ARCHIVUM MATHEMATICUM (BRNO)
Tomus 40 (2004), 287 – 293

PARAMETERIZED CURVE AS ATTRACTORS OF SOME COUNTABLE ITERATED FUNCTION SYSTEMS

NICOLAE-ADRIAN SECELEAN

Abstract. In this paper we will demonstrate that, in some conditions, the attractor of a countable iterated function system is a parameterized curve. This fact results by generalizing a construction of J. E. Hutchinson [3].

1. Preliminary facts

We will present some notions and results used in the sequel (more complete and rigorous treatments may be found in [2], [1]).

1.1. **Hausdorff metric.** Let (X, d) be a complete metric space and $\mathcal{K}(X)$ be the class of all compact non-empty subsets of X.

If we define a function $\delta : \mathcal{K}(X) \times \mathcal{K}(X) \longrightarrow \mathbb{R}_+$,

$$\delta(A, B) = \max\{\mathrm{d}(A, B), \mathrm{d}(B, A)\}\,,$$

where

$$\mathrm{d}(A, B) = \sup_{x\in A}(\inf_{y\in B} \mathrm{d}(x, y))\,, \quad \text{for all} \quad A, B \in \mathcal{K}(X)\,,$$

we obtain a metric, namely the *Hausdorff metric*.

The set $\mathcal{K}(X)$ is a complete metric space with respect to this metric δ.

1.2. **Parameterized curve in the case of iterated function systems.** In this section, we will present the iterated functions system (abbreviated IFS) and the Hutchinson's construction of a continuous function f defined to $[0, 1]$ such that $\mathrm{Im}(f)$ (the image of f) is the attractor of some IFS (for details see [3]).

Let (X, d) be a complete metric space. A set of contractions $(\omega_n)_{n=1}^N$, $N \geq 1$, is called an *iterated function system* (IFS), according to M. Barnsley. Such a system of maps induces a set function $\mathcal{S} : \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$,

$$\mathcal{S}(E) = \bigcup_{n=1}^{N} \omega_n(E)$$

2000 *Mathematics Subject Classification*: Primary 28A80, Secondary 37C70, 47H10.
Key words and phrases: Hausdorff metric, countable iterated function system, attractor.
Received September 10, 2002.

which is a contraction on $\mathcal{K}(X)$ with contraction ratio $r \leq \max_{1\leq n\leq N} r_n$, r_n being the contraction ratio of ω_n, $n = 1, \ldots, N$. According to the Banach contraction principle, there is a unique set $A \in \mathcal{K}(X)$ which is invariant with respect to $\mathcal{S}$, that is

$$A = \mathcal{S}(A) = \bigcup_{n=1}^{N} \omega_n(A)\,.$$

The set $A \in \mathcal{K}(X)$ is called the *attractor* of IFS $(\omega_n)_{n=1}^N$.

Suppose that $(\omega_n)_{n=1}^N$ has the property that

$$\begin{aligned} a = e_1 &\quad \text{is the fixed point of } \omega_1\,, \\ b = e_N &\quad \text{is the fixed point of } \omega_N\,, \\ \omega_i(b) = \omega_{i+1} &\quad \text{if} \quad 1 \leq i \leq N-1\,. \end{aligned}$$

Fix $0 = t_1 < t_2 < \cdots < t_{N+1} = 1$. Define $g_i : [t_i, t_{i+1}] \to [0,1]$ for $1 \leq i \leq N$ by

$$g_i(x) = \frac{x - t_i}{t_{i+1} - t_i}\,.$$

Let

$$\mathcal{F} = \{f : [0,1] \longrightarrow X :\ f \text{ is continuous, and obeys } f(0) = a,\ f(1) = b\}\,.$$

Define $\mathcal{S}(f)$ for $f \in \mathcal{F}$ by

$$\mathcal{S}(f)(x) = \omega_i \circ f \circ g_i(x) \quad \text{for} \quad x \in [t_i, t_{i+1}],\quad 1 \leq i \leq N\,.$$

Theorem 1. *Under the above hypotheses, there is a unique $f \in \mathcal{F}$ such that $\mathcal{S}(f) = f$. Furthermore* $\mathrm{Im}(f) = A$.

1.3. **Countable iterated function systems.** In this section, we will present the compact set invariant with respect to a sequence of contraction maps (for details see [4]).

Suppose that (X, d) is a compact metric space.

A sequence of contractions $(\omega_n)_{n\geq 1}$ on X whose contraction ratios are, respectively, r_n, $r_n > 0$, such that $\sup_n r_n < 1$ is called a *countable iterated function system*, for simplicity CIFS.

Let $(\omega_n)_{n\geq 1}$ be a CIFS.

We define the set function $\mathcal{S} : \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$ by

$$\mathcal{S}(E) = \overline{\bigcup_{n\geq 1} \omega_n(E)}\,,$$

where the bar means the closure of the corresponding set. Then, $\mathcal{S}$ is a contraction map on $(\mathcal{K}(X), \delta)$ with contraction ratio $r \leq \sup_n r_n$. According to the Banach contraction principle, it follows that there exists a unique non-empty compact set $A \subset X$ which is invariant for the family $(\omega_n)_{n\geq 1}$, that is

$$A = \mathcal{S}(A) = \overline{\bigcup_{n\geq 1} \omega_n(A)}\,.$$

The set A is called *the attractor* of CIFS $(\omega_n)_{n\geq 1}$.

2. Parameterized curve

Let (X, d) be a compact and connected metric space and $(\omega_n)_{n\geq 1}$ be a sequence of contraction maps on X whose contraction ratios are, respectively, r_n, $r_n > 0$, such that $\sup\limits_n r_n < 1$ having the following properties:

a) $r_n \xrightarrow{n} 0$;
b) $(e_n)_n$ is a convergent sequence, we denote by $b = \lim\limits_n e_n$ (e_n is the unique fixed point of the contraction map ω_n, $n \in \mathbb{N}^*$);
c) if we denote by $a = e_1$, then $\omega_n(b) = \omega_{n+1}(a)$, $\forall\, n \geq 1$.

We note that there exists a sequence of contractions as above, this fact results from the example which is presented in the sequel.

We shall show that, in the above conditions, there exists a continuous function $h : [0,1] \longrightarrow X$ with $\mathrm{Im}(h) = A$, A being the attractor of CIFS $(\omega_n)_{n\geq 1}$, where we denote by $\mathrm{Im}h = h([0,1])$ the image of h.

We consider a sequence of real numbers $(t_n)_n$ such that

$$0 = t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots < 1 \quad \text{and} \quad \lim_n t_n = 1\,.$$

We define, for each $n \geq 1$, $g_n : [t_n, t_{n+1}] \longrightarrow [0,1]$,

$$g_n(x) = \frac{x - t_n}{t_{n+1} - t_n}\,.$$

We denote by

$$\mathcal{F} = \{f : [0,1] \longrightarrow X : \ f \text{ is continuous, and obeys } \ f(0) = a,\ f(1) = b\}$$

and by $\mathcal{P}$ the uniform metric on $\mathcal{F}$, $\mathcal{P}(f_1, f_2) = \sup\limits_{x\in[0,1]} \mathrm{d}(f_1(x), f_2(x))$.

It is a standard fact that $(\mathcal{F}, \mathcal{P})$ is a complete metric space.

For every $f \in \mathcal{F}$, we define $\mathcal{S}(f)$ by

$$\mathcal{S}(f)(x) = \begin{cases} \omega_n \circ f \circ g_n(x)\,, & x \in [t_n, t_{n+1}]\,, \quad n = 1, 2, \ldots \\ \quad b\,, & x = 1\,. \end{cases}$$

Proposition 1. *The application $\mathcal{S} : \mathcal{F} \longrightarrow \mathcal{F}$ is well-defined and $\mathcal{S}$ is a contraction map with respect to the metric $\mathcal{P}$.*

Proof. First we observe that $g_n(t_n) = 0$, $g_n(t_{n+1}) = 1$ for all n.

Next, if $x = t_{n+1}$, then

$$\begin{aligned} \omega_n \circ f \circ g_n(x) &= \omega_n(f(1)) = \omega_n(b) \overset{c)}{=} \omega_{n+1}(a) \\ &= \omega_{n+1} \circ f \circ g_{n+1}(t_{n+1}) = \omega_{n+1} \circ f \circ g_{n+1}(x)\,, \end{aligned}$$

thus $\mathcal{S}(f)$ is uniquely defined.

I *We shall show that $\mathcal{S}(f) \in \mathcal{F}$*

We consider $x_0 \in [0,1]$ and we will demonstrate that $\mathcal{S}(f)$ is continuous in x_0.

If $x_0 \in (t_n, t_{n+1})$, the assertion is obvious since $\mathcal{S}(f)$ is a composition of three continuous functions.

If $x_0 = t_n$, $n \geq 2$, then

$$\lim_{x \nearrow x_0} \mathcal{S}(f)(x) = \lim_{x \nearrow x_0} \omega_{n-1} \circ f \circ g_{n-1}(x) = \omega_{n-1}(f(1)) = \omega_{n-1}(b) \overset{c)}{=} \omega_n(a)\,,$$
$$\lim_{x \searrow x_0} \mathcal{S}(f)(x) = \mathcal{S}(f)(x_0) = \omega_n \circ f \circ g_n(t_n) = \omega_n(f(0)) = \omega_n(a)\,.$$

We suppose that $x_0 = 1$.

Let $(x_k)_k \subset [0,1]$, $x_k \nearrow 1$. For each $k \in \mathbb{N}^*$, there exists $n_k \in \mathbb{N}^*$ such that $x_k \in [t_{n_k}, t_{n_k+1}]$.

Let $\varepsilon > 0$. Then there exists $k_\varepsilon \in \mathbb{N}$, such that, for all $k \geq k_\varepsilon$, we have

$$r_{n_k} \cdot \operatorname{diam}(X) < \frac{\varepsilon}{2} \qquad \big(\text{by a)}\big)\,; \tag{1}$$

and

$$\mathrm{d}(e_{n_k}, b) < \frac{\varepsilon}{2}\,, \tag{2}$$

where $\operatorname{diam}(X) = \sup\limits_{x,y \in X} \mathrm{d}(x,y)$ is the diameter of the set X.

Thus, for all $k \geq k_\varepsilon$, one has

$$\begin{aligned}
\mathrm{d}(\mathcal{S}(f)(x_k), b) &= \mathrm{d}(\omega_{n_k} \circ f \circ g_{n_k}(x_k), b) \\
&\leq \mathrm{d}(\omega_{n_k} \circ f \circ g_{n_k}(x_k), \omega_{n_k}(e_{n_k})) + \mathrm{d}(\omega_{n_k}(e_{n_k}), b) \\
&\leq r_{n_k} \mathrm{d}(f(g_{n_k}(x_k)), e_{n_k}) + \mathrm{d}(e_{n_k}, b) \\
&\leq r_{n_k} \operatorname{diam}(X) + \mathrm{d}(e_{n_k}, b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.
\end{aligned}$$

It follows that $\mathcal{S}(f)(x_k) \longrightarrow b = \mathcal{S}(f)(1)$, hence $\mathcal{S}(f)$ is continuous.

It is clearly that $\mathcal{S}(f)(0) = a$, $\mathcal{S}(f)(1) = b$. Thus $\mathcal{S}(f) \in \mathcal{F}$.

II $\mathcal{S}$ *is a contraction map with respect to* $\mathcal{P}$*:*

Choose $f_1, f_2 \in \mathcal{F}$ and $x \in [0,1]$.

If $x = 1$, then, it is evident that $\mathrm{d}(\mathcal{S}(f_1)(x), \mathcal{S}(f_2)(x)) = \mathrm{d}(b,b) = 0$.

Assume that $x \in [t_n, t_{n+1}]$, $n \in \mathbb{N}^*$. Then

$$\begin{aligned}
\mathrm{d}(\mathcal{S}(f_1)(x), \mathcal{S}(f_2)(x)) &= \mathrm{d}(\omega_n \circ f_1 \circ g_n(x), \omega_n \circ f_2 \circ g_n(x)) \\
&\leq r_n \mathrm{d}(f_1(g_n(x)), f_2(g_n(x))) \leq r_n \mathcal{P}(f_1, f_2)\,.
\end{aligned}$$

Thus $\mathcal{P}(\mathcal{S}(f_1), \mathcal{S}(f_2)) \leq r\mathcal{P}(f_1, f_2)$, where $r = \sup\limits_n r_n < 1$ $\big(\text{by a)}\big)$. □

Theorem 2. *In the above context, there is a unique function* $h \in \mathcal{F}$ *such that* $\mathcal{S}(h) = h$*. Further* $\operatorname{Im}(h) = A$*.*

Proof. The existence and uniqueness result by the contraction principle.

The second assertion follows by equality

$$\operatorname{Im}\mathcal{S}(f) = \overline{\bigcup_{n \geq 1} \omega_n(\operatorname{Im} f)}\,, \quad \forall\, f \in \mathcal{F}\,. \tag{3}$$

We will demonstrate that equality by using the double inclusion.

"$\subset$"

Choose $y \in \mathrm{Im}\mathcal{S}(f)$. Then there exists $x \in [0,1]$ such that $\mathcal{S}(f)(x) = y$.

A. If $x = 1$, then $\mathcal{S}(f)(x) = \mathcal{S}(f)(1) = b = y$.

For each $b \in \mathrm{Im}f$, we have the relations:

$$\omega_n(b) \in \bigcup_{n=1}^{\infty} \omega_n(\mathrm{Im}f)\,, \quad \forall\, n \in \mathbb{N}\,;$$
$$\mathrm{d}(\omega_n(b), e_n) = \mathrm{d}(\omega_n(b), \omega_n(e_n)) \leq r_n \mathrm{d}(b, e_n)\,.$$

Hence

$$\mathrm{d}(\omega_n(b), b) \leq \mathrm{d}(\omega_n(b), e_n) + \mathrm{d}(e_n, b) \leq (r_n + 1)\mathrm{d}(b, e_n) \xrightarrow{n} 0$$

$\big($by using b)$\big)$.

It follows that $\omega_n(b) \longrightarrow b \in \overline{\bigcup_{n\geq 1} \omega_n(\mathrm{Im}f)}$. Thus

$$y \in \overline{\bigcup_{n\geq 1} \omega_n(\mathrm{Im}f)}\,.$$

B. If $x \in [0,1)$, then there exists $n \geq 1$ such that $x \in [t_n, t_{n+1}]$, hence $\omega_n \circ f \circ g_n(x) = y$. It follows that

$$y = \omega_n(f(g_n(x))) \in \omega_n(\mathrm{Im}f)$$

which implies $y \in \overline{\bigcup_{n\geq 1} \omega_n(\mathrm{Im}f)}$.

"$\supset$"

Choose $y \in \mathcal{S}(\mathrm{Im}f) = \overline{\bigcup_{n\geq 1} \omega_n(\mathrm{Im}f)}$.

Then there exists $(y_k)_k \subset \bigcup_{n=1}^{\infty} \omega_n(\mathrm{Im}f)$, $y_k \longrightarrow y$.

For every fixed $k \in \mathbb{N}^*$, one has:

$$\exists\, n_k \in \mathbb{N}^* \text{ such that } y_k \in \omega_{n_k}(\mathrm{Im}f) \text{ hence } z_k \in \mathrm{Im}f \text{ with } \ y_k = \omega_{n_k}(z_k)\,.$$

Thus there exists $x_k \in [0,1]$ such that

$$f(x_k) = z_k\,.$$

If $x_k = 1$, it follows that

$$z_k = f(x_k) = b = \mathcal{S}(f)(x_k) \in \mathrm{Im}\mathcal{S}(f)\,.$$

Assume that $x_k \in [0,1)$. Then there is $x'_k \in [t_{n_k}, t_{n_k+1}]$ such that

$$g_{n_k}(x'_k) = x_k\,.$$

We deduce that

$$\omega_{n_k} \circ f \circ g_{n_k}(x'_k) = \omega_{n_k}(f(x_k)) = \omega_{n_k}(z_k) = y_k \in \mathrm{Im}\mathcal{S}(f)\,.$$

Thus $(y_k)_k \subset \mathrm{Im}\mathcal{S}(f) = \mathcal{S}(f)([0,1])$, the set $\mathcal{S}(f)([0,1])$ being compact.

Thus $y = \lim\limits_k y_k \in \mathrm{Im}\mathcal{S}(f)$.

Since $h = \mathcal{S}(h)$ by using (3), it follows

$$\mathrm{Im}h = \mathrm{Im}\mathcal{S}(h) = \overline{\bigcup_{n\geq 1} \omega_n(\mathrm{Im}h)} = \mathcal{S}(\mathrm{Im}h)$$

and we conclude that $A = \mathrm{Im}h$ is the attractor of CIFS $(\omega_n)_{n\geq 1}$. $\square$

Example

We consider CIFS von-Koch-infinite given as follows (see [4]).

Let $X = [0,1] \times [0,1] \subset \mathbb{R}^2$ and we consider the contraction maps which are defined as follows: for every $n \in \mathbb{N}^*$, there exists an uniquely $p \in \{0, 1, \dots\}$, $k \in \{1, 2, 3, 4\}$ such that $n = 4p + k$. Then

$$\omega_n(x,y) := \begin{cases} \frac{1}{2^{p+1}}(\frac{1}{3}x + 2^{p+1} - 2, \frac{1}{3}y), & \text{if } k = 1; \\ \frac{1}{2^{p+1}}(\frac{1}{6}x - \frac{\sqrt{3}}{6}y + 2^{p+1} - \frac{5}{3}, \frac{\sqrt{3}}{6}x + \frac{1}{6}y), & \text{if } k = 2; \\ \frac{1}{2^{p+1}}(\frac{1}{6}x + \frac{\sqrt{3}}{6}y + 2^{p+1} - \frac{3}{2}, -\frac{\sqrt{3}}{6}x + \frac{1}{6}y + \frac{\sqrt{3}}{6}), & \text{if } k = 3; \\ \frac{1}{2^{p+1}}(\frac{1}{3}x + 2^{p+1} - \frac{4}{3}, \frac{1}{3}y), & \text{if } k = 4. \end{cases}$$

The attractor of that CIFS is the content of Fig. 1.

Now we shall show that CIFS von-Koch-infinite verifies the conditions a), b), c). Thus

a) Clearly $r_n \xrightarrow{n} 0$ (since $n \to \infty \Leftrightarrow p \to \infty$);

b) It is, also, immediate that

$$\forall\, x, y \in [0,1]\,, \quad \omega_n(x,y) \xrightarrow{n} (1,0)$$

thus $b = (1,0)$;

c) $\omega_1(x,y) = (\frac{1}{6}x, \frac{1}{6}y)$, hence $e_1 = a = (0,0)$.

We will prove that $\omega_n(b) = \omega_{n+1}(a)$, $\forall\, n \geq 1$.

If $p \geq 0$ and $k \in \{1,2,3\}$, it can prove, most difficulty, that

$$\omega_{4p+k}(1,0) = \omega_{4p+k+1}(0,0)\,.$$

Next, if $p \geq 0$ and $k = 4$, we have

$$\omega_{4p+4}(1,0) = \frac{1}{2^{p+1}}(\frac{1}{3} + 2^{p+1} - \frac{4}{3}, 0) = \frac{1}{2^{p+2}}(2^{p+2} - 2, 0) = \omega_{4(p+1)+1}(0,0)\,.$$

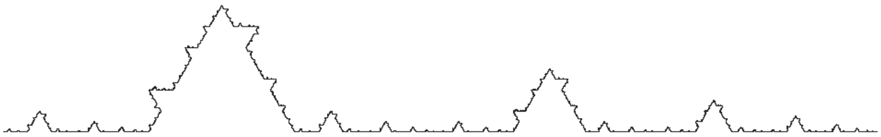

Fig. 1. The attractor von-Koch-infinite

References

[1] Barnsley, M. F., *Fractals everywhere*, Academic Press, Harcourt Brace Janovitch, 1988.

[2] Falconer, K. J., *The Geometry of Fractal Sets*, Cambridge University Press, **85**, 1985.

[3] Hutchinson, J., *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

[4] Secelean, N. A., *Countable Iterated Fuction Systems*, Far East J. Dyn. Syst., Pushpa Publishing House, vol. **3(2)** (2001), 149–167.

Department of Mathematics
"Lucian Blaga" University of Sibiu
Romania
E-mail: secelean@mail.ulbsibiu.ro