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EXISTENCE RESULTS FOR BOUNDARY VALUE PROBLEMS FOR FOURTH-ORDER DIFFERENTIAL INCLUSIONS WITH NONCONVEX VALUED RIGHT HAND SIDE

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Abstract. In this paper a fixed point theorem due to Covitz and Nadler for contraction multivalued maps, and the Schaefer's theorem combined with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued operators with decomposables values, are used to investigate the existence of solutions for boundary value problems of fourth-order differential inclusions.

1. Introduction

In the past few years, by means of fixed point argument and the monotone method, combined with the upper and lower solutions method, several papers have been devoted to the study of boundary value problems for fourth order ordinary differential equations on real compact intervals (see for instance, [1], [2], [4], [7], [9], [14], [15] and [16] and the references cited therein). However very few results are devoted for similar problems with multivalued right hand side. Recently, by means of the Ky Fan fixed point theorem Švec [18] gave an existence result for the fourth order differential inclusion:

$$L_4y(t) + a(t)y(t) \in F(t, y(t))\,, \quad \text{for a.e.} \quad t \in [0, T] \tag{1.1}$$

$$L_iy(0) = L_iy(T), \quad i = 0, 1, 2, 3, \tag{1.2}$$

where

$$L_0y(t) = a_0(t)y(t)\,, \quad L_iy(t) = a_i(t)(L_{i-1}y(t))'\,, \quad i = 1, 2, 3,$$
$$L_4y(t) = \big(a_1(t)\big(a_2(t)\big(a_1(t)(a_0(t)y(t))'\big)'\big)'\big)'$$

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$a_0(t) = 1$, $a(t) \geq 0$, $a_i(t) > 0$, $i = 1, 2$, $a_1(t) = a_3(t)$, continuous on $[0, T]$, $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with nonempty compact and convex values and $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$. The aim of this paper is to generalize Švec's result in considering a nonconvex valued right hand side. This paper will be divided into three sections. In Section 2 we will recall briefly some basic definitions and preliminary facts from multivalued analysis which will be used throughout. In Section 3 we establish two existence theorems for the problem (1.1)-(1.2). The first one relies on a fixed point theorem due to Covitz and Nalder [6] for contraction multivalued maps. In the second we use the Schaefer's fixed point theorem combined with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued operators with nonempty closed and decomposable values.

2. Preliminaries

We will briefly recall some basic definitions and facts from multi valued analysis that we will used throughout this paper.

$C([0, T], \mathbb{R})$ is the Banach space of all continuous functions from $[0, T]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \leq t \leq T\}\,.$$

$L^1([0, T], \mathbb{R})$ denotes the Banach space of functions $y : [0, T] \longrightarrow \mathbb{R}$ which are Lebesgue integrable normed by

$$\|y\|_{L^1} = \int_0^T |y(t)|\, dt\,.$$

$AC^i(J, \mathbb{R}^n)$, $i = 0, 1, 2, 3$ is the space of i-times differentiable functions $y : J \to \mathbb{R}^n$, those i^{th} derivative, $y^{(i)}$, is absolutely continuous.

Let $(X, |\cdot|)$ be a normed space. Let $\mathcal{P}_{\text{cl}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_{\text{b}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_{\text{cp}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$. The multivalued map $F : [0, T] \to \mathcal{P}_{\text{cl}}(X)$ is said to be mesurable, if for every $y \in X$, the function

$$t \longmapsto d(y, F(t)) = \inf\{|y - z| : z \in F(t)\}$$

is measurable.

Let $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C([0, T], \mathbb{R}) \to \mathcal{P}(L^1([0, T], \mathbb{R}))$$

by letting

$$\mathcal{F}(y) = \{w \in L^1([0, T], \mathbb{R}) : w(t) \in F(t, y(t)) \quad \text{for a.e.} \quad t \in [0, T]\}\,.$$

The operator $\mathcal{F}$ is called the *Niemytzki operator* associated to F.

Let (X,d) be a metric space induced from the normed space $(X,|\cdot|)$. Consider $H_d : \mathcal{P}(X)\times\mathcal{P}(X) \longrightarrow \mathbb{R}_+\cup\{\infty\}$, given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$.

Then $(\mathcal{P}_{\mathrm{b,cl}}(X), H_d)$ is a metric space and $(\mathcal{P}_{\mathrm{cl}}(X), H_d)$ is a generalized (complete) metric space [13].

Definition 2.1. A multivalued operator $G : X \to \mathcal{P}_{\mathrm{cl}}(X)$ is called

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(G(x), G(y)) \le \gamma d(x,y), \quad \text{for each} \quad x,\ y \in X;$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Our considerations are based on the following fixed point theorem for contraction multivalued operators given by Covitz and Nadler in 1970 [6] (see also Deimling, [8] Theorem 11.1).

Lemma 2.2. *Let (X,d) be a complete metric space. If $G : X \to \mathcal{P}_{\mathrm{cl}}(X)$ is a contraction, then* $\operatorname{Fix} G \neq \emptyset$.

Let A be a subset of $[0,T]\times\mathbb{R}$. A is $\mathcal{L}\otimes\mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J}\times D$ where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and D is Borel measurable in $\mathbb{R}$. A subset A of $L^1([0,T],\mathbb{R})$ is decomposable if for all $u,v\in A$ and $\mathcal{J}\subset[0,T]$ measurable, $u\chi_{\mathcal{J}} + v\chi_{[0,T]-\mathcal{J}} \in A$, where χ stands for the characteristic function.

Let $G : X \to \mathcal{P}(X)$ a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x\in X : G(x)\cap B \neq \emptyset\}$ is open for any open set B in X.

Definition 2.3. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator. We say N has property (BC) if

1) N is lower semi-continuous (l.s.c.);
2) N has nonempty closed and decomposable values.

Definition 2.4. Let $F : [0,T]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continous and has nonempty closed and decomposable values.

Next we state a selection theorem due to Bressan and Colombo.

Theorem 2.5. *Let Y be separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator which has property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $\tilde{g} : Y \to L^1([0,T],\mathbb{R})$ such that $\tilde{g}(y)\in N(y)$ for every $y\in Y$.*

For more details on multivalued maps see the books of Deimling [8], Gorniewicz [11] and Hu and Papageorgiou [12].

3. Main Result

Before stating and proving our main results we present the following auxiliary lemma which will be used later.

Lemma 3.1 [18]. *The boundary value problem*

$$L_4 y(t) + a(t)y(t) = 0\,, \quad \textit{for all} \quad t \in [0,T]\,. \tag{3.1}$$

$$L_i y(0) = L_i y(T)\,, \quad i = 0,1,2,3\,, \tag{3.2}$$

has only trivial solution $y(t) = 0$, *on* $[0,T]$.

Definition 3.2. A function $y \in AC^3((0,T),\mathbb{R})$ is said to be a solution of (1.1)–(1.2) if $L_i y(0) = L_i y(T)$ $i = 1,2,3$, and there exists $v \in L^1([0,T],\mathbb{R})$ such that $v(t) \in F(t,y(t))$ a.e. $t \in [0,T]$ and

$$L_4 y(t) + a(t)y(t) = v(t) \quad \text{for a.e.} \quad t \in [0,T]\,.$$

The first result of this section is based on the fixed point theorem for contraction multivalued operators given by Covitz and Nadler in 1970 [6] (see also Deimling, [8] Theorem 11.1).

Theorem 3.3. *Assume that the following hypotheses*

(H1) $F : [0,T] \times \mathbb{R} \longrightarrow \mathcal{P}_{\mathrm{cp}}(\mathbb{R})$; $(t,y) \longmapsto F(t,y)$ *is mesurable for each* $y \in \mathbb{R}$;
(H2) *There exists a function* $l \in L^1([0,T],\mathbb{R}^+)$ *such that*

$$H_d(F(t,y),F(t,\overline{y})) \le l(t)|y-\overline{y}|\,, \quad \textit{for a.e.} \quad t \in [0,T] \quad \textit{and all} \quad y,\overline{y} \in \mathbb{R}$$

and

$$H_d(0,F(t,0)) \le l(t) \quad \textit{a.e. on} \quad [0,T]\,,$$

are satisfied. Let $G^* = \sup\{|G(t,s)| : \ (t,s) \in [0,T] \times [0,T]\}$ *and* $l^* = \displaystyle\int_0^T l(s)\,ds$. *If* $G^* l^* < 1$, *then the problem* (1.1)–(1.2) *has at least one solution.*

Proof. For each $y \in C([0,T],\mathbb{R})$ the set $\mathcal{F}(y)$ is nonempty since by (H1) F has a measurable selection (see [5], Theorem III.6) i.e. there exists a function $v \in \mathcal{F}(y)$. Then we seek a solution $y(t)$ of the problem

$$L_4 y(t) + a(t)y(t) = v(t)\,, \quad \text{for a.e} \quad t \in [0,T]\,. \tag{3.3}$$

$$L_i y(0) = L_i y(T)\,, \quad i = 0,1,2,3\,. \tag{3.4}$$

From Lemma 3.1, the solution of the problem (3.3)–(3.4) is given by

$$y(t) = \int_0^T G(t,s)v(s)\,ds\,. \tag{3.5}$$

Consider the multivalued operator $N : C([0,T],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,T],\mathbb{R}))$ defined by:

$$N(y) = \left\{ h \in C([0,T],\mathbb{R}) : h(t) = \int_0^T G(t,s)v(s)\,ds,\ v \in \mathcal{F}(y) \right\}.$$

It is clear that the fixed points of N are solutions to (1.1)–(1.2). We shall show that N satisfies the assumptions of Lemma 2.2. The proof will be given in two steps.

Step 1. *$N(y) \in \mathcal{P}_{\mathrm{cl}}(C([0,T],\mathbb{R}))$ for each $y \in C([0,T],\mathbb{R})$.* Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \longrightarrow \tilde{y}$ in $C([0,T],\mathbb{R})$. Then $\tilde{y} \in C([0,T],\mathbb{R})$ and there exists $g_n \in \mathcal{F}(y)$ such that for each $t \in J$,

$$y_n(t) = \int_0^T G(t,s)g_n(s)\,ds\,.$$

Using the fact that F has compact values and from (H2) we may pass to a subsequence if necessary to get that g_n converges to g in $L^1(J,\mathbb{R})$ and hence $g \in \mathcal{F}(y)$. Then for each $t \in [0,T]$

$$y_n(t) \to \tilde{y}(t) \in \int_0^T G(t,s)g(s)\,ds\,.$$

So $\tilde{y} \in N(y)$, and in particular, $N(y) \in P_{\mathrm{cl}}(C([0,T],\mathbb{R}))$.

Step 2. There exists $\gamma < 1$, such that $H_d(N(y), N(\overline{y})) \leq \gamma\|y-\overline{y}\|_\infty$ for each $y, \overline{y} \in C([0,T],\mathbb{R})$.

Let $y, \overline{y} \in C([0,T],\mathbb{R})$ and $h \in N(y)$. Then there exists $v(t) \in F(t,y(t))$ such that for each $t \in [0,T]$

$$h(t) = \int_0^T G(t,s)v(s)\,ds\,.$$

From (H2) it follows that

$$H_d(F(t,y(t)), F(t,\overline{y}(t)) \leq l(t)|y(t)-\overline{y}(t)|\,.$$

Hence there is $w \in F(t,\overline{y}(t))$ such that

$$|v(t)-w| \leq l(t)|y(t)-\overline{y}(t)|\,, \quad t \in [0,T]\,.$$

Consider $U : [0,T] \to \mathcal{P}(\mathbb{R})$, given by

$$U(t) = \{w \in \mathbb{R} : |v(t)-w| \leq l(t)|y(t)-\overline{y}(t)|\}\,.$$

Since the multivalued operator $V(t) = U(t) \cap F(t,\overline{y}(t))$ is measurable (see Proposition III.4 in [5]), there exists a function $\overline{v}(t)$, which is a measurable selection for V. So, $\overline{v}(t) \in F(t,\overline{y}(t))$ and

$$|v(t)-\overline{v}(t)| \leq l(t)|y(t)-\overline{y}(t)|\,, \quad \text{for each} \quad t \in [0,T]\,.$$

Let us define for each $t \in [0,T]$

$$\overline{h}(t) = \int_0^T G(t,s)\overline{v}(s)\,ds\,.$$

Then we have

$$|h(t)-\overline{h}(t)| \le \int_0^T G^*|v(s)-\overline{v}(s)|\,ds \le G^* \int_0^T l(s)|y(s)-\overline{y}(s)|\,ds \le G^* l^* \|y-\overline{y}\|_\infty\,.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y), N(\overline{y})) \le G^* l^* \|y-\overline{y}\|_\infty\,.$$

Hence N is a contraction and thus, by Lemma 2.2, N has a fixed point y, which is a solution to (1.1)–(1.2). □

We present now a result for the problem (1.1)–(1.2) in the spirit of the Schaefer's theorem.

Theorem 3.4. *Suppose that the folowing hypotheses*

(H3) $F : J \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ *is a nonempty compact valued multivalued map such that:*
- a) $(t,y) \mapsto F(t,y)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable;*
- b) $y \mapsto F(t,y)$ *is lower semi-continuous for a.e.* $t \in J$*;*

(H4) *there exists a function* $h \in L^1(J,\mathbb{R}^+)$ *such that*

$$\|F(t,y)\| := \sup\{|v| : v \in F(t,y)\} \le h(t) \quad \textit{for a.e. } t \in J \textit{ and for } \quad y \in \mathbb{R}$$

are satisfied. Then the boundary value problem (1.1)–(1.2) *has at least one solution.*

Proof. (H3) and (H4) imply by Lemma 2.2 of Frigon [10] that F is of lower semi-continuous type. Then from Theorem 2.5 there exists a continuous function $f : C([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in C([0,T),\mathbb{R})$. Consider the following problem

$$L_4 y(t) + a(t)y(t) = f(y)(t)\,, \quad t \in [0,T] \tag{3.6}$$

$$L_i y(0) = L_i y(T)\,, \quad i = 0,1,2,3\,. \tag{3.7}$$

It is clear that if $y \in AC^3([0,T],\mathbb{R})$ is a solution of the problem (3.6)–(3.7), then y is a solution to the problem (1.1)–(1.2).

Transform the problem (3.6)–(3.7) into a fixed point problem. Consider the operator $N : C([0,T],\mathbb{R}) \longrightarrow C([0,T],\mathbb{R})$ defined by:

$$N(y)(t) = \int_0^T G(t,s) f(y)(s)\,ds\,.$$

We shall show that N is a completely continuous operator. The proof will be given in four steps.

Step 1. *N is continuous.*
Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([0,T],\mathbb{R})$. Then

$$\begin{aligned}|N(y_n)(t) - N(y)(t)| &\le \int_0^T |G(t,s)||f(y_n)(s) - f(y)(s)|\,ds\\ &\le \max_{(t,s)\in[0,T]\times[0,T]} |G(t,s)| \int_0^T |f(y_n)(s) - f(y)(s)|\,ds\,.\end{aligned}$$

Since the funtion f is continuous, then

$$\|N(y_n) - N(y)\|_\infty \le G^*\|f(y_n) - f(y)\|_{L^1} \to 0 \quad \text{as} \quad n \to \infty\,.$$

Step 2. *N maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$.*
Indeed, it is enough to show that for each $q > 0$, there exists a positive constant ℓ such that for each $y \in B_q = \{y \in C([0,T],\mathbb{R}) : \|y\|_\infty \le q\}$ we have $\|N(y)\|_\infty \le \ell$. We have:

$$|N(y)(t)| = \Big|\int_0^T G(t,s)f(y)(s)\,ds\Big| \le G^* \int_0^T |f(y)(s)|\,ds \le G^* \int_0^T h(s)\,ds := \ell\,.$$

Step 3. *N maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$.*

Let $\tau_1,\ \tau_2 \in [0,T]$, $\tau_1 < \tau_2$ and B_q be a bounded set of $C([0,T],\mathbb{R})$. Let $y \in B_q$. Then

$$\begin{aligned}|N(y)(\tau_2) - N(y)(\tau_1)| &\le \Big|\int_0^{\tau_2} G(\tau_2,s)f(y)(s)\,ds - \int_0^{\tau_1} G(\tau_1,s)f(y)(s)\,ds\Big|\\ &\le \int_{\tau_1}^{\tau_2} |G(\tau_1,s)||f(y)(s)|\,ds\\ &\quad + \int_0^{\tau_2} |G(\tau_1,s) - G(\tau_2,s)||f(y)(s)|\,ds\\ &\le G^* \int_{\tau_1}^{\tau_2} h(s)\,ds + \int_0^{\tau_2} |G(\tau_1,s) - G(\tau_2,s)|h(s)\,ds.\end{aligned}$$

As $\tau_2 \longrightarrow \tau_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3 together with the Arzelà-Ascoli theorem we can conclude that N is completely continuous.

Step 4. *Now it remains to show that the set*

$$\mathcal{E}(N) := \{y \in C([0,T],\mathbb{R}) : y = \lambda N(y)\,, \quad \textit{for some} \quad 0 < \lambda < 1\}$$

is bounded.

Let $y \in \mathcal{E}(N)$. Then $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus for each $t \in [0, T]$

$$y(t) = \lambda \int_0^T G(t,s) f(y)(s)\, ds\,.$$

This implies that for each $t \in [0, T]$ we have

$$|y(t)| \leq G^* \int_0^T |f(y(s))|\, ds \leq G^* \int_0^T h(s)\, ds := \widetilde{K}\,.$$

Set $X := C([0, T], \mathbb{R})$. As a consequence of Schaefer's theorem ([17] p. 29) we deduce that N has a fixed point y which is a solution to problem (3.6)–(3.7). Then, y is a solution to the problem (1.1)–(1.2). $\square$

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