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# ALMOST $Q$-RINGS

C. JAYARAM

Abstract. In this paper we establish some new characterizations for $Q$-rings and Noetherian $Q$-rings.

## 1. Introduction

Throughout this paper $R$ is assumed to be a commutative ring with identity. $L(R)$ denotes the lattice of all ideals of $R$. $R$ is said to be a $Q$-ring [4], if every ideal is a finite product of primary ideals. It is well known that if $R$ is a $Q$-ring, then $R_M$ is a $Q$-ring for every maximal ideal $M$ of $R$ [4, Lemma 4]. But in general the converse need not be true. For example, if $R$ is an almost Dedekind domain which is not a Dedekind domain, then $R_M$ is a $Q$-ring, for every maximal ideal $M$ of $R$, but $R$ is not a $Q$-ring. We call a ring $R$ an almost $Q$-ring if $R_M$ is a $Q$-ring, for every maximal ideal $M$ of $R$. The goal of this paper is to characterize those almost $Q$-rings which are also $Q$-rings. We prove that $R$ is an almost $Q$-ring if and only if every non-maximal prime ideal is locally principal (see Theorem 1). Using this result, we characterize $Q$-rings in terms of almost $Q$-rings (see Theorem 2). Finally, we establish some equivalent conditions for Noetherian $Q$-rings (see Theorem 3).

For any $A, B \in L(R)$, we denote $A \setminus B = \{x \in A \mid x \notin B\}$. We use $\subset$ for proper set containment. For any $x \in R$, the principal ideal generated by $x$ is denoted by $(x)$. For any ideal $I \in L(R)$, we denote $\theta(I) = \sum\{(I_1 : I) \mid I_1 \subseteq I$ and $I_1$ is a finitely generated ideal$\}$. Recall that an ideal $I$ of $R$ is called a *multiplication ideal* if for every ideal $J \subseteq I$, there exists an ideal $K$ with $J = KI$. If $I$ is a multiplication ideal, then $I$ is locally principal [1, Theorem 1 and Page 761]. An ideal $M$ of $R$ is called a *quasi-principal ideal* [9, Exercise 10, Page 147] (or a principal element of $L(R)$ [11]) if it satisfies the following identities (i) $(A \cap (B : M))M = AM \cap B$ and (ii) $(A + BM) : M = (A : M) + B$, for all $A$, $B \in L(R)$. Obviously, every quasi-principal ideal is a multiplication ideal. It should be mentioned that every quasi-principal ideal is finitely generated and also a finite product of quasi-principal ideals of $R$ is again a quasi-principal ideal [9, Exercise 10, Page 147]. In fact, an ideal $I$ of $R$ is quasi-principal if and only if it is finitely generated and locally

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principal (see [6, Theorem 4]) or [11, Theorem 2]). A $B_w$-prime of $I$ is a prime ideal $P$ such that $P$ is minimal over $(I : x)$ for some $x \in R$. $R$ is said to be a *Laskerian ring* [8], if every ideal is a finite intersection of primary ideals. It is well known that $R$ is a $Q$-ring if and only if $R$ is a Laskerian ring in which every non-maximal prime ideal is quasi-principal [4, Theorem 13]. $R$ is a $\pi$-ring if every principal ideal is a finite product of prime ideals. We say that $R$ has Noetherian spectrum, if $R$ satisfies the ascending chain condition for radical ideals [12]. It is well known that $R$ has Noetherian spectrum if and only if every prime ideal is the radical of a finitely generated ideal [12, Corollary 2.4]. Also it is well known that if $R$ has Noetherian spectrum, then every ideal has only finitely many minimal primes.

For general background and terminology, the reader is referred to [9].

We shall begin with the following definition.

**Definition 1.** A quasi-local ring $R$ with maximal ideal $M$ is said to satisfy the condition $(*)$ if for each non-maximal prime ideal $P$ with $P = PM$, there exists $t \in M$ such that $P + (t)$ is finitely generated.

Note that valuation rings (i.e., any two ideals are comparable), quasi-local rings in which the maximal ideals are principal and one dimensional quasi-local domains are examples of quasi-local rings satisfying the condition $(*)$.

**Lemma 1.** *Let $R$ be a quasi-local $Q$-ring with maximal ideal $M$. Then $R$ satisfies the condition $(*)$.*

**Proof.** The proof of the lemma follows from [4, Lemma 5]. $\square$

**Lemma 2.** *Let $R$ be a quasi-local ring with maximal ideal $M$ satisfying the condition $(*)$. Suppose every principal ideal is a finite product of primary ideals. If $P$ is a non-maximal prime ideal with $P = PM$, then $P = (0)$.*

**Proof.** Suppose $P$ is a non-maximal prime ideal with $P = PM$. By hypothesis, there exists $a \in M$ such that $P + (a)$ is finitely generated. If $a \in P$, then $P = P + (a)$ is finitely generated, so by Nakayama's lemma, $P = 0$. Suppose $a \notin P$. Since $P+(a)$ is finitely generated, it follows that $P+(a) = P_1+(a)$ for some finitely generated ideal $P_1 \subseteq P$. Since $P = PM$, we have $(P + (a))M = PM + (a)M = P_1M + (a)M$, so $P + (a)M = P_1M + (a)M$ and hence $P + (a) = P_1M + (a)$. Again since $P_1 \subseteq P + (a) = (a) + P_1M$ and $P_1$ is finitely generated, by Nakayama's lemma, it follows that $P_1 \subset (a)$. Therefore $P \subset (a)$. Let $x \in P$. By hypothesis $(x) = QA$ for some primary ideal $Q \subseteq P$ and $A \in L(R)$. Since $Q \subset (a)$, it follows that $Q = (a)Q$. Therefore $(x) = QA = Q(a)A = (x)(a)$ and hence by Nakayama's lemma, $(x) = (0)$. This shows that $P = (0)$. $\square$

**Lemma 3.** *Let $R$ be a quasi-local ring with maximal ideal $M$. Suppose every ideal generated by two elements is a finite product of primary ideals. If $P$ is a non-maximal prime ideal with $P \neq PM$, then $P$ is principal.*

**Proof.** Let $P$ be a non-maximal prime ideal with $P \neq PM$. Choose any element $a \in P$ such that $a \notin PM$. Let $t \in M$ be any element such that $t \notin P$. Suppose

$x \in P$. Then by hypothesis, $(a) + (xt)$ is a finite product of primary ideals. Since $a \notin PM$, it follows that $(a) + (xt)$ is primary. Again since $(xt) \subseteq (a) + (xt)$ and $t \notin \sqrt{(a) + (xt)} \subseteq P$, it follows that $x \in (a) + (xt)$, so by Nakayama's lemma $(x) \subseteq (a)$. Therefore $P = (a)$. □

**Lemma 4.** *Let $R$ be a quasi-local ring with maximal ideal $M$ satisfying the condition $(*)$. Suppose every ideal generated by two elements is a finite product of primary ideals. Then the non-maximal prime ideals are principal. Hence* $\dim R \leq 2$.

**Proof.** By Lemma 2 and Lemma 3, every non-maximal prime ideal is principal. Again as shown in the last paragraph of the proof of Lemma 5 of [4], $\dim R \leq 2$. This completes the proof of the lemma. □

**Lemma 5.** *Suppose $I$ is an ideal of $R$ such that every prime minimal over $I$ is finitely generated. Then $I$ contains a finite product of prime ideals minimal over $I$. Further $I$ has only finitely many minimal primes.*

**Proof.** Suppose $I$ does not contain a finite product of prime ideals minimal over $I$. Let $\Im = \{J \in L(R) \mid I \subseteq J$ and $J$ does not contain a finite product of prime ideals minimal over $I\}$. By Zorn's lemma, $\Im$ has a maximal element, say $P$. It can be easily shown that $P$ is a prime ideal. Again note that $P$ contains a prime ideal $P_0$ which is minimal over $I$, a contradiction. Therefore $I$ contains a finite product of prime ideals minimal over $I$. Consequently, $I$ has only finitely many minimal primes. □

**Lemma 6.** *Suppose $R$ is a quasi-local ring. Then the following statements are equivalent:*

(i) *$R$ is a $Q$-ring.*
(ii) *$R$ satisfies the condition $(*)$ and every ideal generated by two elements is a finite product of primary ideals.*
(iii) *Every non-maximal prime ideal is principal.*

**Proof.** (i)⇒(ii) follows from Lemma 1.

(ii)⇒(iii) follows from Lemma 4.

(iii)⇒(i). Suppose (iii) holds. Then every ideal $I$ is either $M$-primary ($M$ is a maximal ideal of $R$) or by Lemma 5, $I$ has only finitely many minimal primes. Again by the last paragraph of the proof of [4, Lemma 5], $R$ is Laskerian. Now the result follows from [4, Theorem 10]. □

**Lemma 7.** *Let $R$ be an almost $Q$-ring. Suppose every principal ideal is a finite product of primary ideals. Then every non-maximal prime ideal of $R$ is a multiplication ideal.*

**Proof.** Using Lemma 6 and by imitating the proof of [4, Lemma 7], we can get the result. □

**Lemma 8.** *Let* $\dim R \leq 2$ *and let every ideal generated by two elements has only finitely many minimal primes. Then $R$ has Noetherian spectrum.*

**Proof.** First we show that every minimal prime ideal is the radical of a finitely generated ideal. By hypothesis, $R$ has only finitely many minimal primes. Let $P_1, P_2, \ldots, P_n$ be the distinct minimal prime ideals. If $n = 1$, then $P_1$ is the radical of the zero ideal. Suppose $n > 1$. Then $P_1 \not\subseteq \bigcup_{i=2}^{n} P_i$. Choose any $x \in P_1$ such that $x \notin \bigcup_{i=2}^{n} P_i$. Let $Q_1, Q_2, \ldots, Q_m$ be the distinct primes minimal over $(x)$. Then $P_1 = Q_j$ for some $j$, say $P_1 = Q_1$. If $m = 1$, then $P_1$ is the radical of $(x)$. Suppose $m > 1$. Then $P_1 \not\subseteq \bigcup_{i=2}^{m} Q_i$. Choose any $y \in P_1$ such that $y \notin \bigcup_{i=2}^{m} Q_i$. By hypothesis, $(x) + (y)$ has only finitely many minimal primes. Let $Q'_1, Q'_2, \ldots, Q'_k$ be the distinct primes minimal over $(x) + (y)$. Note that $P_1 = Q'_j$ for some $j$, say $P_1 = Q'_1$. If $k = 1$, then $P_1$ is the radical of $(x) + (y)$. Suppose $k > 1$. Observe that any $Q'_j$ different from $P_1$ contains $Q_i$ properly, for some $i \neq 1$, and each $Q_i$ different from $P_1$, is non-minimal. So each $Q'_j$ is maximal, for $j = 2, 3, \ldots, k$. Choose any element $z \in P_1$ such that $z \notin \bigcup_{i=2}^{k} Q'_i$. Now it can be easily shown that $P_1$ is the radical of $(x) + (y) + (z)$. Thus we have shown that every minimal prime ideal is the radical of a finitely generated ideal.

Next we show that every non-minimal prime ideal is the radical of a finitely generated ideal. Let $P$ be a non-minimal prime ideal. Then $P \not\subseteq \bigcup_{i=1}^{n} P_i$. Choose any $x \in P$ such that $x \notin \bigcup_{i=1}^{n} P_i$. Let $Q_1, Q_2, \ldots, Q_m$ be the distinct primes minimal over $(x)$. Then $P \supseteq Q_j$ for some $j$, say $P \supseteq Q_1$. If $m = 1$ and $P = Q_1$, then $P$ is the radical of $(x)$ and so we are through. Suppose $m \geq 1$ and $Q_1 \subset P$. Then $P \not\subseteq \bigcup_{i=1}^{m} Q_i$. Choose any $y \in P$ such that $y \notin \bigcup_{i=1}^{m} Q_i$. Then $(x) + (y)$ has only finitely many minimal primes and every prime minimal over $(x) + (y)$ is a maximal ideal. Therefore there exists a finitely generated ideal $I$ such that $P$ is the radical of $I$. Finally assume that $m > 1$ and $P = Q_1$. Then $P \not\subseteq \bigcup_{i=2}^{m} Q_i$. Choose any $y \in P$ such that $y \notin \bigcup_{i=2}^{m} Q_i$. Let $Q'_1, Q'_2, \ldots, Q'_k$ be the distinct primes minimal over $(x) + (y)$. Note that $P_1 \supseteq Q'_j$ for some $j$, say $P_1 \supseteq Q'_1$. Since $x \in Q'_1$ and $Q_1 = P \supseteq Q'_1$, it follows that $P = Q_1 = Q'_1$. If $k = 1$, then $P_1$ is the radical of $(x) + (y)$. Suppose $k > 1$. Then $P \not\subseteq \bigcup_{i=2}^{k} Q'_i$ and each $Q'_i$ different from $P$, is maximal. Choose any element $z \in P$ such that $z \notin \bigcup_{i=2}^{k} Q'_i$. Then $P$ is the radical of $(x) + (y) + (z)$. Thus every prime ideal is the radical of a finitely generated ideal and hence $R$ has Noetherian spectrum. □

For any $I \in L(R)$ and for any prime ideal $P$ minimal over $I$, we denote $P_I = \cap\{Q \in L(R) \mid Q$ is a $P$-primary ideal containing $I\}$. It can be easily seen that $P_I$ is the smallest $P$-primary ideal containing $I$. For any $x \in R$, and for any prime ideal $P$ minimal over $(x)$, we denote $P_x = \cap\{Q \in L(R) \mid Q$ is a $P$-primary ideal containing $(x)\}$.

For any $x \in R$, we denote $(x)^* = \cap\{P_x \mid P$ is a prime ideal minimal over $(x)\}$.

**Lemma 9.** *Let $P$ be a prime minimal over an ideal $I$ of $R$ and let $P_1$ be a prime properly containing $P$. Then the following statements hold:*

(i) *If $P$ is a multiplication ideal, then $P \subset ((I + PP_I) : P_I)$.*
(ii) *If $P_1$ is a $B_w$-prime of $I$, then $(P_I)_M \neq I_M$ (in $R_M$) for all maximal ideals $M$ containing $P_1$.*

**Proof.** (i) Consider the ideal $((I + PP_I) : P_I)$. Note that $P \subseteq ((I + PP_I) : P_I)$. Suppose $P = ((I + PP_I) : P_I)$. We claim that $I + PP_I$ is $P$-primary. Let $yz \in I + PP_I$ and $z \notin P$. Then $yz \in P_I$, so $y \in P_I$. Since $P$ is a multiplication ideal, by [3, Lemma 1] and [2, Corollary], $P_I$ is a multiplication ideal. As $P_I$ is a multiplication ideal, it follows that $(y) = P_I C$ for some ideal $C$ of $R$. If $C \subseteq P$, then we are through. Suppose $C \not\subseteq P$. Then $(yz) = (z)P_I C \subseteq I + PP_I$, so $zC \subseteq ((I + PP_I) : P_I) = P$, a contradiction. Therefore $I + PP_I$ is $P$-primary and hence $P_I = I + PP_I$. Consequently, $1 \in ((I + PP_I) : P_I) = P$, a contradiction. Therefore $P \subset ((I + PP_I) : P_I)$.

(ii) Suppose $P_1$ is a $B_w$-prime of $I$. Then $P_1$ is minimal over $(I : r)$ for some $r \in R$. Since$(I : r)r \subseteq I \subseteq P_I$, $(I : r) \not\subseteq P$ and $P_I$ is $P$-primary, it follows that $r \in P_I$. If $(P_I)_M = I_M$ for some maximal ideal $M$ containing $P_1$, then $rs \in I$ for some $s \notin M$. So $s \in (I : r) \subseteq M$, a contradiction. Therefore the result is true. □

**Lemma 10.** *Let every non-maximal prime ideal of $R$ be a multiplication ideal. Suppose $P$ is a non-maximal minimal prime and minimal over an ideal $I$ of $R$. Then the following statements hold:*

(i) *Any $B_w$-prime of $I$ which contains $P$ properly, is a rank one maximal ideal and minimal over $((I + PP_I) : P_I)$.*
(ii) *If the maximal ideals of $R$ are finitely generated, then the ideal $((I + PP_I) : P_I)$* has only finitely many minimal primes.

**Proof.** (i) Let $P_1$ be any $B_w$-prime of $I$ which contains $P$ properly. If $M$ is a maximal ideal containing $P_1$, then by Lemma 9(ii), $R_M$ is not a domain. Note that by Lemma 6, $R$ is an almost $Q$-ring. As $R$ is an almost $Q$-ring, by [4, Corollary 6], it follows that rank $M = 1$, so $P_1$ is a rank one maximal ideal. If $((I + PP_I) : P_I) \not\subseteq P_1$, then $(P_I)_{P_1} \subseteq I_{P_1} + (PP_I)_{P_1}$. As $P$ is a multiplication ideal, it follows that $P_I$ is a multiplication ideal, so $P_I$ is locally principal, and hence by Nakayama's lemma, it follows that $(P_I)_{P_1} = I_{P_1}$. But this contradicts the statement of Lemma 9(ii). Therefore $((I + PP_I) : P_I) \subseteq P_1$ and hence by Lemma 9(i), $P_1$ is minimal over $((I + PP_I) : P_I)$.

(ii) Note that by hypothesis, $R$ is an almost $Q$-ring and so dim $R \leq 2$. By Lemma 9(i), every prime minimal over $((I + PP_I) : P_I)$ is either a non-minimal maximal ideal or a rank one non-maximal prime. As every non-maximal prime is a multiplication ideal, by [2, Theorem 3], the rank one non-maximal primes are quasi-principal. By hypothesis, the minimal primes over $((I + PP_I) : P_I)$ are finitely generated and so by Lemma 5, the ideal $((I + PP_I) : P_I)$ has only finitely many minimal primes. □

**Lemma 11.** *Suppose every ideal* (*of* $R$) *generated by two elements has only finitely many minimal primes and the non-maximal prime ideals are multiplication ideals. Then the non-maximal prime ideals are quasi-principal.*

**Proof.** Let $P$ be a non-maximal prime ideal. As $\dim R \leq 2$, it follows that $P$ is either minimal or a rank one prime. If $P$ is non-minimal, then $P$ is quasi-principal [2, Theorem 3]. Suppose $P$ is minimal. By Lemma 8, $P = \sqrt{I}$ for some finitely generated ideal $I$ of $R$. Note that every $B_w$-prime of $I$ contains $P$, and by Lemma 8, the ideal $((I + PP_I) : P_I)$ has only finitely many minimal primes. Therefore by Lemma 10(i), $I$ has only finitely many $B_w$-primes. Again note that by Lemma 10(i), for every finitely generated ideal $I_0$ with $I \subseteq I_0 \subseteq P$, $I_0$ has only finitely many $B_w$-primes. As $\dim R \leq 2$, by [7, Theorem 1.3], $P$ is finitely generated and hence quasi-principal. □

**Lemma 12.** *Suppose every non-maximal prime ideal of* $R$ *is a multiplication ideal, the maximal ideals of* $R$ *are finitely generated and every principal ideal has only finitely many minimal primes. Then every principal ideal is a finite intersection of primary ideals.*

**Proof.** Note that by hypothesis, $R$ is an almost $Q$-ring, so by Lemma 4, $\dim R \leq 2$. Let $x \in R$. Then by hypothesis, $(x)^*$ is a finite intersection of primary ideals. Suppose $(x)$ is not contained in any minimal prime. We show that $(x) = (x)^*$. Let $M$ be a maximal ideal. If $x \notin M$, then $(x)_M = {(x)^*}_M$. Suppose $x \in M$. If $M$ is minimal over $(x)$, then $(x)_M = {(x)^*}_M$. Suppose $M$ is not minimal over $(x)$. Then rank $M = 2$, so by [4, Corollary 6], $R_M$ is a $\pi$-domain. Therefore $(x)_M = {(x)^*}_M$ (see the proof of [10, Theorem 1.2] or [5, Theorem 3]). This shows that $(x)_M = {(x)^*}_M$ for all maximal ideals containing $x$ and hence $(x) = (x)^*$.

Now assume that $P_1, P_2, \ldots, P_m$ be the primes minimal over $(x)$. Let $P_1, P_2, \ldots, P_t$ be the non-maximal minimal primes and let $P_{t+1}, P_{t+2}, \ldots, P_m$ be the primes which are either maximal or rank one non-maximal primes. By Lemma 10(ii), the ideals $((x) + P_i{(P_i)}_x : {(P_i)}_x)$ for $i = 1, 2, \ldots, t$ have only finitely many minimal primes, say $M_1, M_2, \ldots, M_n$. Again by the proof of Lemma 10(ii), these are either non-minimal maximal ideals or rank one non-maximal prime ideals. Without loss of generality, assume that $M_1, M_2, \ldots, M_k$ are the rank one maximal prime ideals and $M_{k+1}, M_{k+2}, \ldots, M_n$ are either rank two maximal ideals or rank one non-maximal prime ideals. Let $M$ be any maximal ideal different from $M_1, M_2, \ldots, M_k$. We claim that $(x)_M = {(x)^*}_M$. Obviously, if $x \notin M$, then $(x)_M = {(x)^*}_M$. Suppose $x \in M$. If either $M$ is minimal over $(x)$ or rank $M = 2$, then $(x)_M = {(x)^*}_M$. Suppose $M$ is not minimal over $(x)$ and rank $M = 1$. Then $M$ is different from $M_1, M_2, ..., M_n$, so $((x) + P_i{(P_i)}_x : {(P_i)}_x) \nsubseteq M$ for $i = 1, 2, \ldots, t$ and hence ${({(P_i)}_x)}_M = (x)_M$ for $i = 1, 2, \ldots, t$. Consequently, $(x)_M = {(x)^*}_M$. If $(x)_{M_i} = {(x)^*}_{M_i}$ for $i = 1, 2, \ldots, k$, then $(x)_M = {(x)^*}_M$ for all maximal ideals, so $(x) = (x)^*$. Suppose $(x)_{M_i} \neq {(x)^*}_{M_i}$ for $i = 1, 2, \ldots, l$ $(1 \leq l \leq k)$. As $R_{M_i}$ is a Laskerian ring, it follows that there exist $M_i$-primary $Q_i$ such that $(x)_{M_i} = ((x)^*)_{M_i} \cap (Q_i)_{M_i}$ for $i = 1, 2, \ldots, l$. Then $(x)_M = ((x)^* \cap Q_1 \cap Q_2 \cap \cdots \cap Q_l)_M$ for all maximal ideals $M$ of $R$. Therefore $(x) = (x)^* \cap Q_1 \cap Q_2 \cap \cdots \cap Q_l$ and hence $(x)$ is a finite intersection of primary ideals. This completes the proof of the lemma. □

**Lemma 13.** *Suppose $R$ is a quasi-local ring in which the maximal ideal $M$ is finitely generated. If every ideal generated by two elements is a finite product of primary ideals, then $R$ is a Noetherian $Q$-ring.*

**Proof.** If $M$ is minimal, then we are through. Suppose $M$ is non-minimal. By Lemma 6, it is enough if we show that $R$ satisfies the condition $(*)$. Let $P$ be a non-maximal prime ideal with $P = PM$. Let $\Psi = \{P_\alpha \mid P \subseteq P_\alpha, P_\alpha$ is prime and $P_\alpha = P_\alpha M\}$. Clearly $\Psi \neq \emptyset$ and by Zorn′s lemma, $\Psi$ has a maximal element, say $P_0$. Note that $P_0 \neq M$. If $P_0 \subset P_1 \subset M$ for some prime ideal $P_1$, then $P_1 \neq P_1 M$, so by Lemma 3, $P_1$ is principal and hence $P$ is contained in a principal ideal. Now assume that $M$ covers $P_0$. Choose any $x \in M$ such that $x \notin P_0$. Then $P_0 + (x)$ is $M$-primary. As $M$ is finitely generated, it follows that $M^k \subseteq P_0 + (x)$ for some positive integer $k$. Again since $P_0 = P_0 M$, it follows that $P_0 \subseteq M^n$ for all positive integers $n$. Therefore $M^k \subseteq P_0 + (x) \subseteq (x) + M^{k+1} = (x) + M^k M$ and hence by Nakayama′s lemma $P_0 \subset M^k \subseteq (x)$. This shows that $P$ is properly contained in $(x)$ and hence $R$ satisfies the condition $(*)$. □

**Lemma 14.** *Suppose every finitely generated ideal of $R$ is a finite product of primary ideals. Suppose $I$ is an ideal of $R$ such that $I$ is locally finitely generated and every prime minimal over $I$ is a maximal ideal. Then $I$ is finitely generated.*

**Proof.** We claim that $\theta(I) = R$. Suppose $\theta(I) \neq R$. Then $\theta(I) \subseteq M$ for some maximal ideal $M$ of $R$. Since $I$ is locally finitely generated, it follows that $I_M = (I_1)_M$ for some finitely generated ideal $I_1$ contained in $I$. By hypothesis, there exist primary ideals $Q_1, Q_2, \ldots, Q_n$ such that $I_1 = Q_1 Q_2 \ldots Q_n$. Without loss of generality, assume that $Q_i \subseteq M$ for $i = 1, 2, \ldots, k$ and $Q_j \nsubseteq M$ for $j = k+1, k+2, \ldots, n$. Then $I_M = (I_1)_M = (Q_1)_M (Q_2)_M \ldots (Q_k)_M$. Since $I_M \subseteq (Q_i)_M$, it follows that $I \subseteq Q_i$ for $i = 1, 2, \ldots, k$. Since $M$ is minimal over $I$, it follows that each $Q_i$ is $M$-primary and hence $Q_1 Q_2 \ldots Q_k$ is $M$-primary. Therefore $I \subseteq Q_1 Q_2 \ldots Q_k$. Choose elements $x_j \in Q_j$ such that $x_j \notin M$ for $j = k+1, k+2, \ldots, n$. Let $z = x_{k+1} x_{k+2} \ldots x_n$. Since $I \subseteq Q_1 Q_2 \ldots Q_k$ and $z \in Q_{k+1} Q_{k+2} \ldots Q_n$, it follows that $Iz \subseteq Q_1 Q_2 \ldots Q_n = I_1$, so $z \in (I_1 : I) \subseteq \theta(I) \subseteq M$, which is a contradiction. Therefore $\theta(I) = R$ and hence $R = \sum_{i=1}^{n} (I_i : I)$, where $I_i$'s are finitely generated ideals contained in $I$. So $I = \sum_{i=1}^{n} I_i$. This shows that $I$ is a finitely generated ideal. □

**Theorem 1.** *$R$ is an almost $Q$-ring if and only if every non-maximal prime ideal is locally principal.*

**Proof.** The result follows from Lemma 6. □

**Corollary 1.** *Suppose every principal ideal is a finite product of primary ideals. Then $R$ is an almost $Q$-ring if and only if every non-maximal prime ideal is a multiplication ideal.*

**Proof.** The proof of the corollary follows from Theorem 1 and Lemma 7. □

**Corollary 2.** *Suppose every principal ideal is a finite intersection of primary ideals. Then $R$ is an almost $Q$-ring if and only if every non-maximal prime ideal is quasi-principal.*

**Proof.** The proof of the corollary follows from Theorem 1 and [4, Theorem 12]. □

**Theorem 2.** *The following statements on $R$ are equivalent:*

(i) *$R$ is a $Q$-ring.*
(ii) *$R$ is an almost $Q$-ring in which every ideal generated by two elements is a finite intersection of primary ideals.*
(iii) *$R$ is an almost $Q$-ring in which every ideal generated by two elements is a finite product of primary ideals.*
(iv) *Every ideal generated by two elements is a finite product of primary ideals and for every maximal ideal $M$ of $R$, $R_M$ satisfies the condition $(*)$.*
(v) *Every non-maximal prime ideal is a multiplication ideal and every ideal generated by two elements has only finitely many minimal primes.*

**Proof.** (i)⇒(ii) and (i)⇒(iii) follow from [4, Lemma 4 and Theorem 10].

(ii)⇒(v) follows from Corollary 2.

(iii)⇒(iv) follows from Lemma 1.

(iv)⇒(v) follows from Lemma 6 and Corollary 1.

(v)⇒(i). Suppose (v) holds. By Lemma 4 and Lemma 6, $\dim R \leq 2$. By Lemma 8, $R$ has Noetherian spectrum. Also by Lemma 11, every non-maximal prime ideal is quasi-principal. Therefore by [4, Lemma 1], every primary ideal whose radical is non-maximal is a power of its radical and hence quasi-principal. Consequently, every primary ideal whose radical is non-maximal is finitely generated. Again by [8, Corollary 2.3], $R$ is Laskerian and hence by [4, Theorem 13], $R$ is a $Q$-ring. □

The following theorem gives some new equivalent conditions for Noetherian $Q$-rings.

**Theorem 3.** *The following statements on $R$ are equivalent:*

(i) *$R$ is a Noetherian $Q$-ring.*
(ii) *The maximal ideals are locally finitely generated and every ideal generated by two elements is a finite product of primary ideals.*
(iii) *$R$ is an almost $Q$-ring in which the maximal ideals are finitely generated and every principal ideal is a finite product of primary ideals.*

**Proof.** (i)⇒(ii) is obvious.

(ii)⇒(iii). Suppose (ii) holds. By Lemma 13, $R$ is locally Noetherian and an almost $Q$-ring. By Theorem 2, $R$ is a $Q$-ring and so by Lemma 14, the maximal ideals are finitely generated. Therefore (iii) holds.

(iii)⇒(i). Suppose (iii) holds. By Corollary 1, Corollary 2 and Lemma 12, $R$ is a Noetherian ring and hence by Theorem 2, $R$ is a Noetherian $Q$-ring. This completes the proof of the theorem. □

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University of Botswana, Department of Mathematics
P/ Bag 00704, Gaborone, Botswana
*E-mail*: chillumu@mopipi.ub.bw