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FINITENESS OF A CLASS OF RABINOWITSCH POLYNOMIALS

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Abstract. We prove that there are only finitely many positive integers m such that there is some integer t such that $|n^2 + n - m|$ is 1 or a prime for all $n \in [t + 1, t + \sqrt{m}]$, thus solving a problem of Byeon and Stark.

In 1913, G. Rabinowitsch [4] proved that for any positive integer m with squarefree $4m-1$, the class number of $\mathbb{Q}(\sqrt{1-4m})$ is 1 if and only if n^2+n+m is prime for all integers $0 \leq n \leq m-3$. Recently, D. Byeon and H. M. Stark [1] proved an analogue statement for real quadratic fields. The polynomial $f_m(x) = x^2 + x - m$ is called a Rabinowitsch polynomial, if there is some integer t such that $|f_m(n)|$ is 1 or a prime for all integral $n \in [t+1, t+\sqrt{m}]$. They proved the following theorem:

Theorem 1.
1. *If f_m is Rabinowitsch, then one of the following equations hold: $m = 1$, $m = 2$, $m = p^2$ for some odd prime p, $m = t^2 + t \pm 1$, or $m = t^2 + t \pm \frac{2t+1}{3}$, where $\frac{2t+1}{3}$ is an odd prime.*
2. *If f_m is Rabinowitsch, then $\mathbb{Q}(\sqrt{4m+1})$ has class number 1.*
3. *There are only finitely many m such that $4m+1$ is squarefree and that f_m is Rabinowitsch.*

They asked whether the finiteness of m holds without the assumption on $4m+1$. It is the aim of this note to show that this is indeed the case.

Theorem 2. *There are only finitely many $m \geq 0$ such that f_m is Rabinowitsch.*

For the proof write $4m + 1 = u^2 D$ with D squarefree and u a positive integer. We distinguish three cases, namely $D = 1$, $1 < D < m^{1/12}$ and $D \geq m^{1/12}$, and formulate each as a seperate lemma. The first two cases are solved elementary, while the last one requires a slight extension of the argument in the case $4m + 1$ squarefree given by Byeon and Stark.

Lemma 1. *If f_m is Rabinowitsch and $D = 1$, then $m = 2$.*

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Proof. We only deal with the case $m = t^2 + t + \frac{2t+1}{3}$, the other cases are similar. Assume that $D = 1$, that is $4t^2 + \frac{20t}{3} + \frac{7}{3} = u^2$. We have

$$4t^2 + 4t + 1 < 4t^2 + \frac{20t}{3} + \frac{7}{3} < 4t^2 + 8t + 4$$

that is, $2t + 1 < u < 2t + 2$, which is impossible for integral t and u. □

Lemma 2. *There are only finitely many m such that f_m is Rabinowitsch and $1 < D < m^{1/12}$.*

Proof. Let p be the least prime with $p \equiv 1 \pmod{4D}$ and $(p, m) = 1$. By Linnik's theorem, we have $p < D^C$ for some absolute constant C, moreover, for D sufficiently large we may take $C = 5.5$, as shown by D. R. Heath-Brown [3]. Hence, there is some constant D_0 such that for $D > D_0$ we have $p < m^{1/2}/6$. By construction of p, in any interval of length p there is some n such that $x - \frac{1+u\sqrt{D}}{2}$ is not coprime to p, i.e. such that p divides $n^2 + n - m$. If f_m is Rabinowitsch, this implies $f_m(n) = \pm p$, since f_m is of degree 2, this cannot happen but for 4 values of n. However, since $p < m^{1/2}/6$, in every interval of length $m^{1/2}$, there are at least five such values of n, hence, f_m is not Rabinowitsch.

Finally we choose a prime number $p_D \equiv 1 \pmod{4D}$ for each $D \leq D_0$, and for $m > 6 \max p_D$ we argue as above. □

Lemma 3. *There are only finitely many m such that f_m is Rabinowitch and that $D \geq m^{1/12}$.*

Proof. We may neglect the case $m = 2$. In each of the other cases, there exists a unit ϵ_m in $\mathbb{Q}(\sqrt{D})$ with $1 < |\epsilon_m| \ll m$, more precisely, such a unit is given by

$$\begin{array}{lcl} m = t^2 & : & \epsilon_m = 2t + \sqrt{4m+1} \\ m = t^2 + t \pm 1 & : & \epsilon_m = \dfrac{2t + 1 + \sqrt{4m+1}}{2} \\ m = t^2 + t \pm \dfrac{2t+1}{3} & : & \epsilon_m = \dfrac{6t + 3 \pm 2 + 3\sqrt{4m+1}}{2} \end{array}$$

Let $\epsilon_D > 1$ be the fundamental unit of $\mathbb{Q}(\sqrt{D})$. Since the group of positive units in $\mathbb{Q}(\sqrt{D})$ is free abelian of rank 1, there is some k such that $\epsilon_m = \epsilon_D^k$, hence we have $\epsilon_D < m$. By the Siegel-Brauer-theorem we have $\log(h(\mathbb{Q}(\sqrt{D})) \log|\epsilon_D|) \sim \log\sqrt{D}$. If f_m is Rabinowitch, then $h(\mathbb{Q}(\sqrt{D})) = 1$, and by assumption we have

$$\log|\epsilon_D| \leq \log|\epsilon_m| < \log m \leq 12 \log D,$$

hence we obtain the inequality

$$12 \log D > D^{1/2+o(1)}$$

which can only be true for finitely many D. Since $m \leq D^{12}$, there are only finitely many m, and our claim follows. □

Note that Lemma 1 and Lemma 2 are effective, while Lemma 3 depends on a bound for Siegel's zero. However, one can deduce that there is an effective constant m_0, such that there exists at most one $m > m_0$ such that f_m is Rabinowitsch.

Note added in proof. In the mean time, D. Byeon and H. M. Stark [2] also obtained a proof of Theorem 1, moreover, they determined all Rabinowitsch polynomials up to at most one exception. The same result has also been obtained independently by S. Louboutin.

References

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