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ON A NEW SET OF ORTHOGONAL POLYNOMIALS

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Abstract. An orthogonal system of polynomials, arising from a second-order ordinary differential equation, is presented.

1. Introduction

Most of the popular families of orthogonal polynomials in mathematical text books have their origin in differential equations occurring in theoretical physics. This is also the case for the polynomials constructed in the present paper. The following second-order linear ordinary differential equation emerges as wave equation for the physical state functions $f(x)$ of a quantized closed Friedmann cosmological model [1]:

$$\left(x\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\frac{\mathrm{d}}{\mathrm{d}x}-x^3+Mx^2\right)f(x)=0 \tag{1}$$

Up to constants, the variable x is the radius of the universe and M is its total mass in units of Planck masses. Here we assume a domain $-\infty<x<\infty$.

The main subject of this paper, another differential equation, is obtained from (1) by splitting off from $f(x)$ an exponential function describing the two possible kinds of asymptotic behaviour of the solutions,

$$f(x)=e^{\pm(\frac{x^2}{2}-\frac{M}{2}x)}\,p(x)\,. \tag{2}$$

From the physical point of view only the exponentially falling functions are interesting.

2. The reduced equation

Inserting (2) (with the negative sign in the exponent) into (1) we obtain

$$x\,p''(x)-(2x^2-Mx+1)\,p'(x)+\left(\frac{M^2}{4}\,x-\frac{M}{2}\right)p(x)=0\,. \tag{3}$$

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According to the standard classification [2] this equation has two singular points: $x = 0$ and $x = \infty$. To determine the type of singularity at $x = 0$ we set

$$p(x) = x^\alpha \sum_{k=0}^{\infty} a_k\, x^k\,, \tag{4}$$

insert (4) into (3) and set the arising coefficients of the powers of x equal to zero. From the lowest power we obtain the indicial equation

$$\alpha(\alpha - 2) = 0\,. \tag{5}$$

The fact that it has two solutions, $\alpha = 0$ and $\alpha = 2$, classifies $x = 0$ as regular singular point. In consequence, there are two linearly independent solutions with the expansions

$$p^{(0)}(x) = \sum_{k=0}^{\infty} a_k\, x^k\,, \qquad \text{respectively,} \qquad p^{(2)}(x) = x^2 \sum_{k=0}^{\infty} b_k\, x^k \tag{6}$$

in a neighbourhood of $x = 0$, which converge for all real x, because the next singular point of the differential equation is at infinity. Inserting the expression for $p^{(0)}$ into (3), we deduce recurrence formulae for the coefficients a_k. From the vanishing of the coefficients of both the zeroth and the first power of x we obtain only one equation,

$$a_1 = -\frac{M}{2}\, a_0\,. \tag{7}$$

a_2 is not restricted at all, the general recurrence relation for $k > 1$ is

$$a_{k+1} = \frac{8(k-1) - M^2}{4(k-1)(k+1)}\, a_{k-1} - \frac{M(2k-1)}{2(k-1)(k+1)}\, a_k\,. \tag{8}$$

There are two free parameters, a_0 and a_2. The second expansion, $p^{(2)}$, is obtained simply by setting $a_0 = 0$, then $a_1 = 0$ automatically and a_2 corresponds to b_0, so (8) allows to calculate both linearly independent solutions from (6).

The second singular point, infinity, is an irregular one, where, according to [2], one can expect an asymptotic behaviour like

$$p(x) = e^{\lambda x^2 + \mu x}\, x^\beta \sum_{k=0}^{\infty} c_k\, x^{-k}\,. \tag{9}$$

Insertion into (3) yields the indicial equation

$$\lambda(\lambda - 1) = 0\,; \tag{10}$$

$\lambda = 1$ leads to the asymptotically growing solution indicated in (2). For $\lambda = 0$ it follows that also $\mu = 0$, so this solution has the asymptotic form of a Laurent series.

3. The polynomials

We insert the asymptotic series, which is not exponentially increasing,

$$p(x) = x^{\beta} \sum_{k=0}^{\infty} c_k \, x^{-k} \tag{11}$$

into (3). Demanding that the coefficient of the highest power vanishes, yields

$$\beta = \frac{M^2}{8}\,, \tag{12}$$

so the asymptotic expansion contains positive powers. As the radius of convergence of the series in (6) is infinite, the positive powers of (11) must agree with $p^{(0)}$ or $p^{(2)}$ and, in consequence, one of these series must be identical with (11). This means further that the principal part of the latter must vanish, as well as that at least one of $p^{(0)}$ and $p^{(2)}$ must be finite, i. e. of order β. The conclusion from this is that (1) has an exponentially falling solution only when $M^2/8$ is a non-negative integer and when the associated solution $p(x)$ of (3) is a polynomial. From the vanishing of the terms proportional to x^0 and x^{-1} we obtain

$$\left(2 + \frac{M^2}{4}\right) c_{\beta+1} = \frac{M}{2}\, c_{\beta} + c_{\beta-1} \tag{13}$$

and

$$\left(4 + \frac{M^2}{4}\right) c_{\beta+2} = \frac{3M}{2}\, c_{\beta+1}\,. \tag{14}$$

So, if

$$c_{\beta-1} = -\frac{M}{2}\, c_{\beta}\,, \tag{15}$$

both $c_{\beta+1}$ and $c_{\beta+2}$ vanish and so do all the negative powers, because, the recurrence relation expresses every c_k by the foregoing two coefficients, analogous to (8). (15) corresponds to (7), this shows that it is $p^{(0)}$ which has a chance to agree with (11). Indeed, as $p^{(0)}(x)$ contains two independent parameters, in the case when $M^2/8$ is equal to a positive integer n, it is possible to adjust the ratio $a_2 : a_0$ in such a way that the series terminates after the n-th power.

The lowest order polynomial, obtained in this way, is of first degree, but it is immediately obvious that also a constant is a solution of (3) if $M = 0$. The first seven polynomials for non-negative values of $M_n = +2\sqrt{2n}$ are given by the following formulae:

$$
\begin{aligned}
&p_0(x) = 1\,,\\
&p_1(x) = 1 - \sqrt{2}\,x\,,\\
&p_2(x) = 1 - 2\,x + \frac{2}{3}\,x^2\,,\\
&p_3(x) = 1 - \sqrt{6}\,x + \frac{10}{7}\,x^2 - \frac{2\sqrt{6}}{21}\,x^3\,,\\
&p_4(x) = 1 - 2\sqrt{2}\,x + \frac{132}{59}\,x^2 - \frac{28\sqrt{2}}{59}\,x^3 + \frac{4}{59}\,x^4\,,\\
&p_5(x) = 1 - \sqrt{10}\,x + \frac{452}{147}\,x^2 - \frac{20\sqrt{10}}{49}\,x^3 + \frac{12}{49}\,x^4 - \frac{4\sqrt{10}}{735}\,x^5\,,\\
&p_6(x) = 1 - 2\sqrt{3}x + \frac{2670}{679}x^2 - \frac{2440\sqrt{3}}{2037}x^3 + \frac{380}{679}x^4 - \frac{88\sqrt{3}}{2037}x^5 + \frac{8}{2037}x^6\,.
\end{aligned}
\tag{16}
$$

For $M_n = -2\sqrt{2n}$ we denote the polynomials by p_{-n}. From (3) it may be seen that

$$p_{-n}(x) = p_n(-x)\,. \tag{17}$$

(The physical application was restricted to positive values of M_n.)

Orthogonality of the functions $f_n(x)$ formed by $p_n(x)$ times the exponential function in (2) is easily shown by dividing (1) by x^2, which transforms it into a hermitian eigenvalue equation with respect to the measure $\mathrm{d}x$,

$$L_x\, f(x) := \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x} - x\right) f(x) = -M f(x)\,. \tag{18}$$

Therefore two eigenfunctions, $f_n(x) = e^{-\frac{x^2}{2}+\frac{M_n}{2}x}\, p_n(x)$ and $f_m(x)$, associated to M_n and M_m, respectively, are orthogonal in the sense of the inner product

$$\langle f_n, f_m\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\, f_n(x) f_m(x) \tag{19}$$

and the polynomials (16) are orthogonal in the sense of the inner product

$$\langle p_n, p_m\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-x^2+(\sqrt{2n}+\sqrt{2m})x}\, p_n(x)\, p_m(x)\,. \tag{20}$$

The operator L_x has a completely nondegenerate spectrum and is self-adjoint on the space of functions with compact support on the positive or on the negative real axis (without 0). As this spectrum is dense in the Hilbert space $L^2(\mathbf{R}, \mathrm{d}x)$, the eigenfunctions provide a basis of $L^2(\mathbf{R}, \mathrm{d}x)$.

For the sake of completeness, given a solution p_n, a second linearly independent solution of (3) (which has the expansion $p^{(2)}(x)$), can be found by standard methods,

$$q_n(x) = p_n(x) \int^x \frac{x'}{p_n^2(x')}\, e^{x'^2 - M_n x'}\, \mathrm{d}x', \tag{21}$$

so that we have finally a fundamental system of solutions in closed form for the equations

$$\left(x\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\mathrm{d}}{\mathrm{d}x} - x^3 \pm 2\sqrt{2n}\,x^2\right) f(x) = 0\,, \qquad n = 0, 1, 2, \dots \tag{22}$$

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