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QUASILINEAR AND QUADRATIC SINGULARLY PERTURBED PERIODIC BOUNDARY VALUE PROBLEM

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ABSTRACT. The problem of existence and asymptotic behavior of solutions of the quasilinear and quadratic singularly perturbed periodic boundary value problem as a small parameter at highest derivative tends to zero is studied.

1. Introduction

In the paper [3] the author established the sufficient conditions for existence and uniform convergence of the solutions of semilinear singularly perturbed differential equation $\epsilon y'' = f(t,y)$ to a solution of the reduced problem $f(t,u) = 0$ as the small positive parameter ϵ tends to zero. The purpose of this paper is an extension of lemma 1 above cited paper on the more general cases. We will consider periodic boundary value problem problem

$$(\mathrm{PBVP}_0) \qquad \begin{gathered} \epsilon y'' = F\left(t, y, y'\right), \qquad a < t < b\,, \\ y(a,\epsilon) - y(b,\epsilon) = 0, \quad y'(a,\epsilon) - y'(b,\epsilon) = 0\,, \end{gathered}$$

where $F \in C^1\left([a,b] \times \mathbb{R}^2\right)$ and ϵ is a small positive parameter. The proofs of the theorems are based upon the method of lower and upper solutions.

As usual, we say that $\alpha \in C^2([a,b])$ is a lower solution for (PBVP_0) if $\alpha(a,\epsilon) - \alpha(b,\epsilon) = 0$, $\alpha'(a,\epsilon) - \alpha'(b,\epsilon) \ge 0$, and $\epsilon\alpha''(t,\epsilon) \ge F\left(t, \alpha(t,\epsilon), \alpha'(t,\epsilon)\right)$ for every $t \in [a,b]$. An upper solution $\beta \in C^2([a,b])$ satisfies $\beta(a,\epsilon) - \beta(b,\epsilon) = 0$, $\beta'(a,\epsilon) - \beta'(b,\epsilon) \le 0$, and $\epsilon\beta''(t,\epsilon) \le F\left(t, \beta(t,\epsilon), \beta'(t,\epsilon)\right)$ for every $t \in [a,b]$.

Definition ([2]). We say that the function F satisfies a Bernstein-Nagumo condition if for each $M > 0$ there exists a continuous function $h_M : [0,\infty) \to [a_M, \infty)$ with $a_M > 0$ and $\int_0^\infty \frac{s}{h_M(s)} \mathrm{d}s = \infty$ such that for all $y, |y| \le M$, all $t \in [a,b]$ and all $z \in \mathbb{R}$

$$|F(t,y,z)| \le h_M(|z|)\,.$$

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Remark 1. As a remark we conclude that the functions of the form $F(t,y,y') = f(t,y)y' + g(t,y)$ and $F(t,y,y') = f(t,y)y'^2 + g(t,y)$ satisfy Bernstein-Nagumo condition.

Lemma ([1],[2]). *If α, β are lower and upper solutions for* (PBVP$_0$) *such that $\alpha(t,\epsilon) \le \beta(t,\epsilon)$ on $[a,b]$ and F satisfies a Bernstein-Nagumo condition, then there exists a solution y of* (PBVP$_0$) *with $\alpha(t,\epsilon) \le y(t,\epsilon) \le \beta(t,\epsilon), a \le t \le b$.*

Notation. Let

$$\begin{aligned}
&D_\delta(u) = \left\{(t,y) \in \mathbb{R}^2 : a \le t \le b, |y - u(t)| < d(t)\right\},\\
&D_{\delta,a}(u) = \left\{(t,y) \in \mathbb{R}^2 : a \le t \le a + \tfrac{\delta}{2}, y \in \mathbb{R}\right\} \cap D_\delta(u),\\
&D_{\delta,b}(u) = \left\{(t,y) \in \mathbb{R}^2 : b - \tfrac{\delta}{2} \le t \le b, y \in \mathbb{R}\right\} \cap D_\delta(u),\\
&D^+_{\delta,a}(u) = \left\{(t,y) \in \mathbb{R}^2 : a \le t \le b, y - u(t) \ge 0\right\} \cap D_{\delta,a}(u),\\
&D^-_{\delta,a}(u) = \left\{(t,y) \in \mathbb{R}^2 : a \le t \le b, y - u(t) \le 0\right\} \cap D_{\delta,a}(u),\\
&D^+_{\delta,b}(u) = \left\{(t,y) \in \mathbb{R}^2 : a \le t \le b, y - u(t) \ge 0\right\} \cap D_{\delta,b}(u),\\
&D^-_{\delta,b}(u) = \left\{(t,y) \in \mathbb{R}^2 : a \le t \le b, y - u(t) \le 0\right\} \cap D_{\delta,b}(u),
\end{aligned}$$

where $d(t)$ is the positive continuous function on $[a,b]$ such that

$$d(t) = |u(a) - u(b)| + \delta \qquad \text{for} \quad a \le t \le a + \frac{\delta}{2} \text{ and } \; b - \frac{\delta}{2} \le t \le b$$

and

$$d(t) = \delta \qquad \text{for } a + \delta \le t \le b - \delta\,,$$

δ is a small positive constant and $u = u(t)$ is a solution of the reduced problem $F(t,u,u') = 0$ defined on $[a,b]$ such that $u \in C^2([a,b])$. Next, we use notation $h(t,y)$ for $F(t,y,u'(t))$.

2. Quasilinear periodic boundary value problem

In this section we consider the quasilinear periodic boundary value problem

$$\text{(PBVP}_1) \qquad \begin{aligned} &\epsilon y'' = f(t,y)y' + g(t,y), \qquad a < t < b\,,\\ &y(a,\epsilon) - y(b,\epsilon) = 0, \quad y'(a,\epsilon) - y'(b,\epsilon) = 0\,, \end{aligned}$$

where $f, g \in C^1(D_\delta(u))$. About the behavior of solutions of (PBVP$_1$) for $\epsilon \to 0^+$ states the following result.

Theorem 1. *Consider the problem* (PBVP$_1$). *Let δ, m be the positive constants such that $\frac{\partial h(t,y)}{\partial y} \ge m$ for every $(t,y) \in D_\delta(u)$. Let $f(t,y) \le 0$ and $f(t,y) \ge 0$ for every $(t,y) \in D_{\delta,a}(u)$ and $(t,y) \in D_{\delta,b}(u)$, respectively. Then there exists ϵ_0 such that for any $\epsilon \in (0,\epsilon_0]$ the problem* (PBVP$_1$) *has solution satisfying the inequality*

$$-v_{11} - v_2 - C\epsilon \le y(t,\epsilon) - u(t) \le v_{12} + v_2 + C\epsilon$$

for $u(a) \ge u(b)$ and

$$v_{12} - v_2 - C\epsilon \le y(t,\epsilon) - u(t) \le -v_{11} + v_2 + C\epsilon$$

for $u(a) \leq u(b)$, *where*

$$v_{11}(t,\epsilon) = (u(a) - u(b)) \frac{\exp\left[-\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}(t-b)\right]}{\exp\left[\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}(b-a)\right] - 1}$$

$$v_{12}(t,\epsilon) = (u(a) - u(b)) \frac{\exp\left[-\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}(a-t)\right]}{\exp\left[\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}(b-a)\right] - 1}$$

$$v_2(t,\epsilon) = |u'(a) - u'(b)| \frac{\exp\left[-\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}(t-a)\right] + \exp\left[-\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}(b-t)\right]}{2\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}\left(1 - \exp\left[-\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}(b-a)\right]\right)},$$

and C *is a positive constant.*

Proof. For $u(a) \geq u(b)$ we define the lower solutions

$$\alpha(t,\epsilon) = u(t) - v_{11}(t,\epsilon) - v_2(t,\epsilon) - \Gamma(\epsilon)$$

and the upper solutions by

$$\beta(t,\epsilon) = u(t) + v_{12}(t,\epsilon) + v_2(t,\epsilon) + \Gamma(\epsilon)$$

(in the case $u(a) \leq u(b)$ we proceed analogously).

Here $\Gamma(\epsilon) = \frac{\epsilon\gamma}{m}$, where γ is a constant which will be defined below. One can easily check that the function α, β satisfy the boundary conditions required for the lower and upper solutions of (PBVP_1) and $\alpha \leq \beta$ on $[a,b]$. Now we show that $\epsilon\alpha''(t,\epsilon) \geq f(t,\alpha(t,\epsilon))\alpha'(t,\epsilon) + g(t,\alpha(t,\epsilon))$ and $\epsilon\beta''(t,\epsilon) \leq f(t,\beta(t,\epsilon))\beta'(t,\epsilon) + g(t,\beta(t,\epsilon))$ on $[a,b]$. By Taylor theorem we obtain

$$\begin{aligned}
\epsilon\alpha'' - F(t,\alpha,\alpha') &= \epsilon\alpha'' - (F(t,\alpha,\alpha') - F(t,u,u')) \\
&= \epsilon\alpha'' - [(F(t,\alpha,u') - F(t,u,u')) + (F(t,\alpha,\alpha') - F(t,\alpha,u'))] \\
&= \epsilon\alpha'' - \left[\frac{\partial h(t,\eta(t,\epsilon))}{\partial y}(\alpha - u) + f(t,\alpha)(\alpha' - u')\right] \\
&= \epsilon u'' - \epsilon v_{11}'' - \epsilon v_2'' + \frac{\partial h(t,\eta)}{\partial y}(v_{11} + v_2 + \Gamma) + f(t,\alpha)(v_{11}' + v_2') \\
&\geq \epsilon u'' - \epsilon v_{11}'' - \epsilon v_2'' + m(v_{11} + v_2 + \Gamma) + f(t,\alpha)(v_{11}' + v_2') \\
&= \epsilon u'' + \epsilon\gamma + f(t,\alpha)(v_{11}' + v_2') \\
&\geq -\epsilon|u''| + \epsilon\gamma + f(t,\alpha)(v_{11}' + v_2')
\end{aligned}$$

and

$$F(t,\beta,\beta') - \epsilon\beta'' \geq -\epsilon|u''| + \epsilon\gamma + f(t,\beta)(v_{12}' + v_2'),$$

where $(t,\eta(t,\epsilon))$ is a point between $(t,\alpha(t,\epsilon))$ and $(t,u(t))$, $(t,\eta(t,\epsilon)) \in D_\delta(u)$ for sufficiently small ϵ.

Let $|u'(a)-u'(b)|>0$. From the above assumptions we obtain that

$$f(t,\alpha)\,(v'_{11}+v'_2)\geq 0 \text{ and } \ f(t,\beta)\,(v'_{12}+v'_2)\geq 0$$

on $\left[a,a+\frac{\tilde{\delta}}{2}\right]\cup\left[b-\frac{\tilde{\delta}}{2},b\right]$ for $\epsilon\in(0,\epsilon_1]$ where $\tilde{\delta}=\min\{\delta,\delta_1\}$, and δ_1,ϵ_1 are such that $v'_{11}+v'_2<0$ $(v'_{12}+v'_2<0)$ $(v'_{11}+v'_2>0$ $(v'_{12}+v'_2>0)$ on $[a,a+\delta_1]$ $([b-\delta_1,b])$ and $(t,\alpha)\subset D_{\tilde{\delta}}(u)$, $(t,\beta)\subset D_{\tilde{\delta}}(u)$ for $\epsilon\in(0,\epsilon_1]$. On the interval $\left[a+\frac{\tilde{\delta}}{2},b-\frac{\tilde{\delta}}{2}\right]$ is $|f(t,\alpha)\,(v'_{11}+v'_2)|\leq c_1\epsilon$ and $|f(t,\beta)\,(v'_{12}+v'_2)|\leq c_1\epsilon$ for sufficiently small ϵ, for instance, when $\epsilon\in(0,\epsilon_0]$, $\epsilon_0\leq\epsilon_1$ and suitable positive constant c_1.

If $|u'(a)-u'(b)|=0$ then $v'_{11}\leq 0$ $(v'_{12}\geq 0)$ and $|f(t,\alpha)v'_{11}|\leq c_1\epsilon$ and $|f(t,\beta)v'_{12}|\leq c_1\epsilon$ on $\left[a+\frac{\tilde{\delta}}{2},b\right]$ $(\left[a,b-\frac{\tilde{\delta}}{2}\right])$.

Thus if we choose a constant $\gamma\geq c_1+\max\{|u''(t)|,t\in[a,b]\}$ then $\epsilon\alpha''(t,\epsilon)\geq f(t,\alpha(t,\epsilon))\alpha'(t,\epsilon)+g(t,\alpha(t,\epsilon))$ and $\epsilon\beta''(t,\epsilon)\leq f(t,\beta(t,\epsilon))\beta'(t,\epsilon)+g(t,\beta(t,\epsilon))$ on $[a,b]$. The existence of solution of (PBVP_1) satisfying the stated inequalities follows from lemma. This completes the proof. □

Example. As an illustrative example we consider the (PBVP_1) for differential equation $\epsilon y''=yy'-\left(t-\frac{1}{2}\right)$ on $[0,1]$. General solution of the reduced problem $uu'-\left(t-\frac{1}{2}\right)=0$ is $u^2=t^2-t+k, k\in\mathbb{R}$; however, only $u(t)=t-\frac{1}{2}$ satisfies the assumptions asked on the solution of reduced problem. On the basis of theorem 1, there is ϵ_0 such that for every $\epsilon\in(0,\epsilon_0]$ has the problem solution satisfying

$$v_{12}-v_2-C\epsilon\leq y(t,\epsilon)-\left(t-\frac{1}{2}\right)\leq -v_{11}+v_2+C\epsilon$$

on $[0,1]$.

3. Quadratic periodic boundary value problem

Now we will consider the quadratic periodic boundary value problem

$$(\mathrm{PBVP}_2)\qquad \begin{aligned} \epsilon y''&=f(t,y)y'^{\,2}+g(t,y), \qquad a<t<b\,,\\ y(a,\epsilon)-y(b,\epsilon)&=0, \quad y'(a,\epsilon)-y'(b,\epsilon)=0\,,\end{aligned}$$

where $f,g\in C^1\left(D_\delta(u)\right)$.

In the following theorem we use notation $f(t,y)$ arb M when we do not impose any restrictions on the sign of the function f in the set $M\subset\mathbb{E}_2$.

Theorem 2. *Consider the problem* (PBVP_2). *Let* δ,m *be the positive constants such that* $\frac{\partial h(t,y)}{\partial y}\geq m$ *for every* $(t,y)\in D_\delta(u)$. *Let* $u(a)\geq u(b)$ *and*

1. $f(t,y)\leq 0,(t,y)\in D_{\delta,a}(u)$ $(f(t,y)\geq 0,(t,y)\in D_{\delta,b}(u))$ *for* $4u'(a)>|u'(a)-u'(b)|>0$ $(4u'(b)>|u'(a)-u'(b)|>0)$.
2. $f(t,y)\equiv 0,(t,y)\in D^+_{\delta,a}(u)$ *and* $f(t,y)\leq 0,(t,y)\in D^-_{\delta,a}(u)$ $(f(t,y)\geq 0,(t,y)\in D^+_{\delta,b}(u)$ *and* $f(t,y)\equiv 0,(t,y)\in D^-_{\delta,b}(u))$ *for* $|u'(a)-u'(b)|>4u'(a)>0$ $(|u'(a)-u'(b)|>4u'(b)>0)$.

3_1. $f(t,y) \equiv 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \equiv 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| = 4u'(a) > 0, 4u'(t) < |u'(a) - u'(b)|, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| = 4u'(b) > 0, 4u'(t) < |u'(a) - u'(b)|, t \in (b-\omega, b), \omega > 0)$.

3_2. $f(t,y) \leq 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \geq 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| = 4u'(a) > 0, 4u'(t) > |u'(a) - u'(b)|, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| = 4u'(b) > 0, 4u'(t) > |u'(a) - u'(b)|, t \in (b-\omega, b), \omega > 0)$.

3_3. $f(t,y) \geq 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| = 4u'(a) > 0, 4u'(t) \equiv |u'(a) - u'(b)|, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| = 4u'(b) > 0, 4u'(t) \equiv |u'(a) - u'(b)|, t \in (b-\omega, b), \omega > 0)$.

4_1. $f(t,y) \equiv 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \equiv 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 4u'(a) = 0, 4u'(t) > 0, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| > 4u'(b) = 0, 4u'(t) > 0, t \in (b-\omega, b), \omega > 0)$.

4_2. $f(t,y) \geq 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \equiv 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \equiv 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 4u'(a) = 0, 4u'(t) < 0, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| > 4u'(b) = 0, 4u'(t) < 0, t \in (b-\omega, b), \omega > 0)$.

4_3. $f(t,y) \geq 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 4u'(a) = 0, 4u'(t) \equiv 0, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| > 4u'(b) = 0, 4u'(t) \equiv 0, t \in (b-\omega, b), \omega > 0)$.

5_1. $f(t,y)$ *arb* $D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y)$ *arb* $D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| = 4u'(a) = 0, u(a) \neq u(b), 4u'(t) > 0, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| = 4u'(b) = 0, u(a) \neq u(b), 4u'(t) > 0, t \in (b-\omega, b), \omega > 0)$.

5_2. $f(t,y)$ *arb* $D^+_{\delta,a}(u)$ *and* $f(t,y) \equiv 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \equiv 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y)$ *arb* $D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| = 4u'(a) = 0, u(a) \neq u(b), 4u'(t) < 0, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| = 4u'(b) = 0, u(a) \neq u(b), 4u'(t) < 0, t \in (b-\omega, b), \omega > 0)$.

5_3. $f(t,y)$ *arb* $D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y)$ *arb* $D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| = 4u'(a) = 0, u(a) \neq u(b), 4u'(t) \equiv 0, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| = 4u'(b) = 0, u(a) \neq u(b), 4u'(t) \equiv 0, t \in (b-\omega, b), \omega > 0)$.

6. $f(t,y)$ *arb* $D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y)$ *arb* $D^-_{\delta,b}(u))$
for $4u'(a) > |u'(a) - u'(b)| = 0, u(a) \neq u(b)$
$(4u'(b) > |u'(a) - u'(b)| = 0, u(a) \neq u(b))$.

7_1. $f(t,y) \geq 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \equiv 0, (t,y) \in D^-_{\delta,a}(u)$ $(f(t,y) \equiv 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,b}(u))$ *for* $|u'(a) - u'(b)| > 0$,

$u'(a) < 0,\ u(a) \neq u(b)\ (|u'(a) - u'(b)| > 0, u'(b) < 0, u(a) \neq u(b))$.

7_{21}. $f(t,y) \geq 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \equiv 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \equiv 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 0, u'(a) < 0, |u'(a) - u'(b)| > -4u'(a), u(a) = u(b)$
$(|u'(a) - u'(b)| > 0, u'(b) < 0, |u'(a) - u'(b)| > -4u'(b), u(a) = u(b))$.

7_{22}. $f(t,y) \geq 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \geq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \leq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 0, u'(a) < 0, |u'(a) - u'(b)| < -4u'(a), u(a) = u(b)$
$(|u'(a) - u'(b)| > 0, u'(b) < 0, |u'(a) - u'(b)| > -4u'(b), u(a) = u(b))$.

7_{231}. $f(t,y) \geq 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \equiv 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \equiv 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 0, u'(a) < 0, |u'(a) - u'(b)| = -4u'(a)$,
$|u'(a) - u'(b)| > -4u'(t), t \in (a, a+\omega), \omega > 0, u(a) = u(b)$
$(|u'(a) - u'(b)| > 0, u'(b) < 0, |u'(a) - u'(b)| = -4u'(b)$,
$|u'(a) - u'(b)| > -4u'(t), t \in (b-\omega, b), \omega > 0, u(a) = u(b))$.

7_{232}. $f(t,y) \geq 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \geq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \leq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 0, u'(a) < 0, |u'(a) - u'(b)| = -4u'(a)$,
$|u'(a) - u'(b)| < -4u'(t), t \in (a, a+\omega), \omega > 0, u(a) = u(b)$
$(|u'(a) - u'(b)| > 0, u'(b) < 0, |u'(a) - u'(b)| = -4u'(b)$,
$|u'(a) - u'(b)| < -4u'(t), t \in (b-\omega, b), \omega > 0, u(a) = u(b))$.

7_{233}. $f(t,y) \geq 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \geq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \leq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 0, u'(a) < 0, |u'(a) - u'(b)| = -4u'(a)$,
$|u'(a) - u'(b)| \equiv -4u'(t), t \in (a, a+\omega), \omega > 0, u(a) = u(b)$
$(|u'(a) - u'(b)| > 0, u'(b) < 0, |u'(a) - u'(b)| = -4u'(b)$,
$|u'(a) - u'(b)| \equiv -4u'(t), t \in (b-\omega, b), \omega > 0, u(a) = u(b))$.

8. $f(t,y)$ *arb* $D^+_{\delta,a}(u)$ *and* $f(t,y) \equiv 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \equiv 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y)$ *arb* $D^-_{\delta,b}(u))$ *for* $|u'(a) - u'(b)| = 0$,
$u'(a) < 0, u(a) \neq u(b)\ (|u'(a) - u'(b)| = 0, u'(b) < 0, u(a) \neq u(b))$.

Then there exists ϵ_0 *such that for any* $\epsilon \in (0, \epsilon_0]$ *the problem* $(PBVP_2)$ *has solution satisfying the inequality*

$$-v_{11} - v_2 - C\epsilon \leq y(t,\epsilon) - u(t) \leq v_{12} + v_2 + C\epsilon$$

on $[a,b]$ *where* v_{11}, v_{12} *and* v_2 *are the functions from theorem 1 and* C *is a positive constant.*

Proof. The idea of the proof is essentially the same as of theorem 1. Let us define the lower solutions by

$$\alpha(t,\epsilon) = u(t) - v_{11}(t,\epsilon) - v_2(t,\epsilon) - \Gamma(\epsilon)$$

and the upper solutions by

$$\beta(t,\epsilon) = u(t) + v_{12}(t,\epsilon) + v_2(t,\epsilon) + \Gamma(\epsilon).$$

Analogously as in theorem 1 we obtain

$$\epsilon\alpha'' - F(t,\alpha,\alpha') \geq -\epsilon\left|u''\right| + \gamma\epsilon - f(t,\alpha)\left(\alpha'^2 - u'^2\right)$$

$$= -\epsilon\left|u''\right| + \gamma\epsilon + f(t,\alpha)\left(v_{11}' + v_2'\right)\left(2u' - v_{11}' - v_2'\right)$$

and

$$F(t,\beta,\beta') - \epsilon\beta'' \geq -\epsilon\left|u''\right| + \gamma\epsilon + f(t,\beta)\left(\beta'^2 - u'^2\right)$$

$$= -\epsilon\left|u''\right| + \gamma\epsilon + f(t,\beta)\left(v_{12}' + v_2'\right)\left(2u' + v_{12}' + v_2'\right).$$

Similarly as in previous theorem we conclude that

$$f(t,\alpha)\left(v_{11}' + v_2'\right)\left(2u' - v_{11}' - v_2'\right) \geq 0$$

$$\left(f(t,\beta)\left(v_{12}' + v_2'\right)\left(2u' + v_{12}' + v_2'\right) \geq 0\right)$$

or

$$\left|f(t,\alpha)\left(v_1' + v_2'\right)\left(2u' - v_{11}' - v_2'\right)\right| \leq c_2\epsilon$$

$$\left(\left|f(t,\beta)\left(v_{12}' + v_2'\right)\left(2u' + v_{12}' + v_2'\right)\right| \leq c_2\epsilon\right)$$

on $[a,b]$ for some positive constant c_2 and sufficiently small ϵ, for instance, when $\epsilon \in (0,\epsilon_0]$. Therefore, for $\gamma \geq c_2 + \max\{|u''(t)|, t \in [a,b]\}$ is $\epsilon\alpha''(t,\epsilon) \geq f(t,\alpha(t,\epsilon))\alpha'^2(t,\epsilon) + g(t,\alpha(t,\epsilon))$ and $\epsilon\beta''(t,\epsilon) \leq f(t,\beta(t,\epsilon))\beta'^2(t,\epsilon) + g(t,\beta(t,\epsilon))$ on $[a,b]$. Hence theorem 2 is proved. □

Remark 2. As a remark we note, that the assumed condition $u(a) \geq u(b)$ is not essential. One can take into consideration the lower and upper solutions for $u(a) \leq u(b)$ and reformulated theorem 2 for this case.

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