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FIXED POINT THEOREMS FOR NONEXPANSIVE MAPPINGS IN MODULAR SPACES

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Abstract. In this paper, we extend several concepts from geometry of Banach spaces to modular spaces. With a careful generalization, we can cover all corresponding results in the former setting. Main result we prove says that if ρ is a convex, ρ-complete modular space satisfying the Fatou property and ρ_r-uniformly convex for all $r > 0$, C a convex, ρ-closed, ρ-bounded subset of X_ρ, $T : C \to C$ a ρ-nonexpansive mapping, then T has a fixed point.

1. Introduction

The theory of modular spaces was initiated by Nakano [15] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [14] in 1959. It is well known that one of the standard proof of Banach's fixed point theorem is based on Cantor's theorem in complete metric spaces [5, 6]. To this end, using some convenient constants in the contraction assumption, we present a generalization of Banach's fixed point theorem in some classes of modular spaces.

In this paper, we extend many concepts and results in normed spaces to modular spaces.

2. Preliminaries

We start by reviewing some basic facts about modular spaces as formulated by Musielak and Orlicz [14]. For more details the reader is refered to [7, 9, 10] and [13].

Definition 2.1 (cf. [7])**.** Let X be an arbitrary vector space.

(a) A function $\rho : X \to [0, \infty]$ is called a *modular* on X if for arbitrary x, y in X,

(i) $\rho(x) = 0$ if and only if $x = 0$,

(ii) $\rho(\alpha x) = \rho(x)$ for every scalar α with $\mid \alpha \mid= 1$, and

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(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$.

(b) If (iii) is replaced by (iii)'$\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$, e say that ρ is a *convex modular*.

(c) A modular ρ defines a corresponding *modular space*, i.e. the vector space X_ρ given by

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\} .$$

X_ρ is a linear subspace of X.

In general the modular ρ is not subadditive and therefore does not behave as a norm or a distance. But one can associate to a modular an F-norm (see [13]).

The modular space X_ρ can be equipped with an F-norm (see [13]) defined by

$$\|x\|_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{x}{\alpha}\right) \leq \alpha\right\} .$$

Namely, if ρ is convex, then the functional $|||x|||_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{x}{\alpha}\right) \leq 1\right\}$ is a norm in X_ρ which is equivalent to the F-norm $\|.\|_\rho$.

Definition 2.2 (cf. [7, 8])**.** Let X_ρ be a modular space.

(a) A sequence $(x_n) \subset X_\rho$ is said to be *ρ-convergent* to $x \in X_\rho$ and write $x_n \xrightarrow{\rho} x$, if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(b) A sequence (x_n) is called *ρ-Cauchy* whenever $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(c) The modular ρ is called *ρ-complete* if any ρ-Cauchy sequence is ρ-convergent.

(d) A subset $B \subset X_\rho$ is called *ρ-closed* if for any sequence $(x_n) \subset B$ ρ-convergent to $x \in X_\rho$, we have $x \in B$.

(e) A ρ-closed subset $B \subset X_\rho$ is called *ρ-compact* if any sequence $(x_n) \subset B$ has a ρ-convergent subsequence.

(f) ρ is said to satisfy the *Δ_2-condition* if $\rho(2x_n) \to 0$ whenever $\rho(x_n) \to 0$ as $n \to \infty$.

(g) We say that ρ has the *Fatou property* if $\rho(x) \leq \liminf_n \rho(x_n)$ whenever $x_n \xrightarrow{\rho} x$.

(h) A subset $B \subset X_\rho$ is said to be *ρ-bounded* if

$$\operatorname{diam}_\rho(B) < \infty ,$$

where $\operatorname{diam}_\rho(B) = \sup\{\rho(x - y); x, y \in B\}$ is called the *ρ-diameter* of B.

(i) Define the *ρ-distance* between $x \in X_\rho$ and $B \subset X_\rho$ as

$$\operatorname{dis}_\rho(x, B) = \inf\{\rho(x - y); y \in B\} .$$

(j) Define the *ρ-Ball*, $B_\rho(x, r)$, centered at $x \in X_\rho$ with radius r as

$$B_\rho(x, r) = \{y \in X_\rho; \rho(x - y) \leq r\} .$$

Let $(X, \|.\|)$ be a normed space. Then $\rho(x) = \|x\|$ is a convex modular on X. One can check that ρ-balls are ρ-closed if and only if ρ has the Fatou property (cf. [8]).

Example 2.3.
(1) The *Orlicz modular* is defined for every measurable real function f by the formula

$$\rho(f) = \int_{\mathbb{R}} \varphi(|f(t)|)\, dm(t)\,,$$

where m denotes the Lebesgue measure in $\mathbb{R}$ and $\varphi : \mathbb{R} \to [0, \infty)$ is continuous. We also assume that $\varphi(u) = 0$ iff $u = 0$ and $\varphi(t) \to \infty$ as $n \to \infty$. The modular space induced by the Orlicz modular ρ_φ is called the *Orlicz space* L^φ.
(2) The *Musielak-Orlicz modular spaces* (see. [17]). Let

$$\rho(f) = \int_{\Omega} \varphi(\omega, f(\omega))d\mu(\omega)\,,$$

where μ is a σ-finite measure on Ω, and $\varphi : \Omega \times \mathbb{R} \to [0, \infty)$ satisfy the following:
(i) $\varphi(\omega, u)$ is a continuous even function of u which is nondecreasing for $u > 0$, such that $\varphi(\omega, 0) = 0, \varphi(\omega, u) > 0$ for $u \neq 0$, and $\varphi(\omega, u) \to \infty$ as $n \to \infty$.
(ii) $\varphi(\omega, u)$ is a measurable function of ω for each $u \in \mathbb{R}$.

The corresponding modular space is called the *Musielak-Orlicz spaces*, and is denoted by L^φ.

Definition 2.4 (cf. [8])**.** A modular space X_ρ is said to have *ρ-normal structure* if for any nonempty ρ-bounded ρ-closed convex subset C of X_ρ not reduced to a one point, there exists a point $x \in C$ such that

$$r_\rho(x, C) := \sup\{\rho(x - y); y \in C\} < \operatorname{diam}_\rho(C)\,.$$

A modular space X_ρ is said to have *ρ-uniformly normal structure* if there exists a constant $c \in (0, 1)$ such that for any subset C as above, there exists $x \in C$ such that

$$r_\rho(x, C) < c \operatorname{diam}_\rho(C)\,.$$

Clearly ρ-uniformly normal structure is ρ-normal structure.

Let X_ρ be a modular space and let C be a nonempty ρ-bounded and ρ-closed convex subset C of X_ρ. We will say that C has the *fixed point property (fpp)* if every ρ-nonexpansive selfmap defined on C (i.e.,$T : C \to C, \rho(T(x) - T(y)) \leq \rho(x - y)$ for every $x, y \in C$) has a fixed point, that is, there eists $x \in C$ such that $T(x) = x$. Also, a modular space X_ρ is said to have the *fixed point property (fpp)* if every nonempty ρ-bounded ρ-closed convex subset of X_ρ has the fixed point property.

In Banach spaces, when we think about reflexivity automatically the dual space is present in our taught. But in modular spaces, it is very hard to conceive the dual space. To circumvent the problem, we use some characterization of reflexivity.

Theorem 2.5 (Smulian 1939, cf. [12])**.** *A normed space X is reflexive if and only if $\bigcap_n C_n \neq \emptyset$ whenever (C_n) is a sequence of nonempty, closed bounded and convex subsets of X such that $C_n \supseteq C_{n+1}$ for each $n \in \mathbb{N}$.*

Definition 2.6 (cf. [8])**.** Let X_ρ be a modular space. We will say that X_ρ or ρ satisfies the *property (R)* if every decreasing sequence of nonempty ρ-closed and ρ-bounded convex subsets of X_ρ, has a nonempty intersection.

The following theorem is known.

Theorem 2.7 (cf. [8]). *Let X_ρ be a ρ-complete modular space. Assume that ρ is convex and satisfies the Fatou property. Moreover, assume that X_ρ has the ρ-normal structure and has the property (R) and C is any ρ-closed ρ-bounded convex nonempty subset of X_ρ. Then any ρ-nonexpansive mapping $T : C \to C$ has a fixed point in C.*

3. Results

We start this chapter with generalizations as well as their corresponding results of uniform convexity and normal structure coefficients in modular spaces.

Definition 3.1. For $r > 0$, a modular space X_ρ is said to be *ρ_r-uniformly convex* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in X_\rho$, the conditions $\rho(x) \leq r, \rho(y) \leq r$ and $\rho(x-y) \geq r\varepsilon$ imply

$$\rho\Big(\frac{x+y}{2}\Big) \leq (1-\delta)r\,.$$

Definition 3.2. Let X_ρ be a Modular space. For any $\varepsilon \geq 0$ and $r > 0$, the *modulus of ρ_r-uniform convexity* of X_ρ is defined by

$$\delta_\rho(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\Big(\frac{x+y}{2}\Big) : \rho(x) \leq r, \rho(y) \leq r, \rho(x-y) \geq r\varepsilon\right\}.$$

Definition 3.3. The normal structure coefficient of X_ρ is the number

$$N(X_\rho) = \inf\Big\{\frac{\operatorname{diam}_\rho(C)}{R_\rho(C)} : C \subset X_\rho\ C \text{ is } \rho\text{-closed convex, } \rho\text{-bounded and } \operatorname{diam}_\rho(C) > 0\Big\},$$

where $R_\rho(C) := \inf\{r_\rho(x, C) : x \in C\}$ which is called the *ρ-Chebyshev radius* of C (cf. [7]).

Remark 3.4.

(1) It is not hard to show that $R_\rho(C) \neq 0$. Indeed, suppose $R_\rho(C) = 0$ and let, $x_0, y_0 \in C$ be such that $x_0 \neq y_0$. Since $R_\rho(C) = \inf_{y\in C} r_\rho(x, C) = 0$, so there exists a sequence (x_n) in C such that $\lim_{n\to\infty} r_\rho(x_n, C) = 0$. Thus

$$\rho\Big(\frac{x_0-y_0}{2}\Big) = \rho\left(\frac{(x_0-x_n)+(x_n-y_0)}{2}\right) \leq \rho(x_0-x_n) + \rho(x_n-y_0) \to 0$$

as $n \to \infty$. Therefore $x_0 = y_0$, a contradiction.

(2) For any $x \in C$ we have $R_\rho(C) \leq r_\rho(x, C) \leq \operatorname{diam}_\rho(C)$.

(3) It is obvious form the definition that X_ρ has ρ-uniform normal structure if and only if $N(X_\rho) > 1$ (see [11]).

Lemma 3.5. *Let $r > 0$. A modular space X_ρ is ρ_r-uniformly convex if and only if $\delta_\rho(r,\varepsilon) > 0$ for all $\varepsilon > 0$.*

Proof. Let $\varepsilon > 0$. If X_ρ is ρ_r-uniformly convex, then there exists $\delta > 0$ such that for any $x, y \in X_\rho$ with $\rho(x) \leq r, \rho(y) \leq r$, and $\rho(x-y) \geq r\varepsilon$. we have $\rho\big(\frac{x+y}{2}\big) \leq (1-\delta)r$. Thus, for these x and y, $\delta \leq 1 - \frac{1}{r}\rho\big(\frac{x+y}{2}\big)$. Hence $\delta_\rho(r,\varepsilon) \geq$

$\delta > 0$. Conversely, suppose $\delta_\rho(r,\varepsilon) \geq \delta > 0$ for some $\varepsilon > 0$ and $\delta > 0$. Take any $x, y \in X_\rho$ such that $\rho(x) \leq r, \rho(y) \leq r$ and $\rho(x-y) \geq r\varepsilon$. By definition of δ_ρ, we get $\delta_\rho(r,\varepsilon) \leq 1 - \frac{1}{r}\rho\big(\frac{x+y}{2}\big)$. Hence

$$\frac{1}{r}\rho\Big(\frac{x+y}{2}\Big) \leq 1 - \delta(r,\varepsilon) \leq 1 - \delta\,.$$

Therefore X_ρ is ρ_r-uniformly convex. □

Lemma 3.6. *The modulus $\delta_\rho(r,.)$ of uniform convexity of X_ρ is increasing on $[0,\infty)$.*

Proof. Let $r > 0$ and $\varepsilon_1 > \varepsilon_2 \geq 0$. Let $x, y \in X_\rho$ be such that $\rho(x) \leq r$ and $\rho(y) \leq r$. If $\rho(x-y) \geq \varepsilon_1 r$, then $\rho(x-y) \geq \varepsilon_2 r$. This show that

$$\left\{1 - \frac{1}{r}\rho\Big(\frac{x+y}{2}\Big) : \rho(x) \leq r, \rho(y) \leq r, \rho(x-y) \geq r\varepsilon_1\right\}$$
$$\subseteq \left\{1 - \frac{1}{r}\rho\Big(\frac{x+y}{2}\Big) : \rho(x) \leq r, \rho(y) \leq r, \rho(x-y) \geq r\varepsilon_2\right\}.$$

This implies that $\delta_\rho(r,\varepsilon_1) \geq \delta_\rho(r,\varepsilon_2)$. □

Theorem 3.7. *If the modulus δ_ρ of convexity of a modular space X_ρ satisfies $\delta_\rho(d,\varepsilon) > 0$ for all $d, \varepsilon > 0$, then X_ρ has ρ-normal structure.*

Proof. Let C be a nonempty ρ-bounded ρ-closed convex subset of X_ρ with $\operatorname{diam}_\rho(C) = d > 0$. Let $\varepsilon \in (0,1)$ there exist $x, y \in C$ such that

$$\rho(x-y) \geq d\varepsilon\,.$$

Let $z = \frac{x+y}{2}$ and $w \in C$. Thus, $z \in C$, $\rho(w-x) \leq d, \rho(w-y) \leq d$ and $\rho((w-x)-(w-y)) = \rho(x-y) \geq d\varepsilon$.
Consequently,

$$\rho\left(w - \Big(\frac{x+y}{2}\Big)\right) = \rho\left(\frac{(w-x)+(w-y)}{2}\right) \leq (1 - \delta_\rho(d,\varepsilon))d\,.$$

Hence

$$\sup_{w\in C} \rho(w-z) \leq \big(1 - \delta_\rho(d,\varepsilon)\big)d\,.$$

Since $\delta_\rho(d,\varepsilon) > 0$, we get

$$\sup_{w\in C} \rho(w-z) < d = \operatorname{diam}_\rho(C)\,.$$

Since this is true for any C, this proves that X_ρ has ρ-normal structure.

Lemma 3.5 and Theorem 3.7 give us immediately

Corollary 3.8. *For a modular space X_ρ, if X_ρ is ρ_r-uniformly convex for all $r > 0$, then X_ρ has ρ-normally structure.*

Corollary 3.9 (cf. [4])**.** *Closed bounded convex subsets of uniformly convex Banach spaces have normal structure.*

Theorem 3.10. *Let X_ρ be a ρ-complete modular space. If ρ is convex and satisfies the Fatou property and X_ρ is ρ_r-uniformly convex for all $r > 0$, then X_ρ has the property (R).*

Proof. Let (C_n) be a decreasing sequence of ρ-bounded, ρ-closed nonempty convex subsets of X_ρ, $z \in X_\rho$ which does not belong to C_1 and

$$r = \lim_{n\to\infty} \operatorname{dis}_\rho(z, C_n).$$

Define $D_n = C_n \cap B_\rho(z,r)$ and let d_n be the diameter of D_n. By the Fatou property of ρ, (D_n) is a decreasing sequence of nonempty ρ-bounded, ρ-closed convex subsets of X_ρ because $B_\rho(z,r)$ is then a ρ-closed set (see [8]).

Let r_n be a sequence of positive number that decreases to zero and $d_n - r_n > 0$ for all n. There exist $x, y \in D_n$ such that $\rho(x-y) \geq d_n - r_n$. Thus, by the definition of $\delta_\rho(r, \frac{d_n - r_n}{r})$, we have

$$\rho(z - \frac{x+y}{2}) = \rho\left(\frac{(z-x)+(z-y)}{2}\right) \leq \left(1 - \delta_\rho\left(r, \frac{d_n - r_n}{r}\right)\right) r.$$

Hence

$$(*) \qquad \frac{1}{r}\operatorname{dis}_\rho(z, C_n) \leq \frac{1}{r}\rho\left(z - \frac{x+y}{2}\right) \leq 1 - \delta_\rho\left(r, \frac{d_n - r_n}{r}\right).$$

Put $d = \lim_{n\to\infty} d_n$ and $a_n = d_n - \frac{1}{n}$, and consider two cases.

Case 1 ($a_n \geq d$, for all n large enough). By δ_ρ being increasing and $(*)$, we have for all n large enough,

$$\frac{1}{r}\operatorname{dis}_\rho(z, C_n) \leq 1 - \delta_\rho\left(r, \frac{a_n}{r}\right) \leq 1 - \delta_\rho\left(r, \frac{d}{r}\right).$$

Letting $n \to \infty$, we get

$$1 \leq 1 - \delta_\rho\left(r, \frac{d}{r}\right),$$

which implies that $\delta_\rho(r, \frac{d}{r}) = 0$. By ρ_r-uniform convexity of X_ρ and Lemma 3.1.6 we have $\delta_\rho(r, \varepsilon) > 0$ for all $\varepsilon > 0$, whence $d = 0$.

Case 2 ($0 < a_n < d$, for infinitely many n). There exists a subsequence $(a_{n'})$ such that $a_{n'} \nearrow d$, whence the limit $\lim_{n'\to\infty} \delta_\rho(r, \frac{a_{n'}}{r})$ exists and by $(*)$, we have

$$1 \leq 1 - \lim_{n'\to\infty} \delta_\rho\left(r, \frac{a_{n'}}{r}\right).$$

Consequently, $\lim_{n'\to\infty} \delta_\rho(r, \frac{a_{n'}}{r}) = 0$. Since $a_{n'} \nearrow d$ and $\delta_\rho(r, \varepsilon) > 0$ for all $\varepsilon > 0$, we have $d = \lim_{n\to\infty} d_n = 0$ as well. Thus, there exists a ρ-Cauchy sequence (x_n), where $x_n \in D_n$ for each n. Since X_ρ is ρ-complete, $(x_n)\rho$-converges to some $x_0 \in X_\rho$. Using the ρ-closeness of D_n, we deduce that $x_0 \in D_n$ for all $n \geq 1$. This implies that $\cap_{n\in\mathbb{N}} D_n \neq \emptyset$ and so $\cap_{n\in\mathbb{N}} C_n \neq \emptyset$ as well. The proof is therefore complete. $\square$

Corollary 3.11 (cf. [4]). *Let X_ρ be a ρ-complete modular space with ρ convex and satisfying the Fatou property. If X_ρ is ρ_r-uniformly convex for all $r > 0$, then X_ρ has the fixed point property.*

Proof. By Corollary 3.8 and Theorem 3.10, X_ρ has ρ-normal structure and property (R). Consequently, Theorem 2.7 can be applied to conclude that X_ρ has the fixed point property. □

Corollary 3.12 (cf. [4]). *If C is a nonempty closed bounded convex subset of a uniformly convex Banach space, then every nonexpansive mapping $T : C \to C$ has a fixed point in C.*

Theorem 3.13. *Let X_ρ be a modular space with modulus of convexity $\delta_\rho(1, \varepsilon) \neq 1$ for some $\varepsilon \in (0, 1)$. If we assume that $\rho(\alpha x) = \alpha\rho(x)$ for all $\alpha > 0$, then*

$$N(X_\rho) \geq \frac{1}{1 - \delta_\rho(1, \varepsilon)}.$$

Proof. Let C be a ρ-closed, ρ-bounded convex subset of X_ρ with $\operatorname{diam}_\rho(C) = d > 0$. Since $\varepsilon \in (0, 1)$, there exist $x, y \in C$ such that

$$\rho(x - y) \geq d\varepsilon\,.$$

Let $z = \frac{x+y}{2} \in C$ and $w \in C$. Then $\rho(\frac{w-x}{d}) = \frac{1}{d}\rho(w - x) \leq 1, \rho(\frac{w-y}{d}) = \frac{1}{d}\rho(w - y) \leq 1$, and

$$\rho\left(\left(\frac{w-x}{d}\right) - \left(\frac{w-y}{d}\right)\right) = \frac{1}{d}\rho(x - y) \geq \varepsilon\,.$$

By the definition of $\delta_\rho(1, \varepsilon)$, we obtain

$$\frac{1}{d}\rho\Big(w - \frac{x+y}{2}\Big) = \frac{1}{d}\rho\left(\frac{(w-x)+(w-y)}{2}\right) \leq 1 - \delta_\rho(1, \varepsilon)\,.$$

Hence it follows that

$$R_\rho(C) \leq \sup_{w \in K} \rho(z - w) \leq d\big(1 - \delta_\rho(1, \varepsilon)\big)\,.$$

Consequently,

$$\frac{\operatorname{diam}_\rho(C)}{R_\rho(C)} \geq \frac{1}{1 - \delta_\rho(1, \varepsilon)}\,.$$

Therefore

$$N(X_\rho) \geq \frac{1}{1 - \delta_\rho(1, \varepsilon)}\,.$$

□

Remark 3.14. If we assume that in Colloray3.8 $\rho(\alpha x) = \alpha\rho(x)$ for all $\alpha > 0$, then X_ρ will have ρ-uniformly normal structure.

Corollary 3.15. *If X_ρ is a modular space with the modulus of convexity $\delta_\rho(1, \varepsilon) \in (0, 1)$ for some $\varepsilon \in (0, 1)$, then X_ρ has ρ-uniformly normal structure.*

Proof. By Theorem 3.13 we have $N(X_\rho) > 1$. Thus, by Remarks 3.4 (3), X_ρ has ρ-uniformly normal structure.

Corollary 3.16. *If X is a Banach space space with modulus of convexity $\delta_X(\varepsilon) \in (0,1)$ for some $\varepsilon \in (0,1)$ and we put $\rho(x) = \|x\|$, then we get that X has uniformly normal structure.*

Corollary 3.16 strongly improves [1] which states that any uniformly convex Banach space has uniformly normal structure.

Note that a Banach space X is uniformly convex if and only if its modulus of convexity satisfies $\delta_X(\varepsilon) > 0$ for all $\varepsilon > 0$ (see [5]).

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