ARCHIVUM MATHEMATICUM (BRNO)
Tomus 40 (2004), 233 – 248

THE CONTACT SYSTEM FOR A-JET MANIFOLDS

R. J. ALONSO-BLANCO AND J. MUÑOZ-DÍAZ

ABSTRACT. Jets of a manifold M can be described as ideals of $\mathcal{C}^\infty(M)$. This way, all the usual processes on jets can be directly referred to that ring. By using this fact, we give a very simple construction of the contact system on jet spaces. The same way, we also define the contact system for the recently considered A-jet spaces, where A is a Weil algebra. We will need to introduce the concept of *derived algebra*.

Although without formalization, jets are present in the work of S. Lie (see, for instance, [6]; § 130, pp. 541) who does not assume a fibered structure on the concerned manifold; on the contrary, this assumption is usually done nowadays in the more narrow approach given by the jets of sections.

It is an old idea to consider the points of a manifold other than the ordinary ones. This can be traced back to Plücker, Grassmann, Lie or Weil. Jets are 'points' of a manifold M and can be described as ideals of its ring of differentiable functions [9, 13]. Indeed, the k-jets of m-dimensional submanifolds of M are those ideals $\mathfrak{p} \subset \mathcal{C}^\infty(M)$ such that $\mathcal{C}^\infty(M)/\mathfrak{p}$ is isomorphic to $\mathbb{R}^k_m \overset{\text{def}}{=} \mathbb{R}[\epsilon_1, \dots, \epsilon_m]/(\epsilon_1, \dots, \epsilon_m)^{k+1}$ (where the ϵ' are undetermined variables).

This point of view was introduced in the Ph. D. thesis of J. Rodríguez, advised by the second author [13]. Subsequently, several applications were done showing the improvement given by this approach with respect to the usual one: formal integrability theory [10], Lie equations and Lie pseudogroups [7, 8], differential invariants [12] and transformations of partial differential equations [3]. Even the present paper may be placed into that series.

The main advantage of considering jets as ideals is the following. All the operations on the space of (m,k)-jets $J^k_m M$ are directly referred to $\mathcal{C}^\infty(M)$, making the usual processes much more transparent and natural. In particular, the tangent space $T_{\mathfrak{p}} J^k_m M$ is given by classes of derivations from $\mathcal{C}^\infty(M)$ to $\mathcal{C}^\infty(M)/\mathfrak{p}$ (where two of these derivations are considered as equivalent if they agree on

---

2000 *Mathematics Subject Classification*: 58A20.
Key words and phrases: jet, contact system, Weil algebra, Weil bundle.
The first author was partially founded by Junta de Castilla y León under contract SA30/00B.
Received July 10, 2002.

$\mathfrak{p} \subset \mathcal{C}^\infty(M)$). As a result, the very functions $f \in \mathfrak{p}$ define canonically $\mathcal{C}^\infty(M)/\mathfrak{p}$-linear maps $\mathfrak{d}_\mathfrak{p} f \colon T_\mathfrak{p} J^k_m M \to \mathcal{C}^\infty(M)/\mathfrak{p}$ whose real components span the cotangent space $T^*_\mathfrak{p} J^k_m M$ (Corollary 1.5).

We will construct the contact system starting from the following remark. Let p be the unique point of M such that $\mathfrak{p} \subset \mathfrak{m}_p$ (where $\mathfrak{m}_p$ denotes the ideal of the functions vanishing on p). When f runs over $\mathfrak{p}$ and $D_\mathfrak{p}$ runs over the tangent spaces to jet prolongations of m-dimensional submanifolds $X \subset M$, the set of the values of $\mathfrak{d}_\mathfrak{p} f(D_\mathfrak{p})$ equals $\mathfrak{m}_p^k/\mathfrak{p}$.

As a consequence, it is natural to define the contact system by composing each $\mathfrak{d}_\mathfrak{p} f$ with the projection $\mathcal{C}^\infty(M)/\mathfrak{p} \to \mathcal{C}^\infty(M)/\mathfrak{p} + \mathfrak{m}_p^k$ (Definition 1.6). The resulting maps annihilate all the tangent vectors to jet prolongations of m-dimensional submanifolds. This way, the basic properties of the contact system are easily established.

On the other hand, for each Weil algebra A (finite local rational commutative $\mathbb{R}$-algebra), we can define an A-jet on M as an ideal $\mathfrak{p} \subset \mathcal{C}^\infty(M)$ such that $\mathcal{C}^\infty(M)/\mathfrak{p}$ is isomorphic to A. The set of A-jets $J^A M$ can be also endowed with an smooth structure [1]. The way we have defined the contact system for (m,k)-jets can be translated into A-jets. All we have to do is looking for a suitable substitute for $\mathcal{C}^\infty(M)/\mathfrak{p} + \mathfrak{m}_p^k$. Such a substitute turns to be the *derived algebra* associated with $\mathcal{C}^\infty(M)/\mathfrak{p}$ (Proposition 3.9). Once this is done, we can proceed as in the case of $A = \mathbb{R}^k_m$.

Notation. Let $\phi\colon A \to B$ be an $\mathbb{R}$-algebra morphism; by $\mathrm{Der}_\mathbb{R}(A,B)_\phi$ we will denote the set of $\mathbb{R}$-derivations from A to B where B is considered as an A-module via ϕ. When ϕ is implicitly assumed, we will omit it. The characters α, β will be reserved to denoting multi-indices $\alpha = (\alpha_1, \dots, \alpha_k)$, $\beta = (\beta_1, \dots, \beta_k) \in \mathbb{N}^k$ (typically, k will be n or m). Besides, we will denote by 1_j the multi-indice $(1_j)_i = \delta_{ij}$.

1. The contact system on Jet spaces

In the whole of this paper, M will be a smooth manifold of dimension n. Besides, 'submanifold' will mean 'locally closed submanifold'. When X is a closed submanifold of M, I_X will be the ideal of $C^\infty(M)$ consisting of the functions vanishing on X. When X is only locally closed, one would replace M by the open set U into which X is a closed submanifold but, for the sake of simplicity in the exposition, that will be implicitly understood.

Let us consider an m-dimensional submanifold $X \subset M$, its associated ideal $I_X \subset \mathcal{C}^\infty(M)$, and a point $p \in X$. The class of the submanifolds having at p a contact of order k with X is naturally identified with the ideal $\mathfrak{p} \overset{\mathrm{def}}{=} I_X + \mathfrak{m}_p^{k+1} \subset \mathcal{C}^\infty(M)$. Moreover, an isomorphism $\mathcal{C}^\infty(M)/\mathfrak{p} \simeq \mathbb{R}^k_m$ is deduced by taking local coordinates $\{x_i, y_j\}$ centered at p and such that $I_X = (y_j)$.

Definition 1.1. A jet of dimension m and order k (or, simply, an (m,k)-jet) of M is, by definition, an ideal $\mathfrak{p} \subset \mathcal{C}^\infty(M)$ such that $\mathcal{C}^\infty(M)/\mathfrak{p} \simeq \mathbb{R}^k_m$. The set of (m,k)-jets of M will be denoted by $J^k_m M$.

Given $\mathfrak{p} \in J^k_m M$, there is a unique point $p \in M$ such that $\mathfrak{p} \subset \mathfrak{m}_p$. This way, it is deduced a map $J^k_m M \to M$, $\mathfrak{p} \mapsto p$.

The smooth structure on $J^k_m M$ is obtained in the following way (see [13, 9]). Let $(U; x_1, \dots, x_n)$ be a local chart of M. Now, let us choose m coordinates, for instance $x_1, \dots, x_m$, and let us consider the subset $\underline{J}^k_m U$ given by those jets $\mathfrak{p} \in J^k_m U$ such that $\mathbb{R}[x_1, \dots, x_m]/\mathfrak{p} \cap \mathbb{R}[x_1, \dots, x_m] \simeq \mathcal{C}^\infty(U)/\mathfrak{p}$. So, with each function $f \in \mathcal{C}^\infty(U)$ we can associate a unique polynomial $P_f(x)$ of degree $\leq k$ such that $f - P_f \in \mathfrak{p}$.

Let us denote by y_j the coordinate x_{m+j}. Then we have

$$P_{y_j}(x) = \sum_{|\alpha| \leq k} y_{j\alpha}(\mathfrak{p}) \frac{(x - x(p))^\alpha}{\alpha!}\,, \tag{1.1}$$

for suitable numbers $y_{j\alpha}(\mathfrak{p})$. Besides, $\mathfrak{p}$ is spanned by the functions $y_j - P_{y_j}$ together with $\mathfrak{m}_p^{k+1}$. So the set of functions $\{x_i, y_j, y_{j\alpha}\}$ provides one with a coordinate system on $\underline{J}^k_m U$.

By taking in the above process all the possible choices of m elements of $\{x_1, \dots, x_n\}$ in all the local charts of M we get an atlas on $J^k_m M$.

The following basic statement was proved in [13] (see also [1, 9]).

Theorem 1.2. *For each $\mathfrak{p} \in J^k_m M$ the following isomorphism holds,*

$$T_\mathfrak{p} J^k_m M \simeq \mathcal{D}_\mathfrak{p} / \mathcal{D}'_\mathfrak{p}$$

where $\mathcal{D}_p = \mathrm{Der}_\mathbb{R}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)/\mathfrak{p})$ and $\mathcal{D}'_\mathfrak{p} = \{D \in \mathcal{D}_\mathfrak{p} \mid Df = 0, \forall f \in \mathfrak{p}\}$.

The correspondence in the above theorem is locally given by

$$\left(\frac{\partial}{\partial x_i}\right)_\mathfrak{p} = \left[\frac{\partial}{\partial x_i}\right]_\mathfrak{p}, \qquad \left(\frac{\partial}{\partial y_{j\alpha}}\right)_\mathfrak{p} = \left[\frac{(x - x(p))^\alpha}{\alpha!} \frac{\partial}{\partial y_j}\right]_\mathfrak{p} \tag{1.2}$$

where $[D]_\mathfrak{p}$ denotes the class of a derivation $D \in \mathcal{D}_\mathfrak{p}$ modulo $\mathcal{D}'_\mathfrak{p}$ (see [9], pp. 744-45, for this calculation).

Remark 1.3. Since Theorem 1.2 it is deduced that the tangent space at a jet $\mathfrak{p} \in J^k_m M$ is naturally provided with the structure of $\mathcal{C}^\infty(M)/\mathfrak{p}$-module.

Corollary 1.4. *Each function $f \in \mathfrak{p}$ defines an $\mathcal{C}^\infty(M)/\mathfrak{p}$-linear map*

$$\mathfrak{d}_\mathfrak{p} f \colon T_\mathfrak{p} J^k_m M \longrightarrow \mathcal{C}^\infty(M)/\mathfrak{p}\,; \qquad D_\mathfrak{p} = [D]_\mathfrak{p} \mapsto [Df]_\mathfrak{p}$$

where $[Df]_\mathfrak{p}$ denotes the class of the function Df modulo $\mathfrak{p}$.

The local expression of $\mathfrak{d}_\mathfrak{p} f$ is given by

$$\mathfrak{d}_\mathfrak{p} f = \sum_i \left[\frac{\partial f}{\partial x_i}\right]_\mathfrak{p} d_\mathfrak{p} x_i + \sum_{j,\alpha} \left[\frac{(x - x(p))^\alpha}{\alpha!} \frac{\partial f}{\partial y_j}\right]_\mathfrak{p} d_\mathfrak{p} y_{j\alpha}\,. \tag{1.3}$$

Corollary 1.5. *For each jet $\mathfrak{p}$, the cotangent space $T^*_\mathfrak{p} J^k_m M$ is spanned by the real components of the $\mathfrak{d}_\mathfrak{p} f$, $f \in \mathfrak{p}$:*

$$T^*_\mathfrak{p} J^k_m M = \textit{Real components of } \;\{\mathfrak{d}_\mathfrak{p} f \mid f \in \mathfrak{p}\}\,.$$

Proof. Given $D_{\mathfrak{p}} \in T_{\mathfrak{p}} J^k_m M$, there exist at least a function $f \in \mathfrak{p}$ such that $\mathfrak{d}_{\mathfrak{p}} f(D_{\mathfrak{p}}) \neq 0$ (elsewhere, $D_{\mathfrak{p}} = 0$); so, also a real component of $\mathfrak{d}_{\mathfrak{p}} f$ is not vanishing on $D_{\mathfrak{p}}$. □

Let us denote by $\mathfrak{d}'_{\mathfrak{p}} f$ the following composition

$$T_{\mathfrak{p}} J^k_m M \xrightarrow{\mathfrak{d}_{\mathfrak{p}} f} \mathcal{C}^\infty(M)/\mathfrak{p} \xrightarrow{\pi'} \mathcal{C}^\infty(M)/\mathfrak{p}' ,$$

where $\mathfrak{p}' \stackrel{\text{def}}{=} \mathfrak{p} + \mathfrak{m}^k_p$.

Definition 1.6. The distribution of tangent vectors $\mathcal{C}$ given by

$$\mathcal{C}_{\mathfrak{p}} \stackrel{\text{def}}{=} \bigcap_{f \in \mathfrak{p}} \ker(\mathfrak{d}'_{\mathfrak{p}} f) \subset T_{\mathfrak{p}} J^k_m M$$

will be called the *contact distribution* on $J^k_m M$. The Pfaffian system associated with $\mathcal{C}$ will be called the *contact system* on $J^k_m M$ and we will denote it by Ω.

In order to get the local expression of Ω let us consider the functions $f_j = y_j - P_{y_j} \in \mathfrak{p}$ (thus, $\mathfrak{p} = (f_j) + \mathfrak{m}^{k+1}_p$). From relations (1.1)-(1.3) we get

$$\begin{aligned} \mathfrak{d}'_{\mathfrak{p}} f_j &= \sum_{|\alpha| \leq k-1} \left[\frac{(x - x(p))^\alpha}{\alpha!} \right]_{\mathfrak{p}'} d_{\mathfrak{p}} y_{j\alpha} - \sum_{i, |\alpha| \leq k} y_{j\alpha}(\mathfrak{p}) \left[\frac{(x - x(p))^{\alpha - 1_i}}{(\alpha - 1_i)!} \right]_{\mathfrak{p}'} d_{\mathfrak{p}} x_i \\ &= \sum_{|\alpha| \leq k-1} \left[\frac{(x - x(p))^\alpha}{\alpha!} \right]_{\mathfrak{p}'} \big(d_{\mathfrak{p}} y_{j\alpha} - \sum_i y_{j\alpha + 1_i}(\mathfrak{p}) d_{\mathfrak{p}} x_i\big) . \end{aligned}$$

Because $\mathfrak{d}'_{\mathfrak{p}} \mathfrak{m}^{k+1}_p = 0$, we deduce that the contact system Ω is generated by the 1-forms

$$\omega_{j\alpha} \stackrel{\text{def}}{=} dy_{j\alpha} - \sum_i y_{j\alpha + 1_i} dx_i \tag{1.4}$$

which are the real components of the $\mathfrak{d}'_{\mathfrak{p}} f_j$.

Since (1.4) it is obvious that Ω is the usual contact system. Nevertheless, in the rest of this section we will explain why Ω is well behaved.

Let X be an m-dimensional submanifold of M; each (m, k)-jet $\mathfrak{q} \in J^k_m X$ is necessarily of the form $\mathfrak{q} = \overline{\mathfrak{m}}^{k+1}_p$, where $\overline{\mathfrak{m}}_p \subset \mathcal{C}^\infty(X)$ denotes the maximal ideal of a point $p \in X$. Accordingly, an identification $J^k_m X \approx X$ arises. Moreover, if $I_X \subset \mathcal{C}^\infty(M)$ denotes the ideal associated with X, we can consider the inclusion $X \approx J^k_m X \hookrightarrow J^k_m M$ by $p \mapsto I_X + \mathfrak{m}^{k+1}_p$. That defines an immersion which will be called the *k-jet prolongation* of X. The ideal of $J^k_m X$ into $J^k_m M$ is the prolongation of I_X to $\mathcal{C}^\infty(J^k_m M)$ (see [9]). As a result, and taking into account the characterization of the tangent spaces given in Theorem 1.2, we obtain item (2) of the following statement (item (1) is easy).

Theorem 1.7. *Let X be an m-dimensional submanifold of M and consider its jet prolongation $J^k_m X$ immersed into $J^k_m M$.*

(1) *A jet $\mathfrak{p} \in J^k_m M$ belongs to $J^k_m X$ if and only if $\mathfrak{p} \supset I_X$.*

(2) *A vector* $D_{\mathfrak{p}} \in T_{\mathfrak{p}} J^k_m M$ *is tangent to* $J^k_m X$ *if and only if*

$$\mathfrak{d}_{\mathfrak{p}} f(D_{\mathfrak{p}}) = 0, \qquad \forall f \in I_X.$$

Let us suppose $\mathfrak{p} = (y_j) + \mathfrak{m}_p^{k+1}$ where $\{x_i, y_j\}$ are local coordinates around $p \in M$. All the submanifolds X such that $I_X \subset \mathfrak{p}$ are locally given by equations $y_j = P_j(x)$; $j = 1, \ldots, n-m$, where $P_j(x) \in \mathfrak{m}_p^{k+1}$. As a consequence,

Lemma 1.8. *Given a jet* $\mathfrak{p} \in J^k_m M$*, the set of the values* $\mathfrak{d}_{\mathfrak{p}} f(D_{\mathfrak{p}})$ *when* f *runs over* $\mathfrak{p}$ *and* $D_{\mathfrak{p}}$ *runs over the tangent spaces to jet prolongations of* m*-dimensional submanifolds* $X \subset M$*, equals* $\mathfrak{m}_p^k/\mathfrak{p}$.

According to the above lemma, if $f \in \mathfrak{p}$ then $\mathfrak{d}'_{\mathfrak{p}} f$ annihilates each vector which is tangent to a jet prolongation $J^k_m X$. This is why Definition 1.6 gives the usual contact system.

From this point the basic properties of Ω could be deduced. However, we have preferred to do it in the more general context of A-jets where a similar construction of the contact system will be carried on (see below).

2. A-jets

It is well known that a manifold M can be recovered as the set of $\mathbb{R}$-algebra morphisms $\mathcal{C}^\infty(M) \to \mathbb{R}$; also the tangent bundle TM is obtained by taking the morphisms with values in $\mathbb{R}[\epsilon]/\epsilon^2$. In general we can consider the morphisms taking values in an algebra A. This concept comes back to Weil [14], who called them 'points A-proches' of M.

Definition 2.1. A commutative $\mathbb{R}$-algebra A is called a *Weil algebra* if it is finite dimensional, local and rational. Let us denote by $\mathfrak{m}_A$ the maximal ideal of A. The integer k such that $\mathfrak{m}_A^{k+1} = 0$, $\mathfrak{m}_A^k \neq 0$, will be called the *order* of A and denoted by $o(A)$. The dimension of $\mathfrak{m}_A/\mathfrak{m}_A^2$ will be called the *width* of A and denoted by $w(A)$.

The main examples of Weil algebras are the rings of truncated polynomials $\mathbb{R}^k_m$ (here, $o(\mathbb{R}^k_m) = k$ and $w(\mathbb{R}^k_m) = m$). On the other hand, if $\mathfrak{m}_p$ denotes the maximal ideal associated to a point p in a manifold M, the quotient $\mathcal{C}^\infty(M)/\mathfrak{m}_p^{k+1}$ is also a Weil algebra isomorphic to $\mathbb{R}^k_n$ where n is the dimension of M (an isomorphism is induced by taking local coordinates).

Definition 2.2. Let M be a manifold and A a Weil algebra. An $\mathbb{R}$-algebra morphism

$$p^A \colon \mathcal{C}^\infty(M) \longrightarrow A$$

is called an *A-point* (or *A-velocity*) of M. The set of A-points of M will be called the *Weil bundle of A-points* of M and denoted by M^A. We will say that an A-point p^A is *regular* if it is surjective. The set of regular A-points of M will be denoted by $\check{M}^A$.

To simplify notation, when $A = \mathbb{R}^k_m$ we will write M^k_m instead of $M^{\mathbb{R}^k_m}$. For instance, $M^0_m = M$ (for any m) and $M^1_1 = TM$.

If we compose an A-point p^A with the canonical projection $A \to A/\mathfrak{m}_A = \mathbb{R}$ we obtain an $\mathbb{R}$-point of M, that is, an ordinary point $p \in M$; this defines a projection $M^A \longrightarrow M$.

On the other hand, each function $f \in \mathcal{C}^\infty(M)$ defines a map

$$f^A \colon M^A \longrightarrow A,$$

by the tautological rule $f^A(p^A) \overset{\text{def}}{=} p^A(f)$.

For the proof of the following statement see [5] or [9].

Theorem 2.3. *There exists a differentiable structure on M^A determined by the condition that the maps f^A are smooth ($\check{M}^A$ is a dense open set of M^A). Furthermore, $M^A \to M$ is a fiber bundle with typical fiber* $\operatorname{Hom}(\mathbb{R}_n^k, A)$*, where $n = \dim M$ and $k = o(A)$.*

Remark 2.4. If $\{y_j\}$ is a local chart on M and $\{a_\alpha\}$ is a basis of A, then the collection of functions $y_{j\alpha}$ determined by the rule $y_j^A(p^A) = \sum_\alpha y_{j\alpha}(p^A) a_\alpha$, define a local chart on M^A.

The next proposition is straightforward (see, for instance, [2]).

Proposition 2.5. (1) *Let $\psi \colon A \longrightarrow B$ be a morphism of Weil algebras. For each smooth manifold M, ψ induces a differentiable map*

$$\psi_M \colon M^A \longrightarrow M^B\,; \qquad p^A \mapsto \psi_M(p^A) \overset{\text{def}}{=} \psi \circ p^A$$

(2) *Let $\phi \colon M \longrightarrow N$ be an smooth map between the smooth manifolds M and N. For each Weil algebra A, ϕ induces a differentiable map*

$$\phi^A \colon M^A \longrightarrow N^A\,; \qquad p^A \mapsto \phi^A(p^A) \overset{\text{def}}{=} p^A \circ \phi^*$$

where ϕ^* stands for the map induced between the rings of functions of M and N.

The following theorem was given in [9].

Theorem 2.6. *There is a natural identification*

$$T_{p^A} M^A \simeq \operatorname{Der}_{\mathbb{R}}(\mathcal{C}^\infty(M), A)_{p^A}$$

where each $X \in T_{p^A} M^A$ is related to the derivation $X' \in \operatorname{Der}_{\mathbb{R}}(\mathcal{C}^\infty(M), A)_{p^A}$ determined by $X'(f) = X(f^A) \in A$, $f \in \mathcal{C}^\infty(M)$ (where X derives componentwise the vector-valued function f^A).

Remark 2.7. According with this theorem, the tangent maps corresponding with Proposition 2.5 are given respectively by

$$\begin{aligned} (\psi_M)_* D_{p^A} &= \psi \circ D'_{p^A} \ \in T_{\psi_M(p^A)} M^B\,, \\ (\phi^A)_* D_{p^A} &= D'_{p^A} \circ \phi^* \in T_{\phi^A(p^A)} N^A\,. \end{aligned}$$

Next, we will generalize the notion of jet for any Weil algebra A.

Definition 2.8. An *A-jet* on M is, by definition, an ideal $\mathfrak{p} \subset \mathcal{C}^\infty(M)$ such that $\mathcal{C}^\infty(M)/\mathfrak{p} \simeq A$. The space of A-jets of M will be denoted by $J^A M$.

We have a surjective map $\operatorname{Ker}\colon \check{M}^A \to J^A M$ which associates with each A-point its kernel. The group $\operatorname{Aut}(A)$ acts on $\check{M}^A$ by composition and there is an obvious equivalence between the set of orbits of this action and $J^A M$.

The proof of the following two theorems was given in [1].

Theorem 2.9. *On $J^A M$ there exists an smooth structure such that*

$$\operatorname{Ker}\colon \check{M}^A \longrightarrow J^A M$$

is a principal fiber bundle with structure group $\operatorname{Aut}(A)$*.*

Remark 2.10. In particular, $J^k_m M$ is the quotient manifold of $\check{M}^k_m$ under the action of $\operatorname{Aut}(\mathbb{R}^k_m)$.

Theorem 2.11. *For each $\mathfrak{p} \in J^A M$, the following isomorphism holds,*

$$T_{\mathfrak{p}} J^A M \simeq \mathcal{D}_{\mathfrak{p}}/\mathcal{D}'_{\mathfrak{p}}$$

where $\mathcal{D}_{\mathfrak{p}} = \operatorname{Der}_{\mathbb{R}}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)/\mathfrak{p})$ and $\mathcal{D}'_{\mathfrak{p}} = \{D \in \mathcal{D}_{\mathfrak{p}} \mid Df = 0, \forall f \in \mathfrak{p}\}$.

As a result and similarly to the case of (m,k)-jets, each function $f \in \mathfrak{p}$ defines a $\mathcal{C}^\infty(M)/\mathfrak{p}$-linear map

$$\mathfrak{d}_{\mathfrak{p}} f\colon T_{\mathfrak{p}} J^A M \longrightarrow \mathcal{C}^\infty(M)/\mathfrak{p}$$

and Corollaries 1.4 – 1.5 also hold for A-jets with the same proof.

On the other hand, each smooth map $\phi\colon M \to N$ induces a new map between the corresponding A-Weil bundles, $\phi^A\colon M^A \to N^A$ (Definition 2.5). However, the condition of regularity of an A-point is not, in general, preserved, that is, $\phi^A(\check{M}^A) \nsubseteq \check{N}^A$. This is why we give the following definition (see [2]).

Definition 2.12. Let $\phi\colon M \longrightarrow N$ be a differentiable map. An A-point $p^A \in M^A$ will be called *ϕ-regular* if $\phi^A(p^A) = p^A \circ \phi^* \in \check{N}^A$. The set of ϕ-regular A-points of M^A will be denoted by $\check{M}^A_\phi$.

The proof of the following propositions is not difficult (see [2]).

Proposition 2.13. *The set of ϕ-regular A-points, $\check{M}^A_\phi$, is an open subset of M^A (eventually the empty set).*

The set of jets of ϕ-regular A-points will be denoted by $J^A_\phi M$. In particular, we have a principal fiber bundle $\operatorname{Ker}\colon \check{M}^A_\phi \to J^A_\phi M$.

Proposition 2.14. *The map $\phi\colon M \to N$ induces maps*

$$\check{M}^A_\phi \xrightarrow{\phi^A} \check{N}^A\,, \qquad p^A \mapsto \phi^A(p^A) \overset{\text{def}}{=} p^A \circ \phi^*$$
$$J^A_\phi M \xrightarrow{j^A\phi} J^A N\,, \qquad \mathfrak{p} \mapsto j^A\phi(\mathfrak{p}) \overset{\text{def}}{=} (\phi^*)^{-1}\mathfrak{p}$$

in such a way that $\operatorname{Ker} \circ \phi^A = j^A\phi \circ \operatorname{Ker}$*.*

Example 2.15. 1) If $\gamma\colon X \to M$ is an immersion, then $\gamma^*\colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(X)$ is surjective on germs; so $\check{X}^A_\gamma = \check{X}^A$ and $J^A_\gamma X = J^A X$. In particular, J^A defines a functor in the category of differentiable manifolds with immersions (see [4]). When γ is the inclusion of an m-dimensional submanifold $X \subset M$ and $A = \mathbb{R}^k_m$, $j^A\gamma$ gives the jet prolongation of X.

2) If $\pi\colon M \longrightarrow X$ is a fiber bundle and s is a section of π, then we have induced maps π^A, s^A, $j^A\pi$ and j^As such that $\pi^A \circ s^A = id_{\check{X}^A}$ and $j^A\pi \circ j^A s = id_{J^A X}$. When $A = \mathbb{R}^k_m$ and $m = \dim X$, $J^A_\pi M$ equals the well known bundle of *k-jets of sections* of π.

Proposition 2.16. *Let $\phi\colon M \to N$, A be as above. The tangent map corresponding to $j^A\phi$ at a point $\mathfrak{p} \in J^A_\phi M$ sends each $D_\mathfrak{p} = [D]_\mathfrak{p} \in T_\mathfrak{p} J^A_\phi M$ to*

$$(j^A\phi)_* D_\mathfrak{p} = \left[[\phi^*]^{-1} \circ D \circ \phi^*\right]_{j^A\phi(\mathfrak{p})} \in T_{j^A\phi(\mathfrak{p})} J^A N\,,$$

where $[\phi^*]$ denotes the isomorphism $\mathcal{C}^\infty(M)/\mathfrak{p} \simeq \mathcal{C}^\infty(N)/j^A\phi(\mathfrak{p})$ induced by ϕ^*.

Proof. It follows from Remark 2.7 and Theorem 2.11 (see [2] for details). □

Definition 2.17. Let $i\colon X \hookrightarrow M$ be an m-dimensional submanifold of M, where $m = w(A)$; then $j^A i\colon J^A X \hookrightarrow J^A M$ will be called the *A-jet prolongation* of X.

Theorem 1.7 remain valid for A-jet prolongations. It can be shown by means of an attentive inspection of the definitions. There is just a difference: as a rule, $J^A X$ can not be identified with X.

Definition 2.18. Let $\mathfrak{p} \in J^A M$, and $p \in M$ be its projection (that is, $\mathfrak{p} \subset \mathfrak{m}_p \subset \mathcal{C}^\infty(M)$) and let us denote by $\mathfrak{m}$ the maximal ideal of $\mathcal{C}^\infty(M)/\mathfrak{p}$ (i.e., $\mathfrak{m} = \mathfrak{m}_p/\mathfrak{p}$).

A local chart $\{x_1, \dots, x_m, y_1, \dots, y_{n-m}\}$ (where $m = w(A)$) in a neighborhood of p will be called *adapted* to the jet $\mathfrak{p}$ if it holds

1. The classes of $\{x_i\}$ modulo $\mathfrak{m}^2$ generate $\mathfrak{m}/\mathfrak{m}^2$.
2. The functions y_j belong to $\mathfrak{p}$ and they are linearly independent modulo $\mathfrak{m}_p^2$.

It is easily deduced the existence of local charts adapted to a given jet.

Lemma 2.19. *Let $\{x_i, y_j\}$ be a local chart adapted to a jet $\mathfrak{p} \in J^A M$; then, there exists polynomials $Q_s(x)$, $deg(Q_s) \le o(A) = k$ such that*

$$\mathfrak{p} = (y_j) + (Q_s(x)) + \mathfrak{m}_p^{k+1}\,.$$

Proof. By hypothesis we have an epimorphism

$$\mathbb{R}[x_1, \dots, x_m]/(x_1, \dots, x_m)^{k+1} \hookrightarrow \mathcal{C}^\infty(M)/\mathfrak{m}_p^{k+1} \longrightarrow \mathcal{C}^\infty(M)/\mathfrak{p}\,,$$

whose kernel is generated by a finite number polynomials $Q_s(x)$. This way we get an isomorphism

$$\mathbb{R}[x_1, \dots, x_m]/(Q_s) + (x_1, \dots, x_m)^{k+1} \simeq \mathcal{C}^\infty(M)/\mathfrak{p}$$

from which we deduce the statement. □

Remark 2.20. We have $Q_s \in \mathfrak{m}_p^2$, elsewhere $w(A)$ could not be m, but lower.

The proof of Corollaries 2.21 and 2.22 below is straightforward.

Corollary 2.21. *Let X be an m-dimensional submanifold of M, and $\mathfrak{p} \in J^A M$ be an A-jet containing I_X. There exists local coordinates $\{x_i, y_j\}$ such that the local equations of X into M are*

$$y_j = P_j(x)\,,$$

for suitable functions $P_j(x) \in \mathfrak{p}$.

Corollary 2.22. *Let $X \overset{i}{\hookrightarrow} M$ be as above, and $\mathfrak{p} = j^A i(\mathfrak{q})$ where $\mathfrak{q} \in J^A X$ and $j^A i \colon J^A X \longrightarrow J^A M$ is the jet prolongation of i. Besides, let $\{x_i, y_j\}$ be a local chart adapted to $\mathfrak{p}$. Then the tangent map is given by*

$$T_{\mathfrak{q}} J^A X \xrightarrow{(j^A i)_*} T_{\mathfrak{p}} J^A M\,; \qquad \left[\frac{\partial}{\partial x_i}\right]_{\mathfrak{q}} \mapsto \left[\frac{\partial}{\partial x_i} + \sum_j \frac{\partial P_j(x)}{\partial x_i}\frac{\partial}{\partial y_j}\right]_{\mathfrak{p}}\,.$$

3. Derived algebra of a Weil algebra

Each Weil algebra A has several canonically defined ideals; examples of which are the powers of its maximal ideal. We show here two more of them which are a key point in order to obtain a contact system for A-jet spaces.

Definition 3.1. Let $\mathcal{W}$ be the category whose objects are the Weil algebras and whose morphims are the Weil algebra isomorphisms.

A functor $\mathcal{W} \xrightarrow{F} \mathcal{W}$ will be called an *equivariant projection* of Weil algebras if for each $A \in \mathcal{W}$ there is an epimorphism $A \xrightarrow{\pi_F} F(A)$ such that for any isomorphim $A \xrightarrow{\psi} B$ of Weil algebras we have

$$\pi_F \circ \psi = F(\psi) \circ \pi_F\,.$$

Example 3.2. For each positive integer j we define the functor $F_j \colon \mathcal{W} \to \mathcal{W}$ which maps a Weil algebra A to $F_j(A) \overset{\text{def}}{=} A_j = A/\mathfrak{m}_A^{j+1}$, where $\mathfrak{m}_A$ is the maximal ideal of A; because any isomorphism $A \overset{\psi}{\simeq} B$ holds $\psi(\mathfrak{m}_A) = \mathfrak{m}_B$, we deduce that F_j is an equivariant projection (A_j is the j-th *underlying algebra* of A, see [4]).

The proof of the following lemma is straightforward.

Lemma 3.3. *Each equivariant projection F defines a group morphism*

$$\mathrm{Aut}(A) \xrightarrow{F} \mathrm{Aut}(F(A))\,; \quad g \mapsto F(g)\,.$$

The projections $\pi_F \colon A \to F(A)$ induce maps of Weil bundles $\pi_F \colon \check{M}^A \to \check{M}^{F(A)}$ for each smooth manifold M (Proposition 2.5). The equivariance property of π_F ensures that we have an induced map at the level of jet spaces. This way,

Theorem 3.4. *Given an smooth manifold M and a Weil algebra A, each equivariant projection F defines a differentiable map*

$$\pi_F \colon J^A M \to J^{F(A)} M\,.$$

Remark 3.5. By the very definition, $F\left(\mathcal{C}^\infty(M)/\mathfrak{p}\right) = \mathcal{C}^\infty(M)/\pi_F(\mathfrak{p})$.

Corollary 3.6. *Under the identification in Theorem 2.11, the tangent map corresponding to π_F is given by*

$$T_{\mathfrak{p}} J^A M \xrightarrow{(\pi_F)_*} T_{\pi_F(\mathfrak{p})} J^{F(A)} M \,; \qquad D_{\mathfrak{p}} = [D]_{\mathfrak{p}} \mapsto (\pi_F)_* D_{\mathfrak{p}} = [\pi_F \circ D]_{\pi_F(\mathfrak{p})}$$

where $\pi_F \circ D \in \operatorname{Der}_{\mathbb{R}}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)/\pi_F(\mathfrak{p}))$.

Let $w(A) = m$ and $o(A) = k$.

Definition 3.7. For each given epimorphism $H\colon \mathbb{R}^{k+1}_m \to A$ we define

$$I'_H \stackrel{\text{def}}{=} \{D_H P \mid P \in \ker H, D_H \in \operatorname{Der}_{\mathbb{R}}(\mathbb{R}^{k+1}_m, A)_H\}\,.$$

Lemma 3.8. *Let $\psi\colon A \xrightarrow{\sim} B$ be an isomorphism of Weil algebras and let $H\colon \mathbb{R}^{k+1}_m \to A$, $\overline{H}\colon \mathbb{R}^{k+1}_m \to B$ be algebra epimorphisms. Then $\psi(I'_H) = I'_{\overline{H}}$.*

Proof. By Lemma 1 in the Appendix there exists an automorphism $g \in \operatorname{Aut}(\mathbb{R}^{k+1}_m)$ such that $\overline{H} \circ g = \psi \circ H$. Moreover, g establishes an isomorphism

$$\psi_g\colon\ \operatorname{Der}_{\mathbb{R}}(\mathbb{R}^{k+1}_m, A)_H \xrightarrow{\ \sim\ } \operatorname{Der}_{\mathbb{R}}(\mathbb{R}^{k+1}_m, B)_{\overline{H}}\,,$$

defined by $\psi_g(D_{\overline{H}}) \stackrel{\text{def}}{=} \psi \circ D_{\overline{H}} \circ g^{-1}$.

If $D_H P \in I'_H$, then $\psi(D_H P) = \psi_g(D_{\overline{H}})(gP) \in I'_{\overline{H}}$. So that $\psi(I'_H)$ is included into $I'_{\overline{H}}$. By symmetry the proof is finished. □

From this lemma it follows that I'_H is not depending on H; let us denote it by I'_A. By using again the lemma above we also deduce

Proposition 3.9. *If $\psi\colon A \xrightarrow{\sim} B$ is a Weil algebra isomorphism, then $\psi(I'_A) = I'_B$. This way,*

$$F'(A) \stackrel{\text{def}}{=} A' = A/I'_A$$

defines an equivariant projection (Definition 3.1) where $\pi_{F'} \stackrel{\text{def}}{=} \pi'$ is the natural epimorphism $A \to A'$. We will call A' the derived algebra *of A.*

Remark 3.10. The ideal I'_A is just the first Fitting ideal of the module of differentials $\Omega_{A/\mathbb{R}}$.

Computation of A'. Let $A = \mathbb{R}[\epsilon_1, \dots, \epsilon_m]/I$ where $I = (Q_s(\epsilon)) + (\epsilon_1, \dots, \epsilon_m)^{k+1}$ (the Q_s are suitable polynomials of degree lower than $k+1$). Let us consider the projection

$$\mathbb{R}^{k+1}_m = \mathbb{R}[\epsilon_1, \dots, \epsilon_m]/(\epsilon_1, \dots, \epsilon_m)^{k+2} \xrightarrow{\ H\ } \mathbb{R}[\epsilon_1, \dots, \epsilon_m]/I\,,$$

in such a way that $\ker H = (Q_s(\epsilon)) + (\epsilon_1, \dots, \epsilon_m)^{k+1} \bmod (\epsilon_1, \dots, \epsilon_m)^{k+2}$. On the other hand, $\operatorname{Der}_{\mathbb{R}}(\mathbb{R}^{k+1}_m, A)_H$ is spanned by the partial derivatives $\partial/\partial\epsilon_i$. So we see that $I' = (\partial Q_s/\partial\epsilon_i) + (\epsilon_1, \dots, \epsilon_m)^k \bmod I$ and then

$$A' = \mathbb{R}[\epsilon_1, \dots, \epsilon_m]/((Q_s) + (\partial Q_s/\partial\epsilon_i) + (\epsilon_1, \dots, \epsilon_m)^k)\,. \tag{3.1}$$

In particular, $(\mathbb{R}^k_m)' = \mathbb{R}^{k-1}_m$ and $(\mathbb{R}^k_m \otimes \mathbb{R}^l_n)' = \mathbb{R}^{k-1}_m \otimes \mathbb{R}^{l-1}_n$.

Remark 3.11. Because $(\mathbb{R}^k_m)' = \mathbb{R}^{k-1}_m$, the notation π' used here, is compatible with that of Section 1. Indeed, we think that A', better than $A/\mathfrak{m}^k_A$, is the natural generalization of $\mathbb{R}^{k-1}_m$.

By applying Proposition 3.5 we have an induced map

$$\pi' \colon J^A M \longrightarrow J^{A'} M$$

which takes each $\mathfrak{p} \in J^A M$ to the kernel of the composition

$$\mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)/\mathfrak{p} \xrightarrow{\pi'} (\mathcal{C}^\infty(M)/\mathfrak{p})' \, .$$

Corollary 3.12. *If $\{x_i, y_j\}$ is a local chart adapted to a jet $\mathfrak{p} \in J^A M$ such that $\mathfrak{p} = (y_j) + (Q_s(x)) + \mathfrak{m}^{k+1}_p$ for suitable polynomials Q_s (Lemma 2.19), then*

$$\pi'(\mathfrak{p}) = (y_j) + (Q_s(x)) + (\partial Q_s/\partial x_i) + \mathfrak{m}^k_p \, .$$

There is a second ideal canonically associated to any Weil algebra A. Let us take en epimorphism $H \colon \mathbb{R}^{k+1}_m \to A$ as above and define the following set

$$\widehat{I}_H \overset{\text{def}}{=} \{H(P) \in I'_A \mid D_H(P) \in I'_A, \ \forall\, D_H \in \mathrm{Der}_{\mathbb{R}}(\mathbb{R}^{k+1}_m, A)_H\} \, .$$

It is straightforward to check that $\widehat{I}_H$ is an ideal of A. A similar reasoning like that used for I'_A, shows that $\widehat{I}_H$ is not depending on H. Let us denote this ideal by $\widehat{I}_A$. Then we also have

Proposition 3.13. *If $\psi \colon A \xrightarrow{\sim} B$ is an isomorphism of Weil algebras, then $\psi(\widehat{I}_A) = \widehat{I}_B$. In particular,*

$$\widehat{F}(A) \overset{\text{def}}{=} \widehat{A} = A/\widehat{I}_A \, ;$$

defines an equivariant projection.

Example 3.14. The algebras $A = \mathbb{R}^k_n = \mathbb{R}[\epsilon_1, \dots, \epsilon_n]/(\epsilon_1, \dots, \epsilon_n)^{k+1}$ hold $\widehat{I}_A = 0$. Indeed, let $\mathbb{R}^{k+1}_n \xrightarrow{H} \mathbb{R}^k_n$ be the natural projection and denote by $\mathfrak{m}$ the ideal $(\epsilon_1, \dots, \epsilon_n)$, then $I'_{\mathbb{R}^k_n} = \mathfrak{m}^k$. On the other hand, if a polynomial $P \in \mathbb{R}^{k+1}_n$ verifies $\frac{\partial P}{\partial \epsilon_i} \in I'_{\mathbb{R}^k_n} = \mathfrak{m}^k$, $i = 1, \dots, m$, then necessarily P belongs to $\mathfrak{m}^{k+1}$ and so $H(P) = 0$. However, $\epsilon_1 \epsilon_2$ defines a non trivial element of $\widehat{I}_A$ when $A = \mathbb{R}[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2)$.

4. The contact system on A-jets

In this section we will construct the contact system on A-jet spaces. The way we have defined the contact system for (m,k)-jets (Section 1) can be mostly translated to the new context. However, there is a number of necessary modifications we will focuses ourselves on.

Let $\mathfrak{p}$ be an A-jet on M and $\{x_i, y_j\}$ a local chart adapted to $\mathfrak{p}$ such that $\mathfrak{p} = (y_j) + (Q_s(x)) + \mathfrak{m}^{k+1}_p$. Taking into account Corollaries 2.21 and 2.22, the set of values $\mathfrak{d}_{\mathfrak{p}} f(D_{\mathfrak{p}}) \in \mathcal{C}^\infty(M)/\mathfrak{p}$, where f runs over $\mathfrak{p}$ and $D_{\mathfrak{p}}$ runs over the tangent spaces to m-dimensional submanifolds of M, equals to

$$((\partial Q_s/\partial x_i) + \mathfrak{m}^k_p)/\mathfrak{p}$$

(compare with Lemma 1.8).

Let us consider the epimorphim

$$\mathcal{C}^\infty(M)/\mathfrak{p} \longrightarrow \mathcal{C}^\infty(M)/(\mathfrak{p} + (\partial Q_s/\partial x_i) + \mathfrak{m}_p^k))$$

and observe that $\mathfrak{p} + (\partial Q_s/\partial x_i) + \mathfrak{m}_p^k$ equals $\pi'(\mathfrak{p})$ (see computation (3.1)).

As in the case of (m,k)-jets, if $f \in \mathfrak{p}$ we can define

$$\mathfrak{d}'_\mathfrak{p} f \overset{\text{def}}{=} \pi' \circ \mathfrak{d}_\mathfrak{p} f \colon T_\mathfrak{p} J^A M \longrightarrow \mathcal{C}^\infty(M)/\pi'(\mathfrak{p})$$

where π' denotes the canonical projection of $\mathcal{C}^\infty(M)/\mathfrak{p}$ onto $\mathcal{C}^\infty(M)/\pi'(\mathfrak{p})$. From the above discussion it follows that $\mathfrak{d}'_\mathfrak{p} f$ vanishes on the tangent subspaces $T_\mathfrak{p} J^A X \subset T_\mathfrak{p} J^A M$.

Remark 4.1. For each tangent vector $D_\mathfrak{p} \in T_\mathfrak{p} J^A M$, we have

$$\mathfrak{d}'_\mathfrak{p} f(D_\mathfrak{p}) = \mathfrak{d}_{\mathfrak{p}'} f(\pi'_* D_\mathfrak{p})$$

where $\mathfrak{p}'$ denotes $\pi'(\mathfrak{p}) \in J^{A'} M$.

Remark 4.2. By the very definition and using the above notation we have $\mathfrak{d}'_\mathfrak{p}(Q_s) = 0$ and $\mathfrak{d}'_\mathfrak{p} \mathfrak{m}_p^{k+1} = 0$ (i.e., $\mathfrak{d}'_p f = 0$ if $f \in (Q_s) + \mathfrak{m}_p^{k+1}$).

Definition 4.3. The distribution of tangent vectors $\mathcal{C}$ given by

$$\mathcal{C}_\mathfrak{p} \overset{\text{def}}{=} \bigcap_{f \in \mathfrak{p}} \ker(\mathfrak{d}'_\mathfrak{p} f) \subset T_\mathfrak{p} J^A M$$

will be called the *contact distribution* on $J^A M$. The Pfaffian system associated with $\mathcal{C}$ will be called the *contact system* on $J^A M$ and we will denote it by Ω.

Remark 4.4. Let $\phi \colon N \longrightarrow M$ be a differentiable map. It is deduced from the definition of the contact system that the jet prolongation $j^A\phi \colon J^A_\phi N \to J^A M$ is a contact transformation, i.e., $(j^A\phi)_* \mathcal{C}_\mathfrak{q} \subseteq \mathcal{C}_{(j^A\phi)\mathfrak{q}}$ for each $\mathfrak{q} \in J^A_\phi N$.

Since the construction of $\mathcal{C}$ and the discussion before Remark 4.1 we have,

Proposition 4.5. *Let X be a submanifold of M with $\dim X = w(A) = m$. The prolongation $J^A X \subset J^A M$ is a solution of the contact distribution.*

Lemma 4.6. *Let $\mathfrak{p}_0 \in J^A M$, $f \in \mathfrak{p}_0$ and $\{x_i, y_j\}$ a local chart adapted to $\mathfrak{p}_0$. For each jet $\mathfrak{p}$ in a neighborhood of $\mathfrak{p}_0$ there exists a polynomial $P_{f,\mathfrak{p}} = P_{f,\mathfrak{p}}(x)$ of degree $\leq o(A)$, such that*

$$f - P_{f,\mathfrak{p}} \in \mathfrak{p}\,.$$

Moreover, the coefficients of $P_{f,\mathfrak{p}}$ can be choosen in such a way that they depend smoothly on $\mathfrak{p}$.

Proof. Let p_0^A be a regular A-velocity with $\ker p_0^A = \mathfrak{p}_0$ and let Λ be a set of multi-indices such that $\{p_0^A(x)^\alpha\}_{\alpha \in \Lambda}$ is a basis of A.

Now, let us consider $a_i = p_0^A(x_i)$ and $b_i = p^A(x_i - x_i(p))$ in Lemma 2 of the Appendix. We deduce the existence of differentiable functions Φ_β in a neighborhood of p_0^A such that

$$p^A(f) = \sum_{\alpha \in \Lambda} \Phi_\alpha(p^A) p^A(x - x(p))^\alpha$$

provided that p^A is near enough of p_0^A. So, $f - \sum \Phi_\alpha(p^A)(x - x(p))^\alpha \in \ker p^A$.

Finally, by taking a local section s of Ker: $\check{M}^A \to J^A M$ defined around $\mathfrak{p}_0$ and such that $s(\mathfrak{p}_0) = p_0^A$, we can choose the polynomials in the statement to be

$$P_{f,\mathfrak{p}} \stackrel{\text{def}}{=} \sum_{\alpha \in \Lambda} \Phi_\alpha(s(\mathfrak{p}))(x - x(p))^\alpha . \qquad \square$$

Theorem 4.7. *The contact distribution is smooth.*

Proof. Let $\mathfrak{p}_0 \in J^A M$. The incident subspace to $\mathcal{C}_{\mathfrak{p}_0}$ is generated by the real components of the $\mathfrak{d}'_{\mathfrak{p}_0} f$ when f runs over $\mathfrak{p}_0$.

This way, the theorem follows if each $\mathfrak{d}'_{\mathfrak{p}_0} f$, $f \in \mathfrak{p}_0$, can be extended in the following sense: for all jet $\mathfrak{p}$ in a neighborhood of $\mathfrak{p}_0$, there is a suitable $\mathcal{C}^\infty(M)/\mathfrak{p}$-linear map $\omega_\mathfrak{p} \colon T_\mathfrak{p} J^A M \to \mathcal{C}^\infty(M)/\mathfrak{p}'$ fulfilling

(1) $\omega_\mathfrak{p}$ annihilates each vector $D_p \in \mathcal{C}_\mathfrak{p}$.
(2) $\omega_\mathfrak{p}$ depends smoothly on $\mathfrak{p}$.
(3) $\omega_{\mathfrak{p}_0} = \mathfrak{d}'_{\mathfrak{p}_0} f$.

Since the lema above, items (1) and (2) hold if we take $\omega_\mathfrak{p} \stackrel{\text{def}}{=} \mathfrak{d}'_\mathfrak{p}(f - P_{f,\mathfrak{p}})$. Item (3) also holds because $\mathfrak{d}'_{\mathfrak{p}_0} P_{f,\mathfrak{p}_0} = 0$ (see Remark 4.2). $\square$

Proposition 4.8. *The vector subspace $\mathcal{C}_\mathfrak{p}$ equals the linear span of the tangent spaces at $\mathfrak{p}$ of the m-dimensional submanifolds X such that $I_X \subset \mathfrak{p}$:*

$$\mathcal{C}_\mathfrak{p} = \sum_{I_X \subset \mathfrak{p}} T_\mathfrak{p} J^A X .$$

Proof. Let a fix local coordinates $\{x_i, y_j\}$ adapted to $\mathfrak{p}$. A tangent vector $D_\mathfrak{p} = \left[\sum_i a_i \frac{\partial}{\partial x_i} + \sum_j b_j \frac{\partial}{\partial y_j}\right]_\mathfrak{p}$ (where we can assume that a_i, b_j are polynomials in the x_i) belongs to $\mathcal{C}_\mathfrak{p}$ if and only if $\pi'(b_j) = 0$. So, $b_j \in \mathfrak{p}' \stackrel{\text{def}}{=} \pi'(\mathfrak{p})$ and therefore $b_j = \sum_{si} b^j_{si} \frac{\partial Q_s}{\partial x_i}$, for suitable polynomials $b^j_{si}(x)$ (see Corollary 3.12). If we denote by H^j_i the sum $\sum_s b^j_{si} Q_s$ we will have $D_\mathfrak{p} = \left[\sum_i a_i \frac{\partial}{\partial x_i} + \sum_{ij} \frac{\partial H^j_i}{\partial x_i} \frac{\partial}{\partial y_j}\right]_\mathfrak{p}$.

Next, let us consider the following submanifolds: $X_0 = \{y_j = 0\} \stackrel{i_0}{\hookrightarrow} M$, $X_h = \{y_j = H^j_h(x)\} \stackrel{i_h}{\hookrightarrow} M$, $h = 1, \ldots, m$. Then, a calculation gives

$$D_\mathfrak{p} = (j^A i_0)_* \left[\sum_i a_i \frac{\partial}{\partial x_i}\right]_\mathfrak{p} + \sum_h \left((j^A i_h)_* \left[\frac{\partial}{\partial x_h}\right]_\mathfrak{p} - (j^A i_0)_* \left[\frac{\partial}{\partial x_h}\right]_\mathfrak{p} \right)$$

which belongs to $T_\mathfrak{p} J^A X_0 + \sum_h T_\mathfrak{p} J^A X_h$. $\square$

Remark 4.9. An easy consequence follows. Let $\pi' \colon J^A M \to J^{A'} M$ be the natural projection and $\mathfrak{p}' = \pi'(\mathfrak{p})$, $\mathfrak{p} \in J^A M$. If $w(A') = w(A) = m$, then $\pi'_* \mathcal{C}_\mathfrak{p} \subset \mathcal{C}_{\mathfrak{p}'}$.

Lemma 4.10. *Let $U \subset J^A M$ be a solution of the contact system and $\mathfrak{p}$ a jet in U. Then $\dim \pi'_* T_\mathfrak{p} U \leq \dim J^{A'} \mathbb{R}^m$ where $m = w(A)$. Moreover, if $\mathfrak{p} \supset I_X$, where I_X is the ideal of a given m-dimensional submanifold X, then $\pi'_* T_\mathfrak{p} U \subseteq T_{\mathfrak{p}'} J^{A'} X$.*

Proof. It is sufficient to show the second part in the claim. If $\mathfrak{p} \supset I_X$, also we have $\mathfrak{p}' \supset I_X$ (that is, $\mathfrak{p}' \in J^{A'}X$). Then, by using Remark 4.1, for each given tangent vector $D_\mathfrak{p} \in T_\mathfrak{p}U \subset \mathcal{C}_\mathfrak{p}$ we have

$$\mathfrak{d}_\mathfrak{p} f(\pi'_* D_\mathfrak{p}) = \mathfrak{d}'_\mathfrak{p} f(D_\mathfrak{p}) = 0\,, \qquad \forall f \in I_X\,.$$

From the version of Theorem 1.7 in the case of A-jets, it follows that $\pi'_* D_\mathfrak{p} \in T_{\mathfrak{p}'}J^{A'}X$. $\square$

Lemma 4.11. *Let $U \subset J^AM$ be a solution of the contact system which contains J^AX, where X is an m-dimensional submanifold of M. If $\mathfrak{p} \in J^AX$, there exist a neighborhood of $\pi'(\mathfrak{p}) = \mathfrak{p}'$ where*

$$\pi'(U) = J^{A'}X\,.$$

Proof. By applying the lema above to the inclusion $J^AX \subseteq U$ we have

$$T_{\mathfrak{p}'}J^{A'}X \subseteq \pi'_* T_\mathfrak{p} U \subseteq T_{\mathfrak{p}'}J^{A'}X\,.$$

So, the equality holds and the dimension of $\pi'_* T_\mathfrak{p} U$ is the highest possible. Therefore, the rank of $\pi'|_U$ is constant in a neighborhood of $\mathfrak{p}$. We deduce that, in a neighborhood of $\mathfrak{p}'$, $\pi'(U)$ is a submanifold. Moreover, also locally, $\pi'(U)$ contains $J^{A'}X$ and $\dim \pi'(U) = \dim J^{A'}X$. As a consequence, near of $\mathfrak{p}'$, $\pi'(U) = J^{A'}X$. $\square$

Finally, the proof of the maximality of the solutions J^AX requires an additional hypothesis on the algebra A.

Theorem 4.12. *Let us suppose that $\widehat{I}_A = 0$. The prolongations $J^AX \subseteq J^AM$ (with $\dim X = m = w(A)$) are maximal solutions of the contact system. In other words, if $J^AX \subseteq U \subseteq J^AM$ where U is a solution of the contact system, then $\dim J^AX = \dim U$.*

Proof. Let $\mathfrak{p} \in J^AX \subseteq U$ with $\mathfrak{p}' = \pi'(\mathfrak{p})$ and let us suppose that $\overline{\mathfrak{p}} \in U$ is another jet such that $\pi'(\overline{\mathfrak{p}}) = \pi'(\mathfrak{p}) = \mathfrak{p}'$ and $\overline{\mathfrak{p}} \notin J^AX$.

In a suitable local chart $\{x_i, y_j\}$ we have $I_X = (y_j)$ and

$$\overline{\mathfrak{p}} = (y_j - P_j(x)) + (\overline{Q}_s(x)) + \mathfrak{m}_p^{k+1}\,,$$

for certain polynomials $P_j(x), \overline{Q}_s(x)$, where at least one among the P', say $P_{j_0}(x)$, is not in $\overline{\mathfrak{p}}$ (elsewhere, $\overline{\mathfrak{p}} \supset I_X$, and then $\overline{\mathfrak{p}} \in J^AX$, in contradiction with the above assumption).

For each given index i, let us pick a tangent vector $D_{\overline{\mathfrak{p}}} = [D]_{\overline{\mathfrak{p}}} \in T_{\overline{\mathfrak{p}}}U$ such that $\pi'_* D_{\overline{\mathfrak{p}}} = [\frac{\partial}{\partial x_i}]_{\mathfrak{p}'} \in T_{\mathfrak{p}'}J^{A'}X$, which is always possible according to Lemma 4.11. From U being a solution of the contact system, we get

$$0 = \mathfrak{d}'_{\overline{\mathfrak{p}}}(y_{j_0} - P_{j_0})(D_{\overline{\mathfrak{p}}}) = \mathfrak{d}_{\overline{\mathfrak{p}}}(y_{j_0} - P_{j_0})(\pi'_* D_{\overline{\mathfrak{p}}}) = -\left[\frac{\partial P_{j_0}}{\partial x_i}\right]_{\mathfrak{p}'}\,.$$

It is deduced that $\frac{\partial P_{j_0}}{\partial x_i} \in \pi'(\overline{\mathfrak{p}}) = \mathfrak{p}'$. Moreover, $P_{j_0} \in \pi'(\overline{\mathfrak{p}})$ because $y_{j_0} - P_{j_0} \in \overline{\mathfrak{p}} \subset \pi'(\overline{\mathfrak{p}})$ and $y_{j_0} \in \mathfrak{p} \subset \mathfrak{p}' = \pi'(\overline{\mathfrak{p}})$. This way, we have a polynomial $P_{j_0} \notin \overline{\mathfrak{p}}$ but

$P_{j_0}, \frac{\partial P_{j_0}}{\partial x_i} \in \pi'(\overline{\mathfrak{p}})$, $i = 1, \dots, m$. As a consequence, P_{j_0} belongs to the ideal $\widehat{I}$ of $\mathcal{C}^\infty(M)/\overline{\mathfrak{p}} \simeq A$ and then $\widehat{I}_A \neq 0$. $\square$

Corollary 4.13. *On the spaces $J^k_m M$ the prolongations of m-dimensional submanifolds of M are maximal solutions of the contact system.*

Proof. It is sufficient to taking into account Example 3.14. $\square$

Appendix

Lemma 1. *Let $H, \overline{H} \colon \mathbb{R}^k_n \to A$ be $\mathbb{R}$-algebra epimorphisms; then there exists an automorphism $g \in \mathrm{Aut}(\mathbb{R}^k_n)$ such that $H = \overline{H} \circ g$.*

Proof. If the classes of $a_1, \dots, a_m$ generate $\mathfrak{m}_A/{\mathfrak{m}_A}^2$, one easily deduces that each element in A can be obtained as a polynomial on $a_1, \dots, a_m$. It is not difficult to see that elements $x_1, \dots, x_n$ can be chosen in $\mathbb{R}^k_n$ such that they generate the maximal ideal and we have $H(x_i) = a_i$ if $i \leq m$ and $H(x_{m+j}) = 0$. Analogously, we can choose a elements $\overline{x}_1, \dots, \overline{x}_n$ which hold the same property with respect to $\overline{H}$. Finally, we define g by the condition of mapping the first basis to the second one. $\square$

Lemma 2. *Let $\{a_i\}$ be a basis of $\mathfrak{m}_A$ modulo $\mathfrak{m}_A^2$ and let us choose a collection of multi-indices Λ such that the set $\{a^\alpha\}_{\alpha\in\Lambda}$ is a basis of $\mathfrak{m}_A$. Then, there exist rational functions $\Psi_{\alpha\beta}$, $\alpha, \beta \in \Lambda$ such that for any other basis $\{b_i\}$ of $\mathfrak{m}_A$ modulo $\mathfrak{m}_A^2$, near enough of $\{a_i\}$ we have*

$$a^\alpha = \sum_{\beta\in\Lambda} \Psi_{\alpha\beta}(\lambda_{i\sigma}) b^\beta \,, \qquad \alpha \in \Lambda \,,$$

where $b_i = \sum_{i\sigma\in\Lambda} \lambda_{i\sigma} a^\sigma$.

Proof. Let us suppose the multiplication law on A being $a^\alpha a^\sigma = \sum_{\gamma\in\Lambda} c^\gamma_{\alpha\sigma} a^\gamma$, $c^\gamma_{\alpha\sigma} \in \mathbb{R}$ (structure constants).

Because each b_i is near enough of a_i, $i = 1, \dots, m$ we deduce that the set of powers $\{b^\beta\}_{\beta\in\Lambda}$ is also a basis of $\mathfrak{m}_A$.

From $b_i = \sum_{i\sigma\in\Lambda} \lambda_{i\sigma} a^\sigma$ we can write each b^β as a linear combination of the a^α, $\alpha \in \Lambda$ whose coefficients are polynomials in the $\lambda_{i\sigma}$ (multiplication law of A). These linear relations can be inverted and we get the required expressions for a^α. $\square$

Acknowledgements. We would like to thank Prof. P. Arias, S. Jiménez and J. Rodríguez for useful discussions and suggestions.

References

[1] Alonso Blanco, R. J., *Jet manifold associated to a Weil bundle*, Arch. Math. (Brno) **36** (2000), 195–199.

[2] ———, *On the local structure of A-jet manifolds*, In: Proceedings of Diff. Geom. and its Appl. (Opava, 2002), Math Publ. **3**, Silesian Univ. Opava 2001, 51–61.

[3] Jiménez, S., Muñoz, J., Rodríguez, J., *On the reduction of some systems of partial differential equations to first order systems with only one unknown function*, In: Proceedings of Diff. Geom. and its Appl. (Opava, 2002), Math. Publ. **3**, Silesian Univ. Opava 2001, 187–195.

[4] Kolář, I., *Affine structure on Weil bundles*, Nagoya Math. J. **158** (2000), 99–106.

[5] Kolář, I., Michor, P. W. and Slovák, J., Natural Operations in Differential Geometry, Springer-Verlag, 1993.

[6] Lie, S., *Theorie der Transformationsgruppen*, Leipzig, 1888. (Second edition in Chelsea Publishing Company, New York 1970).

[7] Muñoz, J., Muriel, J., Rodríguez, J., *The canonical isomorphism between the prolongation of the symbols of a nonlinear Lie equation and its attached linear Lie equation*, in Proccedings of Diff. Geom. and its Appl. (Brno, 1998), Masaryk Univ., Brno 1999, 255–261.

[8] ———, *Integrability of Lie equations and pseudogroups*, J. Math. Anal. Appl. **252** (2000), 32–49.

[9] ———, *Weil bundles and jet spaces*, Czechoslovak Math. J. **50** (125) (2000), no. 4, 721–748.

[10] ———, *A remark on Goldschmidt's theorem on formal integrability*, J. Math. Anal. Appl. **254** (2001), 275–290.

[11] ———, *The contact system on the (m, l)-jet spaces*, Arch. Math. (Brno) **37** (2001), 291–300.

[12] ———, *On the finiteness of differential invariants*, J. Math. Anal. Appl. **284** (2003), No. 1, 266–282.

[13] Rodríguez, J., *Sobre los espacios de jets y los fundamentos de la teoría de los sistemas de ecuaciones en derivadas parciales*, Ph. D. Thesis, Salamanca, 1990.

[14] Weil, A., *Théorie des points proches sur les variétés différentiables*, Colloque de Géometrie Différentielle, C. N. R. S. (1953), 111–117.

Departamento de Matemáticas, Universidad de Salamanca
Plaza de la Merced 1-4, E-37008 Salamanca, Spain
E-mail: `ricardo@usal.es` `clint@usal.es`