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A CHARACTERIZATION OF ESSENTIAL SETS OF FUNCTION ALGEBRAS

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Abstract. In the present note, we characterize the essential set E of a function algebra A defined on a compact Hausdorff space X in terms of local properties of functions in A at the points off E.

Let X be a compact Hausdorff topological space. Denote by $C(X)$ the commutative Banach algebra, consisting of all continuous complex-valued functions on X (with respect to usual point-wise algebraic operations) endowed with the sup-norm.

By a *function algebra* on X we mean any closed subalgebra of $C(X)$ which contains constant functions on X and which separates points of X.

Definition. A function algebra A on X is said to be a *maximal* one if it is a proper subset (i.e., a proper subalgebra) of $C(X)$ and has the following property: whenever B is a function algebra on X, $B \supset A$, then either $B = A$ or $B = C(X)$.

A being a function algebra on X, a closed subset $E \subset X$ is said to be an *essential set of* A if the following conditions are fulfilled:

(1) A consists of *all* continuous prolongations of functions in the algebra of restrictions A/E (i.e., the algebra of all restrictions of functions in A from the set X to its subset E).

(2) Whenever a closed subset F of X has the same property as E in (1), then $E \subset F$ (or, E is a unique minimal closed subset of X satisfying the condition (1)).

The notion "essential set" is due to Bear, who proved in [1] that any maximal algebra on X has an essential set.

Hoffman and Singer in [2] found an essential set of any, not necessarily maximal, function algebra on X.

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Denote by $M(X)$ the space of all complex Borel regular measures on X, i.e., by the Riesz Representation Theorem, the dual space of $C(X)$.

The *annihilator* $A^{\perp}$ of a function algebra A is defined to be the set of all measures $m \in M(X)$ such that $\int f\,dm = 0$ for any $f \in A$, or the set of all measures *orthogonal* to A. The dual space A' of A is then canonically isomorphic to the quotient space $M(X)/A^{\perp}$.

Now endow $M(X)$ with the weak-star topology: it is well known that $M(X)$ becomes a locally convex topological linear space with the dual space $C(X)$.

Definition. Let A be a function algebra on X. A (closed nonvoid) set $F \subset X$ is said to be a *peak set* (of A) if there exists a function $f \in A$ with the following properties:

(1) $f(x) = 1$ for any $x \in F$;
(2) $|f(y)| < 1$ for any $y \in X \setminus F$.

In this case we say that f *peaks* on F.

In [3], we have proved the following

Theorem 1. *Let A be a function algebra on X. Denote by E the closure of the union of all closed supports of measures in $A^{\perp}$. Then E is the essential set of A.*

Our aim here is to characterize the essential set E of a function algebra A in terms of local properties of functions in A at the points off E. More precisely, we shall prove the following

Theorem 2. *Let A be a function algebra on X. Denote by E its essential set. Let $x \in X$. Then $x \in X \setminus E$ if and only if there exists a closed neigbourhood V of x in X such that the following two conditions are fulfilled:*

(3) *$A/V = C(V)$, where A/V means the algebra of all restrictions of functions from A to the set V;*
(4) *V is an intersection of peak sets of A.*

Proof. Let at first $x \in X \setminus E$.

Take as V such an closed neighbourhood which does not meet E.

Condition (3) follows immediately from the definition of the essential set.

For any $y \in X \setminus (E \cup V)$ let f_y^0 be a function defined on the set $H_y = E \cup \{y\} \cup V$ such that it is equal to 1 on V and to 0 on $E \cup \{y\}$. We can, by the classical Tietze Theorem, construct a function $\tilde{f}_y \in C(X)$ which is equal to f_y^0 on the set H_y. Finally, put $f_y = \min(1, \tilde{f}_y)$. Then $f_y \in C(X)$ and f is equal to 0 on E; it follows from the definition of the essential set that $f_y \in A$.

Denote the set on which f_y peaks by F_y. Then $F_y \supset V$ and F_y does not meet $E \cup \{y\}$. It follows that

$$V = \bigcap_{y \in X \setminus (E \cup V)} F_y\,,$$

the condition (4).

Let, on the contrary, be V such closed neighborhood of x that the conditions (3), (4) are fulfilled. Let m is a measure on X such that spt m, its closed support, has nonvoid intersection with int V, the interior of V. We shall prove that m is not in $A^{\perp}$; it will follow from Theorem 1 that $x \notin E$.

Let $f \in C(V)$ be such that

$$\text{spt} f \ \subset \text{int}\, V\,, \qquad \int_V f\ dm \neq 0\,. \tag{5}$$

It follows from (3) that there exists a function $g \in A$ such that $g/V = f$. It is $f = g = 0$ on the boundary of V and then the the sets

$$U_n \equiv V \cup \{y \in X; |g(y)| < \frac{1}{n}, n = 1, 2, \dots\} \tag{6}$$

containing V are open.

The set $X \setminus U_n$ is a compact one; the system S of all peak sets of A containing V is a system of compact sets whose intersection is V by (4). It follows that there is a finite subsystem $F_1, F_2, \dots, F_k$ of S such that

$$V_n \equiv \bigcap_{j=1}^{k} F_j \subset U_n\,. \tag{7}$$

But the (nonvoid) intersection of peak sets is a peak set: if f_j peaks on F_j, then $\prod f_j$ peaks on $\cap F_j$. We have proved: there exists a sequence $V_n, n = 1, 2, \dots$ of peak sets of A such that

$$V \subset V_n \subset U_n\,, \quad n = 1, 2, \dots \tag{8}$$

It is easy to see that the intersection $W \equiv \bigcap_{n=1}^{\infty} V_n$ is a peak set of A: if $h_n \in A$ peaks on V_n, then the function

$$h \equiv \sum_{n=1}^{\infty} 2^{-n} h_n$$

peaks on W. It follows from (7) and (8) that

$$V \subset W \subset V \cup \{y \in X; g(y) = 0\}\,. \tag{9}$$

We have

$h^n(y) = 1$ for $y \in W$,

$h^n(y) \to 0$ for $n \to \infty$, $y \in X \setminus V$.

It follows from (9) that

$(g \cdot h^n)(y) = g(y) \leq f(y)$ for $y \in V$,

$(g \cdot h^n)(y) \to 0$ for $n \to \infty$, $y \in X \setminus W$.

Since $|g \cdot h^n| = |g| \cdot |h|^n = |g|$, we have by the Lebesgue Dominated Convergence Theorem and by (9) and (5)

$$\int g \cdot h^n \; dm \to \int_W g \; dm = \int_V g \; dm = \int_V f \; dm \neq 0 \, .$$

But $g \cdot h^n \in A$ for $n = 1, 2, \ldots$ and then the measure m is not in $A^\perp$. □

At first look at Theorem 1 and 2 it would appear that the condition (4) in Theorem 2 is superfluous and could be omitted. The next example shows that it is not the case.

Example. Let A be the *classical disk algebra*, i.e. the algebra of all functions continuous on the closed unit disk K in the complex plane which are holomorphic on the interior of K.

Let B be the restriction of A to the set

$$F \equiv \{z \in K; |z| = 1 \text{ or } z = 0\} \, .$$

Zero is an isolated point of F; it follows that $B/\{0\} = C(\{0\})$.

Let μ be such a measure on F that for any $f \in C(F)$

$$\int f \; d\mu = \frac{1}{2\pi i} \int\limits_{|z|=1} \frac{f(z) \; dz}{z} - f(0) \, .$$

Then $\mu \in B^\perp$ by Cauchy Formula. There is $|\mu|(0) = 1$, so 0 is in the essential set of B, by Theorem 1.

References

[1] Bear, H. S., *Complex function algebras*, Trans. Amer. Math. Soc. **90** (1959), 383–393.

[2] Hoffman, K., Singer, I. M., *Maximal algebras of continuous functions*, Acta Math. **103** (1960), 217–241.

[3] Čerych, J., *On essential sets of function algebras in terms of their orthogonal measures*, Comment. Math. Univ. Carolin. **36**, 3 (1995), 471–474.

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