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ON OSCILLATION OF DIFFERENTIAL SYSTEMS OF NEUTRAL TYPE

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Abstract. We study oscillatory properties of solutions of the systems of differential equations of neutral type.

1. Introduction

In this paper we consider the neutral differential systems of the form

$$\text{(S)}\qquad \begin{aligned}&[y_1(t) - a(t)y_1(g(t))]' = p_1(t)y_2(t)\\ &\qquad\qquad y_2'(t) = p_2(t)f(y_1(h(t)))\,, \quad t \in R_+ = [0,\infty)\,.\end{aligned}$$

The following conditions are assumed to hold throughout this paper:

(a) $a : R_+ \to (0,\infty)$ is a continuous function;

(b) $g : R_+ \to R_+$ is a continuous and increasing function and $\lim\limits_{t\to\infty} g(t) = \infty$;

(c) $p_i : R_+ \to R_+$, $i = 1,2$ are continuous functions not identically equal to zero in every neighbourhood of infinity,

$$\int^\infty p_1(t)\,dt = \infty\,;$$

(d) $h : R_+ \to R_+$ is continuous and increasing function and $\lim\limits_{t\to\infty} h(t) = \infty$;

(e) $f : R \to R$ is a continuous function, $uf(u) > 0$ for $u \neq 0$,

and $|f(u)| \geq K|u|$, where $0 < K = \text{const}$.

Let $p_1(t) \equiv 1$ on R_+ and $f(u) = u$, $u \in R$. Then the system (S) is equivalent to the equation

$$\frac{d^2}{dt^2}[y_1(t) - a(t)y_1(g(t))] - p_2(t)y_1(h(t)) = 0\,, \quad t \in R_+\,.$$

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The oscillatory properties of the solutions of the equation

$$\frac{d^2}{dt^2}[y_1(t) - a(t)y_1(g(t))] + p_2(t)y_1(h(t)) = 0\,, \quad t \in R_+\,.$$

are studied in the paper [8].

The oscillatory theory of neutral differential systems have been studied for example in the papers [1-5], [7], [10,11] and in the references given therein. The more detailed list of publication of the presented topic is given in the monography [6], where the problem of existence of the solutions of neutral differential systems is also studied. The purpose of this paper is to establish some new criteria for the oscillation of the systems (S). Our results are new and extend and improve the know criteria for the oscillation of the differential systems of neutral type.

Let $t_0 \ge 0$. Denote

$$\tilde{t}_0 = \min\{t_0,\, g(t_0),\, h(t_0)\}\,.$$

A function $y = (y_1, y_2)$ is a solution of the system (S) if there exists a $t_0 \ge 0$ such that y is continuous on $[\tilde{t}_0, \infty)$, $y_1(t) - a(t)y_1(g(t))$, $y_2(t)$, are continuously differentiable on $[t_0, \infty)$ and y satisfies (S) on $[t_0, \infty)$.

Denote by W the set of all solutions $y = (y_1, y_2)$ of the system (S) which exist on some ray $[T_y, \infty) \subset R_+$ and satisfy

$$\sup\{|y_1(t)| + |y_2(t)| : t \ge T\} > 0 \quad \text{for any} \quad T \ge T_y\,.$$

A solution $y \in W$ is nonoscillatory if there exists a $T_y \ge 0$ such that its every component is different from zero for all $t \ge T_y$. Otherwise a solution $y \in W$ is said to be oscillatory.

Denote

$$P_1(t) = \int_0^t p_1(x)\,dx\,, \quad t \ge 0\,.$$

For any $y_1(t)$ we define $z_1(t)$ by

$$z_1(t) = y_1(t) - a(t)y_1(g(t))\,. \tag{1}$$

2. Some basic lemmas

The next Lemma 1 can be derived on the base of Lemma 1 in [5].

Lemma 1. *Let $y \in W$ be a solution of the system (S) with $y_1(t) \neq 0$ on $[t_0, \infty)$, $t_0 \ge 0$. Then y is nonoscillatory, $z_1(t)$, $y_2(t)$ are monotone on some ray $[T, \infty)$, $T \ge t_0$ and $z_1(t) \neq 0$ on $[T, \infty)$.*

Lemma 2 [9, Lemma 2]**.** *In addition to the conditions* (a) *and* (b) *suppose that*

$$1 \le a(t) \quad \textit{for} \quad t \ge 0 .$$

Let $y_1(t)$ *be a continuous nonoscillatory solution of the functional inequality*

$$y_1(t)[y_1(t) - a(t)y_1(g(t))] > 0$$

defined in a neighbourhood of infinity. Suppose that $g(t) > t$ *for* $t \ge 0$*. Then* $y_1(t)$ *is bounded.*

Lemma 3 [9, Lemma 3]**.** *Assume that*

$$q : R_+ \to R_+ , \quad \delta : R_+ \to R \quad \textit{are continuous functions,} \quad \lim_{t\to\infty} \delta(t) = \infty$$

and

$$\delta(t) < t \quad \textit{for} \quad t \ge 0 , \qquad \liminf_{t\to\infty} \int_{\delta(t)}^{t} q(s)\, ds > \frac{1}{e} .$$

Then the functional inequality

$$x'(t) + q(t)x(\delta(t)) \le 0 , \quad t \ge 0$$

cannot have an eventually positive solution and

$$x'(t) + q(t)x(\delta(t)) \ge 0 , \quad t \ge 0$$

cannot have an eventually negative solution.

3. Oscillation theorems

In this section we shall study the oscillation of the solutions of the system (S). In the next theorems $g^{-1}(t)$ and $h^{-1}(t)$ will denote the inverse functions of $g(t)$, $h(t)$ and $\alpha : R_+ \to R$ is a continuous function.

Theorem 1. *Suppose that*

$$h(t) \le g(t) , \quad t < \alpha(t) , \quad h(\alpha(t)) < t \quad \textit{for} \quad t \ge 0$$

and

$$\liminf_{t\to\infty} \int_{h(\alpha(t))}^{t} Kp_1(s) \int_{s}^{\alpha(s)} p_2(v)\, dv\, ds > \frac{1}{e} , \tag{2}$$

$$\int^{\infty} \frac{p_2(s)\, ds}{a(g^{-1}(h(s)))} < \infty , \quad \limsup_{t\to\infty} \left\{ KP_1(t) \int_{h^{-1}(g(t))}^{\infty} \frac{p_2(s)\, ds}{a(g^{-1}(h(s)))} \right\} > 1 . \tag{3}$$

Then every solution $y \in W$ of (S) *with $y_1(t)$ bounded is oscillatory.*

Proof. Let $y = (y_1, y_2) \in W$ be a nonoscillatory solution of (S) with $y_1(t)$ bounded. Without loss of generality we may suppose that $y_1(t)$ is positive and bounded for $t \geq t_0$. From the second equation of (S), (c), (d), (e) we get

$$y_2'(t) \geq 0 \quad \text{for sufficiently large} \quad t_1 \geq t_0 \,.$$

In view of Lemma 1 we have two cases for sufficiently large $t_2 \geq t_1$:
1) $y_2(t) > 0$, $t \geq t_2$;
2) $y_2(t) < 0$, $t \geq t_2$.

Case 1. Because $y_2(t)$ is positive and nondecreasing we have

$$y_2(t) \geq L \,, \quad t \geq t_2 \,, \quad 0 < L - \text{const.} \tag{4}$$

Integrating the first equation of (S) from t_2 to t and using (1) and (4) we get

$$z_1(t) - z_1(t_2) \geq L \int_{t_2}^{t} p_1(s)\, ds \,, \quad t \geq t_2 \,. \tag{5}$$

From (5) and (c) we have $\lim\limits_{t\to\infty} z_1(t) = \infty$. From (1) we have

$$z_1(t) < y_1(t) \,, \quad t \geq t_2$$

and this contradicts the fact that $y_1(t)$ is bounded. The Case 1 cannot occur.

Case 2. We can consider two possibilities.

(A) Let $z_1(t) > 0$ for $t \geq t_3$, where $t_3 \geq t_2$ is sufficiently large. We have $z_1(t) < y_1(t)$ and using (e) we get

$$p_2(t) z_1(h(t)) \leq \frac{p_2(t) f(y_1(h(t)))}{K} \,, \quad t \geq t_4 \,,$$

where $t_4 \geq t_3$ is sufficiently large.

Integrating the second equation of (S) from t to $\alpha(t)$ and then using the last inequality and $y_2(\alpha(t)) < 0$ we obtain

$$-y_2(t) \geq K \int_{t}^{\alpha(t)} p_2(s) z_1(h(s))\, ds \,, \quad t \geq t_4 \,.$$

Multiplying the last inequality by $p_1(t)$ and then using the monotonicity of $z_1(t)$ we have

$$z_1'(t) + \left(K p_1(t) \int_{t}^{\alpha(t)} p_2(s)\, ds \right) z_1(h(\alpha(t))) \leq 0 \,, \quad t \geq t_4 \,. \tag{6}$$

By condition (2) and Lemma 3 the inequality (6) cannot have an eventually positive solution. This is a contradiction.

(B) Let $z_1(t) < 0$ for $t \geq t_3$. From (1) and (e) we have

$$z_1(t) > -a(t)y_1(g(t)), \quad t \geq t_3$$

and

$$-\frac{Kp_2(t)z_1(g^{-1}(h(t)))}{a(g^{-1}(h(t)))} \leq Kp_2(t)y_1(h(t)) \leq p_2(t)f(y_1(h(t))), \quad t \geq t_4, \tag{7}$$

where $t_4 \geq t_3$ is sufficiently large.

In view of the second equation of (S) inequality (7) implies

$$y_2'(t) + \frac{Kp_2(t)z_1(g^{-1}(h(t)))}{a(g^{-1}(h(t)))} \geq 0, \quad t \geq t_4. \tag{8}$$

Integrating (8) from t to $t^\star$ and then letting $t^\star \to \infty$ we get

$$y_2(t) \leq \int_t^\infty \frac{Kp_2(s)z_1(g^{-1}(h(s)))\,ds}{a(g^{-1}(h(s)))}, \quad t \geq t_4. \tag{9}$$

With regard to (3) we get

$$\frac{1}{K} < \limsup_{t\to\infty} \left\{ P_1(t) \int_{h^{-1}(g(t))}^\infty \frac{p_2(s)\,ds}{a(g^{-1}(h(s)))} \right\} \leq \limsup_{t\to\infty} \int_t^\infty \frac{P_1(s)p_2(s)\,ds}{a(g^{-1}(h(s)))}. \tag{10}$$

We claim that the condition (3) implies

$$\int_T^\infty \frac{P_1(s)p_2(s)\,ds}{a(g^{-1}(h(s)))} = \infty, \quad T \geq 0. \tag{11}$$

Otherwise if

$$\int_T^\infty \frac{P_1(s)p_2(s)\,ds}{a(g^{-1}(h(s)))} < \infty,$$

we can choose $T_1 \geq T$ such large that

$$\int_{T_1}^\infty \frac{P_1(s)p_2(s)\,ds}{a(g^{-1}(h(s)))} < \frac{1}{K},$$

which is a contradiction with (10).

Integrating $\int_T^t P_1(s)y_2'(s)\,ds$ by parts we have

$$\int_T^t P_1(s)y_2'(s)\,ds = P_1(t)y_2(t) - P_1(T)y_2(T) - z_1(t) + z_1(T)\,. \tag{12}$$

In this case

$$z_1(t) \le -M\,, \quad 0 < M - \text{const.} \tag{13}$$

Using the second equation of (S), (7) and (13) from (12) we get

$$\begin{aligned}\int_T^t P_1(s)y_2'(s)\,ds &= \int_T^t P_1(s)p_2(s)f(y_1(h(s)))\,ds\\ &\ge KM\int_T^t \frac{P_1(s)p_2(s)\,ds}{a(g^{-1}(h(s)))}\,, \quad t \ge T \ge t_4\,.\end{aligned}$$

The last inequality togethet with (12) implies

$$MK\int_T^t \frac{P_1(s)p_2(s)\,ds}{a(g^{-1}(h(s)))} \le P_1(t)y_2(t) - P_1(T)y_2(T) - z_1(t) + z_1(T)\,, \quad t \ge T \ge t_4\,. \tag{14}$$

Combining (11) with (14) we get $\lim\limits_{t\to\infty}(P_1(t)y_2(t) - z_1(t)) = \infty$ and

$$-z_1(t) \ge -P_1(t)y_2(t)\,, \quad t \ge t_5\,, \quad \text{where} \quad t_5 \ge t_4 \quad \text{is sufficiently large.}$$

The last inequality together with (9) and the monotonicity of $z_1(t)$ implies

$$\begin{aligned}-z_1(t) &\ge -KP_1(t)\int_t^\infty \frac{p_2(s)z_1(g^{-1}(h(s)))\,ds}{a(g^{-1}(h(s)))}\\ &\ge -KP_1(t)z_1(t)\int_{h^{-1}(g(t))}^\infty \frac{p_2(s)\,ds}{a(g^{-1}(h(s)))}\,, \quad t \ge T \ge t_5\end{aligned}$$

and

$$1 \ge KP_1(t)\int_{h^{-1}(g(t))}^\infty \frac{p_2(s)\,ds}{a(g^{-1}(h(s)))}\,, \quad t \ge t_5\,,$$

which contradicts (3). This case cannot occur. The proof is complete. □

Theorem 2. *Suppose that*

$$1 \le a(t)\,, \quad t < g(t)\,, \quad t < \alpha(t)\,, \quad h(\alpha(t)) < t \quad \textit{for} \quad t \ge 0$$

and the conditions (2), (3) *are satisfied. Then all solutions of* (S) *are oscillatory.*

Proof. Let $y = (y_1, y_2) \in W$ be a nonoscillatory solution of (S). Without loss of generality we may suppose that $y_1(t)$ is positive for $t \ge t_0$. As in the proof of Theorem 1 we get two cases — Case 1 and Case 2.

Case 1. Analogously as in the Case 1 of the proof of Theorem 1 we can show that $\lim\limits_{t\to\infty} z_1(t) = \infty$. By Lemma 2 $y_1(t)$ is bounded and from (1) $z_1(t) < y_1(t)$ for sufficiently large t. Then $z_1(t)$ is bounded, which is a contradiction. The Case 1 cannot occur.

Case 2. We can treat this case in the same way as in the proof of Theorem 1 we only remind that $h(t) < g(t)$ follows from the above conditions. The proof is complete. □

Theorem 3. *Suppose that*

$$t < g(t)\,, \quad t < \alpha(t)\,, \quad h(\alpha(t)) < t\,, \quad t < g(h(t)) \quad \textit{for} \;\; t \ge 0\,,$$

$$\limsup_{t\to\infty} \int\limits_{h^{-1}(g^{-1}(t))}^{t} K(P_1(t) - P_1(s))p_2(s)a(h(s))\,ds > 1\,, \tag{15}$$

and conditions (2) *and* (3) *hold. Then all solutions of* (S) *are oscillatory.*

Proof. Let $y = (y_1, y_2) \in W$ be a nonoscillatory solution of (S). Without loss of generality we may suppose that $y_1(t)$ is positive for $t \ge t_0$. As in the proof of Theorem 1 we get two cases — Case 1 and Case 2.

Case 1. In this case

$$y_1(t) > a(t)y_1(g(t))\,, \qquad y_1(t) > z_1(t)\,,$$
$$y_1(h(t)) > a(h(t))y_1(g(h(t))) > a(h(t))z_1(g(h(t)))$$

and

$$p_2(t)f(y_1(h(t))) \ge Kp_2(t)y_1(h(t)) > Kp_2(t)a(h(t))z_1(g(h(t)))\,, \tag{16}$$

for $t \ge t_3$, where $t_3 \ge t_2$ is sufficiently large.

Combining the integral identity

$$z_1(t) = z_1(\xi) + (P_1(t) - P_1(\xi))y_2(\xi) + \int\limits_{\xi}^{t} (P_1(t) - P_1(s))y_2'(s)\,ds$$

with (16) we get

$$z_1(t) \geq \int_{\xi}^{t} K(P_1(t) - P_1(s))p_2(s)a(h(s))z_1(g(h(s)))\, ds\,, \quad t > \xi \geq t_3\,.$$

Putting $\xi = h^{-1}(g^{-1}(t))$ and using the monotonicity of $z_1(t)$ from the last inequality we get

$$1 \geq \int_{h^{-1}(g^{-1}(t))}^{t} K(P_1(t) - P_1(s))p_2(s)a(h(s))\, ds\,,$$

which contradicts the condition (15).

Case 2. We can treat this case in the same way as in the proof of Theorem 1. The proof is complete. □

Remark 1. Theorems 1-3 remain true if we change the condition (3) by the condition

$$(3') \qquad \int^{\infty} \frac{p_2(s)\, ds}{a(g^{-1}(h(s)))} = \infty$$

because the conditions (3′) implies (11).

Example 1. We consider the system

$$(17) \qquad \begin{aligned} \left[y_1(t) - \frac{1}{4}\, y_1(8t)\right]' &= t\, y_2(t) \\ y_2'(t) &= \frac{c}{t^3}\, y_1\left(\frac{t}{4}\right), \quad t > 0\,, \end{aligned}$$

where c is a positive constant. In this example $a(t) = \dfrac{1}{4}$, $g(t) = 8t$, $p_1(t) = t$, $P_1(t) = \dfrac{t^2}{2}$, $p_2(t) = \dfrac{c}{t^3}$, $h(t) = \dfrac{t}{4}$, $f(t) = t$ and $K = 1$. We choose $\alpha(t) = 2t$ and calculate the conditions (2), (3) and (15) as follows

$$\liminf_{t\to\infty} \int_{\frac{t}{2}}^{t} s \int_{s}^{2s} \frac{c}{v^3}\, dv\, ds = \frac{3\, c\, \ln 2}{8}\,,$$

$$\limsup_{t\to\infty} \left\{ \frac{t^2}{2} \int_{32t}^{\infty} \frac{4c\, ds}{s^3} \right\} = \frac{c}{1024}\,,$$

$$\limsup_{t\to\infty} \int_{\frac{t}{2}}^{t} \left(\frac{t^2}{2} - \frac{s^2}{2}\right) \frac{c\, ds}{4s^3} = \frac{c}{8}\left(\frac{3}{2} - \ln 2\right)\,.$$

For $c > 1024$ all conditions of Theorem 3 are satisfies and so all solutions of (17) are oscillatory.

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