteq \mathrm{Ann}(a)_P$ *for each $a \in I$ and each positive integer k.*

Proof. (i) The ideal I is multiplication by [13, Theorem 10], and by Lemma 3.11, $\mathrm{Ann}N = \mathrm{Ann}I$. Now, from [3, Corollary 2.4] and Lemma 3.11 we have

$$\mathrm{Ann}N = \mathrm{Ann}I = \mathrm{Ann}I^k = \mathrm{Ann}(I^k M) = \mathrm{Ann}N^k$$

(ii) By [3, Lemma 4.2], $\mathrm{Ann}(I_P^k) \subseteq \mathrm{Ann}(a)_P$ for each $a \in I$. It follows from (i) and [5, Lemma 2] that

$$\mathrm{Ann}N_P^k = \mathrm{Ann}((IM)_P)^k = \mathrm{Ann}(I_P M_P)^k = \mathrm{Ann}(I_P^k M_P) = \mathrm{Ann}I_P^k \subseteq \mathrm{Ann}(a)_P$$

□

Proposition 3.13. *Let $N = IM$ and $K = JM$ be submodules of M such that $N + K$ is a multiplication module. If $I + J$ contains no non-zero nilpotent element and $N^m = K^m$ for some positive integer m, then the following statements are true:*

(i) $N + \mathrm{Ann}(a)M = K + \mathrm{Ann}(a)M$ *for each $a \in I + J$.*

(ii) $\mathrm{Ann}N = \mathrm{Ann}K$.

Proof. (i) As $N^m = K^m$, we get $I^m = J^m$ since M is cancellation. Suppose that $a \in I + J$. Then by [3, Proposition 4.3], we have

$$N+\mathrm{Ann}M = IM+\mathrm{Ann}(a)M = (I+\mathrm{Ann}(a))M = (J+\mathrm{Ann}(a))M = K+\mathrm{Ann}(a)M\,.$$

(ii) This follows from 3.11 and [3, Proposition 4.3]. □

Proposition 3.14. *Let $N = IM$ and $K = JM$ be submodules of M such that K and $N + K$ are multiplication modules. Then for each positive integer m and each $a \in J^m$ we have $(N : K)^m M + \mathrm{Ann}(a)M = (K : N)^m M + \mathrm{Ann}(a)M$. Moreover, if J has no non-zero nilpotent elements, then for each $a \in J$ we have $(N : K)^m M + \mathrm{Ann}(a)M = (K : N)^m M + \mathrm{Ann}(a)M$.*

Proof. This follows from Lemma 3.1 and [3, Proposition 4.4]. □

References

[1] Ali, M. M., *The Ohm type properties for multiplication ideals*, Beiträge Algebra Geom. **37** (2) (1996), 399–414.

[2] Anderson D. D. and Al-Shaniafi, Y., *Multiplication modules and the ideal $\theta(M)$*, Comm. Algebra **30** (7) (2002), 3383–3390.

[3] Anderson, D. D., *Some remarks on multiplication ideals, II*, Comm. Algebra **28** (2000), 2577–2583.

[4] Ameri, R., *On the prime submodules of multiplication modules*, Int. J. Math. Math. Sci. **27** (2003), 1715–1724.

[5] Barnard, A, *Multiplication modules*, J. Algebra **71** (1981), 174–178.

[6] El-Bast Z. A. and Smith, P. F., *Multiplication modules*, Comm. Algebra **16** (1988), 755–779.

[7] Ebrahimi Atani, S., *Submodules of secondary modules*, Int. J. Math. Math. Sci. **31** (6) (2002), 321–327.

[8] Gilmer, R. and Grams, A., *The equality* $(A \cap B)^n = A^n \cap B^n$ *for ideals*, Canad. J. Math. **24** (1972), 792–798.

[9] Low, G. H. and Smith, P. F., *Multiplication modules and ideals*, Comm. Algebra **18** (1990), 4353–4375.

[10] Lu, C-P, *Spectra of modules*, Comm. Algebra **23** (10) (1995), 3741–3752.

[11] Macdonald, I. G., *Secondary representation of modules over commutative rings*, Symposia Matematica 11 (Istituto Nazionale di alta Matematica, Roma, (1973), 23–43.

[12] Naoum, A. G., *The Ohm type properties for finitely generated multiplication ideals*, Period. Math. Hungar. **18** (1987), 287–293.

[13] Smith, P. F., *Some remarks on multiplication module*, Arch. Math. **50** (1988), 223–235.

[14] Schenzel, S., *Asymptotic attached prime ideals to injective modules*, Comm. Algebra **20** (2) (1992), 583–590.

Department of Mathematics, University of Guilan
P.O. Box 1914 Rasht, Iran
E-mail: ebrahimi@guilan.ac.ir