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CHARACTERIZATIONS OF LAMBEK-CARLITZ TYPE

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Abstract. We give Lambek-Carlitz type characterization for completely multiplicative reduced incidence functions in Möbius categories of full binomial type. The q-analog of the Lambek-Carlitz type characterization of exponential series is also established.

1. An arithmetical function f is called multiplicative if

$$f(mn) = f(m)f(n) \quad \text{whenever} \quad (m,n) = 1 \tag{1.1}$$

and it is called completely multiplicative if

$$f(mn) = f(m)f(n) \quad \text{for all} \quad m \quad \text{and} \quad n\,. \tag{1.2}$$

Lambek [5] proved that the arithmetical function f is completely multiplicative if and only if it distributes over every Dirichlet product:

$$f(g *_D h) = fg *_D fh\,, \quad \text{for all arithmetical functions } g \text{ and } h. \tag{1.3}$$

$\big(g *_D h$ is defined by: $(g *_D h)(n) = \sum\limits_{d|n} g(d)h\left(\frac{n}{d}\right)\big)$.

Problems of Carlitz [1] and Sivaramakrishnan [12] concern the equivalence between the complete multiplicativity of the function f and the way it distributes over certain particular Dirichlet products. For example, Carlitz's Problem E 2268 [1] asks us to show that f is completely multiplicative if and only if

$$f(n)\tau(n) = \sum_{d|n} f(d)f\left(\frac{n}{d}\right) \quad (\forall n \in \mathbb{N}^*)\,, \tag{1.4}$$

that is if and only if f distributes over $\zeta *_D \zeta = \tau$, where $\zeta(n) = 1$, $\forall n \in \mathbb{N}^*$, and $\tau(n)$ is the number of positive divisors of $n \in \mathbb{N}^*$.

2. Möbius categories were introduced in [7] to provide a unified setting for Möbius inversion. We refer the reader to [2] and [8] for the definitions of a Möbius category and of a Möbius category of full binomial type, respectively. In the

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incidence algebra $A(\mathcal{C})$ of a Möbius category $\mathcal{C}$ the convolution of two incidence function f and g is defined by:

$$(f*g)(\alpha)=\sum_{\alpha'\alpha''=\alpha} f(\alpha')g(\alpha'') \quad \forall\alpha\in \operatorname{Mor}\mathcal{C}\,. \tag{2.1}$$

The incidence function f is called completely multiplicative (see [11]) if for any morphism $\alpha\in\operatorname{Mor}\mathcal{C}$

$$f(\alpha)=f(\alpha')f(\alpha'') \quad \text{whenever} \quad \alpha'\alpha''=\alpha\,. \tag{2.2}$$

Lambek's characterization can be generalized to the convolution of the incidence functions: $f\in A(\mathcal{C})$ is completely multiplicative if and only if

$$f(g*h)=fg*fh \quad \forall g,h\in A(\mathcal{C})\,, \tag{2.3}$$

but if $\zeta(\alpha)=1$, $\forall\alpha\in\operatorname{Mor}\mathcal{C}$, and $\zeta*\zeta=\tau_{\mathcal{C}}$, then the condition (Carlitz's characterization)

$$f\tau_{\mathcal{C}}=f*f \tag{2.4}$$

is not sufficient for $f\in A(\mathcal{C})$ to be completely multiplicative (see [11]).

3. Let $\mathcal{C}$ be a Möbius category of full binomial type with the surjective "length function" $l:\operatorname{Mor}\mathcal{C}\to\mathbb{N}$ (see [2], [8]) and with the parameters $B(n)$ ($B(n)$ represent the total number of decompositions into indecomposable factors of length 1 of a morphism of length n). If $\alpha\in\operatorname{Mor}\mathcal{C}$ and $k\le l(\alpha)$ then $|\{\alpha',\alpha'')|\alpha'\alpha''=\alpha,\ l(\alpha')=k\}|$ is denoted by $\binom{\alpha}{k}$ and for any $\alpha,\beta\in\operatorname{Mor}\mathcal{C}$ with $l(\alpha)=l(\beta)=n$, the following holds

$$\begin{gathered}\binom{\alpha}{k}=\binom{\beta}{k}\ \left(\text{not}\binom{n}{k}\right) \quad \text{and}\\ \binom{n}{k}_l=\frac{B(n)}{B(k)B(n-k)} \quad (\forall k\in\mathbb{N},\ k\le n)\,.\end{gathered} \tag{3.1}$$

If $A(\mathcal{C})$ is the incidence algebra of $\mathcal{C}$ (with the usual pointwise addition and scalar multiplication and the convolution defined by (2.1)) then

$$A_l(\mathcal{C})=\{f\in A(\mathcal{C})\mid l(\alpha)=l(\beta)\Rightarrow f(\alpha)=f(\beta)\} \tag{3.2}$$

is a subalgebra of $A(\mathcal{C})$, called the reduced incidence algebra of $\mathcal{C}$. For $f,g\in A_l(\mathcal{C})$ considered as arithmetical functions ($f(n)=f(\alpha)$ if $l(\alpha)=n$), the convolution $f*g$ is given by

$$(f*g)(n)=\sum_{k=0}^{n}\binom{n}{k}_l f(k)g(n-k)\,, \quad (\forall n\in\mathbb{N}) \tag{3.3}$$

and $\mathcal{X}_{\mathcal{C}} : \mathbb{C}\,[\![X]\!] \to A_l(\mathcal{C})$ defined by

$$\mathcal{X}_{\mathcal{C}}\left(\sum_{n=0}^{\infty} a_n X^n\right)(\alpha) = a_{l(\alpha)} B(l(\alpha))\,, \quad \forall \alpha \in \operatorname{Mor}\mathcal{C}$$

(3.4.)

$$\left(\mathcal{X}_{\mathcal{C}}\left(\sum_{n=0}^{\infty} a_n X^n\right)(m) = a_m B(m)\,, \quad \forall m \in \mathbb{N}\right)$$

is a $\mathbb{C}$-algebra isomorphism.

4. In general, a completely multiplicative reduced incidence function f of $\mathcal{C}$ (that is an element of the subalgebra $A_l(\mathcal{C})$), is not completely multiplicative as arithmetical function. We have:

Theorem 1. *Let $\mathcal{C}$ be a Möbius category of full binomial type. The reduced incidence function $f \in A_l(\mathcal{C})$, with $f(1_A) = 1$ for an identity morphism 1_A, is completely multiplicative if and only if the arithmetical function $f \circ \omega$ is multiplicative, where $\omega(n)$ denotes the number of distinct prime factors of n.*

Proof. Suppose that f is completely multiplicative as incidence function. Let m and n be positive integers with $(m, n) = 1$ and let $\alpha, \alpha', \alpha''$ morphisms of $\mathcal{C}$ such that $\alpha'\alpha'' = \alpha$, $l(\alpha') = \omega(m)$ and $l(\alpha'') = \omega(n)$. Since $\mathcal{C}$ is of binomial type, $l(\alpha) = \omega(m) + \omega(n)$ and therefore:

$$(f \circ \omega)(mn) = f(\alpha) = f(\alpha')f(\alpha'') = (f \circ \omega)(m) \cdot (f \circ \omega)(n)\,.$$

Conversely, suppose that the arithmetical function $f \circ \omega$ is multiplicative. Let α be a morphism of $\mathcal{C}$ with a factorization $\alpha = \alpha'\alpha''$, $l(\alpha') = m$ and $l(\alpha'') = n$ and let the primes p of $\mathbb{N}^*$ be listed in any definite order $p_1, p_2, p_3, \ldots$ Then

$$\begin{aligned} f(\alpha) &= (f \circ \omega)(p_1 \ldots p_m p_{m+1} \ldots p_{m+n}) \\ &= (f \circ \omega)(p_1 \ldots p_m)(f \circ \omega)(p_{m+1} \ldots p_{m+n}) = f(\alpha')f(\alpha'')\,. \end{aligned}$$ □

5. Let us see now a Lambek-Carlitz type characterization of completely multiplicative reduced incidence functions of a Möbius category of full binomial type.

Theorem 2. *Let $\mathcal{C}$ be a Möbius category of full binomial type and f a reduced incidence function with $f(\overline{\alpha}) = a \neq 0$ for a non-identity indecomposable morphism $\overline{\alpha}$. Then the following statements are equivalent:*

(1) *$f \in A_l(\mathcal{C})$ is completely multiplicative;*

(2) *$f(\alpha) = a^n$ if $l(\alpha) = n$;*

(3) *$f(g * h) = fg * fh$, for all $g, h \in A_l(\mathcal{C})$;*

(4) *$f\tau_{\mathcal{C}} = f * f$, where $\tau_{\mathcal{C}}(\alpha) = \sum_{k=0}^{l(\alpha)} \binom{l(\alpha)}{k}_l$.*

Proof. (1) $\Leftrightarrow$ (2). Since $a \neq 0$ and since the identity morphism 1_A is a morphism of length 0, we have $f(1_A) = 1$, $\forall A \in \operatorname{Ob}\mathcal{C}$, and by induction on the length of α it follows both (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3).

$$[f(g*h)](\alpha) = f(\alpha)\sum_{\alpha'\alpha''=\alpha} g(\alpha')h(\alpha'') = \sum_{\alpha'\alpha''=\alpha} f(\alpha')g(\alpha')f(\alpha'')h(\alpha'')$$
$$= (fg*fh)(\alpha)\,, \quad \forall \alpha \in \operatorname{Mor}\mathcal{C}\,.$$

(3) $\Rightarrow$ (4).

$$\tau_{\mathcal{C}}(\alpha) = \sum_{k=0}^{l(\alpha)} \binom{l(\alpha)}{k}_l = |(\alpha',\alpha'') : \alpha'\alpha'' = \alpha\}|$$
$$= \sum_{\alpha'\alpha''=\alpha} \zeta(\alpha')\zeta(\alpha'') = (\zeta*\zeta)(\alpha)\,, \quad \forall \alpha \in \operatorname{Mor}\mathcal{C}\,,$$

and so (4) follows by using (3) for $g = \zeta$ and $h = \zeta$.

(4)$\Rightarrow$ (2). It follows by induction on the length of α using (3.3).

6. Note that Theorem 2, via the (inverse of the) $\mathbb{C}$-algebra isomorphism $\mathcal{X}_{\mathcal{C}} : \mathbb{C}[\![X]\!] \to A_l(\mathcal{C})$ defined by (3.4), gives rise to characterizations of Lambek-Carlitz type for special classes of formal power series (see also [11, Theorem 3.3.]).

Let $\mathcal{C}$ be a Möbius category of full binomial type and

(6.1.)

$$S(\mathcal{C}) = \left\{ \sum_{n=0}^{\infty} a_n X^n \in \mathbb{C}[\![X]\!] \mid \mathcal{X}_{\mathcal{C}}\Big(\sum_{n=0}^{\infty} a_n X^n\Big) \begin{array}{l} \text{are completely multiplicative} \\ \text{as incidence functions} \end{array} \right\}$$

We remark:

(i) If $\sum_{n=0}^{\infty} a_n X^n \in S(\mathcal{C})$ and if $\overline{\alpha}$ is a non-identity indecomposable morphism than $\mathcal{X}_{\mathcal{C}}\Big(\sum_{n=0}^{\infty} a_n X^n\Big)(\overline{\alpha}) = a_1$. Thus, for $\alpha \in \operatorname{Mor}\mathcal{C}$ with $l(\alpha) = m$ we have $\mathcal{X}_{\mathcal{C}}\Big(\sum_{n=0}^{\infty} a_n X^n\Big)(\alpha) = a_1^m$ and using (3.4), $\mathcal{X}_{\mathcal{C}}\Big(\sum_{n=0}^{\infty} a_n X^n\Big)(\alpha) = a_m B(m)$, where $B(m)$, $m \in \mathbb{N}$, are the parameters of $\mathcal{C}$. It follows that $\sum_{n=0}^{\infty} a_n X^n \in S(\mathcal{C})$ if and only if $a_m = \dfrac{a_1^m}{B(m)}$, $\forall m \in \mathbb{N}$.

(ii) If $\odot_{\mathcal{C}}$ denotes the corresponding binary operation on $\mathbb{C}[\![X]\!]$ of the usual multiplication of incidence functions $\Big($that is $\mathcal{X}_{\mathcal{C}}\Big(\sum_{n=0}^{\infty} a_n X^n \odot_{\mathcal{C}} \sum_{n=0}^{\infty} b_n X^n\Big) = \mathcal{X}_{\mathcal{C}}\Big(\sum_{n=0}^{\infty} a_n X^n\Big) \cdot \mathcal{X}_{\mathcal{C}}\Big(\sum_{n=0}^{\infty} b_n X^n\Big)\Big)$ then, by (3.4), we have
$\sum_{n=0}^{\infty} a_n X^n \odot_{\mathcal{C}} \sum_{n=0}^{\infty} b_n X^n = \sum_{n=0}^{\infty} B(n) a_n b_n X^n$.

In the following section we use these remarks to obtain the q-analog of the Lambek-Carlitz type characterization of exponential series.

7. In [10], using an embedding of the algebra $\mathbb{C}[\![X]\!]$ into the unitary algebra of arithmetical functions, it is proved the following Lambek-Carlitz type characterization of exponential series:

Theorem 3 ([10]). *Let $\sum_{n=0}^{\infty} a_n X^n \in \mathbb{C}[\![X]\!]$ such that $a_1 \neq 0$. The following statements are equivalent:*

(i) $a_n = \dfrac{a_1^n}{n!}, \quad \forall n \in \mathbb{N}$;

(ii) $\sum_{n=0}^{\infty} a_n X^n \odot \Big(\sum_{n=0}^{\infty} b_n X^n \cdot \sum_{n=0}^{\infty} c_n X^n \Big) = \Big(\sum_{n=0}^{\infty} a_n X^n \odot \sum_{n=0}^{\infty} b_n X^n \Big)$ $\cdot \Big(\sum_{n=0}^{\infty} a_n X^n \odot \sum_{n=0}^{\infty} c_n X^n \Big), \forall \sum_{n=0}^{\infty} b_n X^n,\ \sum_{n=0}^{\infty} c_n X^n \in \mathbb{C}\ [\![X]\!]$ (*distributivity over the product of series*);

(iii) $\sum_{n=0}^{\infty} 2^n a_n X^n = \sum_{n=0}^{\infty} a_n X^n \cdot \sum_{n=0}^{\infty} a_n X^n$,

where $\sum_{n=0}^{\infty} a_n X^n \odot \sum_{n=0}^{\infty} b_n X^n = \sum_{n=0}^{\infty} n! a_n b_n X^n$.

The aim of this section is to establish a q-analog of Theorem 3.

Let K be a finite field with $|K| = q$. Then the matrix $A = (a_{ij})_{m\times n}$ over K is called reduced matrix if:

(1) rang $A = m$,
(2) for any i the first nonzero element (called pivot) of the line i equals 1: $a_{ih_i} = 1$, $a_{ij} = 0$ if $j < h_i$,
(3) $h_1 < h_2 < \cdots < h_m$,
(4) pivot columns contain only 0 with the exception of the pivot.

We denote the category of reduced matrices by $\mathcal{R}$. The objects of $\mathcal{R}$ are the non-negative integers with 0 as initial object, the set of morphisms from n to m is the set of reduced $m \times n$ matrices over K, and the composition of morphisms is the matrix multiplication. $\mathcal{R}$ is a Möbius category of full binomial type with $\binom{n}{k}_l = \begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]_q!}{[k]_q![n-k]_q!}$ and $B(n) = [n]_q!$, where $[0]_q! = 1$ and $[n]_q! = (1+q)(1+q+q^2)\dots(1+q+\cdots+q^{n-1})$ (see [8]). Now, from Theorem 2 and the remarks of Section 6 we obtain the following Lambek-Carlitz type characterization:

Theorem 4. *Let $\sum_{n=0}^{\infty} a_n X^n \in \mathbb{C}\ [\![X]\!]$ such that $a_1 \neq 0$. The following statements are equivalent:*

(1) $\sum_{n=0}^{\infty} a_n X^n \in S(\mathcal{R})$;

(2) $a_n = \dfrac{a_1^n}{[n]_q!}, \quad \forall n \in \mathbb{N}$;

(3) $\sum_{n=0}^{\infty} a_n X^n \odot_{\mathcal{R}} \left(\sum_{n=0}^{\infty} b_n X^n \cdot \sum_{n=0}^{\infty} c_n X^n \right) = \left(\sum_{n=0}^{\infty} a_n X^n \odot_{\mathcal{R}} \sum_{n=0}^{\infty} b_n X^n \right)$
$\cdot \left(\sum_{n=0}^{\infty} a_n X^n \odot_{\mathcal{R}} \sum_{n=0}^{\infty} c_n X^n \right), \forall \sum_{n=0}^{\infty} b_n X^n, \sum_{n=0}^{\infty} c_n X^n \in \mathbb{C} [\![X]\!]$
(distributivity over the product of series);

(4) $\sum_{n=0}^{\infty} G_n(q) a_n X^n = \sum_{n=0}^{\infty} a_n X^n \cdot \sum_{n=0}^{\infty} a_n X^n,$

where $G_n(q)$ are the Galois numbers and $\sum_{n=0}^{\infty} a_n X^n \odot_{\mathcal{R}} \sum_{n=0}^{\infty} b_n X^n = \sum_{n=0}^{\infty} [n]_q! a_n b_n X^n$.

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