ARCHIVUM MATHEMATICUM (BRNO)
Tomus 36 (2000), 71 – 75

THE NATURAL TRANSFORMATIONS $TT^{(r)} \to TT^{(r)}$

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Abstract. For natural numbers $r \geq 2$ and n a complete classification of natural transformations $A : TT^{(r)} \to TT^{(r)}$ over n-manifolds is given, where $T^{(r)}$ is the linear r-tangent bundle functor.

0. In [1], Gancarzewicz and Kolář obtained a classification of all natural affinors on the extended linear r-tagent bundle functor $E^{(r)}M = (J^r(M, \mathbf{R}))^*$ over n-manifolds. From the mentioned classification one can easily deduce that any natural affinor $A : TT^{(r)}M \to TT^{(r)}M$ on the linear r-tangent bundle functor $T^{(r)}M = (J^r(M, \mathbf{R})_0)^*$ over n-manifolds is a linear combination (with real coefficients) of the identity affinor $id_{TT^{(r)}M} : TT^{(r)}M \to TT^{(r)}M$ and the affinor being the composition $TT^{(r)}M \to T^{(r)}M \times_M TM \subset T^{(r)}M \times_M T^{(r)}M \tilde{=} VT^{(r)}M \subset TT^{(r)}M$, where the arrow is the system $(\pi^T, T\pi)$, $\pi^T : TT^{(r)}M \to T^{(r)}M$ is the tangent bundle projection, $\pi : T^{(r)}M \to M$ is the bundle projection and the inclusion $i : TM \subset T^{(r)}M$ is given by the dualization of the jet projection $J^r(M, \mathbf{R})_0 \to J^1(M, \mathbf{R})_0$.

Clearly, any natural affinor A on $T^{(r)}M$ is a natural transformation $A : TT^{(r)}M \to TT^{(r)}M$ such that A is a tensor field of type $(1,1)$ on $T^{(r)}M$.

If $r = 1$, the natural transformations $TTM \to TTM$ are in bijection with the Weil algebra homomorphisms $TT\mathbf{R} \to TT\mathbf{R}$, see [2].

The purpose of this note is to give a complete classification of natural transformations $A : TT^{(r)}M \to TT^{r)}M$ over n-manifolds in the case where $r \geq 2$.

In Item 1, we prove that any natural transformation $A : TT^{(r)}M \to T^{(r)}M$ over n-manifold is a linear combination of $\pi^T : TT^{(r)}M \to T^{(r)}M$ and $i \circ T\pi : TT^{(r)}M \to TM \subset T^{(r)}M$.

In Item 2, as a corollary of the result of Item 1, we prove that if $r \geq 2$, then any natural transformation $A : TT^{(r)}M \to TM$ over n-manifolds is proportional to $T\pi : TT^{(r)}M \to TM$.

If $\underline{A} : TT^{(r)}M \to T^{(r)}M$ is a natural transformation, then a natural transformation $A : TT^{(r)}M \to TT^{(r)}M$ is called to be over $\underline{A}$ iff $\pi^T \circ A = \underline{A}$. In Item 3, we

1991 *Mathematics Subject Classification*: 58A20, 53A55.
Key words and phrases: bundle functors, natural transformations.
Received May 18, 1999.

define two natural transformations (of vertical type) $\underline{A}^{\pi^T} := (\underline{A}, \pi^T) : TT^{(r)}M \to T^{(r)}M \times_M T^{(r)}M = VT^{(r)}M \subset TT^{(r)}M$ and $\underline{A}^{i\circ T\pi} := (\underline{A}, i\circ T\pi) : TT^{(r)}M \to T^{(r)}M \times_M T^{(r)}M = VT^{(r)}M \subset TT^{(r)}M$ over $\underline{A}$. Then, as a corollary of the result of Item 1, we prove that any natural transformation $A : TT^{(r)}M \to VT^{(r)}M \subset TT^{(r)}M$ over $\underline{A}$ is a linear combination of $\underline{A}^{\pi^T}$ and $\underline{A}^{i\circ T\pi}$.

In Item 4, if $r \geq 2$ and $\lambda, \mu \in \mathbf{R}$, we construct a natural transformation $A^{(\lambda,\mu)} : TT^{(r)}M \to TT^{(r)}M$ over $\underline{A} = \lambda\pi^T + \mu(i \circ T\pi)$ of not vertical type.

In Item 5, applying the result of Items 2 and 3, we prove that if $r \geq 2$ and $\lambda, \mu \in \mathbf{R}$, then any natural transformation $A : TT^{(r)}M \to TT^{(r)}M$ over $\underline{A} = \lambda\pi^T + \mu(i \circ T\pi)$ is a linear combination of $\underline{A}^{\pi^T}$, $\underline{A}^{i\circ T\pi}$ and $A^{(\lambda,\mu)}$.

Throughout this note the usual coordinates on $\mathbf{R}^n$ are denoted by $x^1, ..., x^n$ and $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, ..., n$.

All manifolds and maps are assumed to be of class C^∞.

1. The tangent bundle projection $\pi^T : TT^{(r)}M \to T^{(r)}M$ is a simple example of a natural transformation $TT^{(r)}M \to T^{(r)}M$ over n-manifolds. Another example is $i \circ T\pi : TT^{(r)}M \to TM \subset T^{(r)}M$, where $\pi : T^{(r)}M \to M$ is the bundle projection and the inclusion $i : TM \tilde{=} T^{(1)}M \to T^{(r)}M$ is the dual map of the jet projection $J^r(M, \mathbf{R})_0 \to J^1(M, \mathbf{R})_0$.

Proposition 1. *Any natural transformation $A : TT^{(r)}M \to T^{(r)}M$ over n-manifolds is a linear combination (with real coefficients) of π^T and $i \circ T\pi$.*

Proof. Any natural transformation A as in the proposition is uniquely determined by the $\langle A(u), j^r_0\gamma\rangle \in \mathbf{R}$ for any $\gamma : \mathbf{R}^n \to \mathbf{R}$ with $\gamma(0) = 0$ and any $u \in (TT^{(r)}\mathbf{R}^n)_0 \tilde{=} \mathbf{R}^n \times (VT^{(r)}\mathbf{R}^n)_0 \tilde{=} \mathbf{R}^n \times T^{(r)}_0\mathbf{R}^n \times T^{(r)}_0\mathbf{R}^n$, where $\tilde{=}$ denotes the standard trivialization and the canonical identification. By the rank theorem $j^r_0x^1$ has dense orbit in $J^r_0(\mathbf{R}^n, \mathbf{R})_0$. Then , by the naturality, A is uniquely determined by the $\langle A(u), j^r_0x^1\rangle$ for any $u \in (TT^{(r)}\mathbf{R}^n)_0 \tilde{=} \mathbf{R}^n \times T^{(r)}_0\mathbf{R}^n \times T^{(r)}_0\mathbf{R}^n$.

Any element from $T^{(r)}_0\mathbf{R}^n$ is a linear combination of the $(j^r_0x^\alpha)^*$ for all $\alpha \in (\mathbf{N}\cup\{0\})^n$ with $1 \leq |\alpha| \leq r$, where the $(j^r_0x^\alpha)^*$ form the basis of $T^{(r)}_0\mathbf{R}^n$ dual to the basis $j^r_0x^\alpha \in J^r_0(\mathbf{R}^n, \mathbf{R})_0$. By the naturality of A with respect to the homotheties $a_t = (t^1x^1, ..., t^nx^n)$, $t = (t^1, ..., t^n) \in \mathbf{R}^n_+$, we have $\langle A(TT^{(r)}(a_t)(u)), j^r_0x^1\rangle = t^1\langle A(u), j^r_0x^1\rangle$ for any $t = (t^1, ..., t^n) \in \mathbf{R}^n_+$. For any $t \in \mathbf{R}^n$ and any $\alpha \in (\mathbf{N} \cup \{0\})^n$ we have $T^{(r)}(a_t)((j^r_0x^\alpha)^*) = t^\alpha(j^r_0x^\alpha)^*$. Then by the homogeneous function theorem, see [2], we deduce easily that

$$(*) \qquad \langle A(u), j^r_0x^1\rangle = \lambda u^1_1 + \mu u_{2,e_1} + \nu u_{3,e_1}$$

for some real numbers λ, μ, ν, where $u = (u_1, u_2, u_3) \in (T(T^{(r)}\mathbf{R}^n))_0 \tilde{=} \mathbf{R}^n \times T^{(r)}_0\mathbf{R}^n \times T^{(r)}_0\mathbf{R}^n$, $u_1 = (u^1_1, ..., u^n_1) \in \mathbf{R}^n$, $u_{2,\alpha}$ is the coefficient (with respect to the basis) of $u_2 \in T^{(r)}_0\mathbf{R}^n$ corresponding to $(j^r_0x^\alpha)^*$ and $u_{3,\alpha}$ is the coefficient of $u_3 \in T^{(r)}_0\mathbf{R}^n$ corresponding to $(j^r_0x^\alpha)^*$, $e_1 = (1, 0, ..., 0) \in (\mathbf{N} \cup \{0\})^n$.

Replacing A by $A - \lambda i \circ T\pi - \mu\pi^T$ we can assume that $\lambda = \mu = 0$. Then (in particular)

$$(**) \qquad \langle A(\partial_1^C{}_{|\omega}), j_0^r x^1\rangle = \langle A(e_1, \omega, 0), j_0^r x^1\rangle = 0$$

for any $\omega \in T_0^{(r)}\mathbf{R}^n$, where $(\ )^C$ is the complete lifting of vector fields to $T^{(r)}$.

It remains to show that $\nu = 0$, i.e. that $\langle A(0, 0, (j_0^r x^1)^*), j_0^r x^1\rangle = 0$.

For showing this, we prove

$$\begin{aligned} 0 &= \langle A((\partial_1 + (x^1)^r\partial_1)^C_{|\omega}), j_0^r x^1\rangle = \langle A(((x^1)^r\partial_1)^C_{|\omega}), j_0^r x^1\rangle \\ &= \langle A(0, \omega, (j_0^r x^1)^*), j_0^r x^1\rangle = \langle A(0, 0, (j_0^r x^1)^*), j_0^r x^1\rangle\,, \end{aligned}$$

where $\omega = (j_0^r (x^1)^r)^*$.

The second and the fourth equalities are clear as in the formula (*) $\lambda = \mu = 0$.

We can prove the first equality as follows. Vector fields $\partial_1 + (x^1)^r\partial_1$ and ∂_1 have the same $(r-1)$-jets at 0. Then, by the result of Zajtz [3], there exists a diffeomorphism $\varphi : \mathbf{R}^n \to \mathbf{R}^n$ such that $j_0^r\varphi = id$ and $\varphi_*\partial_1 = \partial_1 + (x^1)^r\partial_1$ near 0. Clearly, φ preserves $j_0^r x^1$ because of the jet argument. Then, using the naturality of A with respect to φ, from (**) it follows the first equality for any $\omega \in T_0^{(r)}\mathbf{R}^n$.

It remains to prove the third equality. Let φ_t be the flow of $(x^1)^r\partial_1$. For any $\beta \in (\mathbf{N} \cup \{0\})^n$ with $1 \le |\beta| \le r$ we have

$$\begin{aligned} \langle ((x^1)^r\partial_1)^C_{|\omega}, j_0^r x^\beta\rangle &= \langle \frac{d}{dt}_{|t=0} T^{(r)}(\varphi_t)(\omega), j_0^r x^\beta\rangle \\ &= \frac{d}{dt}_{|t=0}\langle T^{(r)}(\varphi_t)(\omega), j_0^r x^\beta\rangle = \frac{d}{dt}_{|t=0}\langle \omega, j_0^r(x^\beta \circ \varphi_t)\rangle \\ &= \langle \omega, j_0^r(\frac{d}{dt}_{|t=0} x^\beta \circ \varphi_t)\rangle = \langle \omega, j_0^r(((x^1)^r\partial_1)x^\beta)\rangle\,. \end{aligned}$$

Because of the definition of ω, the last term is equal to 1 if $j_0^r x^\beta = j_0^r x^1$ and it is equal to 0 in the other cases. Then $((x^1)^r\partial_1)^C_{|\omega} = (j_0^r x^1)^*$ under the isomorphism $V_\omega(T^{(r)}\mathbf{R}^n) \tilde{=} T_0^{(r)}\mathbf{R}^n$. It implies the third equality. □

2. The tangent map $T\pi : TT^{(r)}M \to TM$ of the bundle projection $\pi : T^{(r)}M \to M$ is a natural transformation over n-manifolds.

Proposition 2. *If $r \ge 2$, then any natural transformation $A : TT^{(r)}M \to TM$ over n-manifolds is proportional (by a real number) to $T\pi$.*

Proof. Applying the inclusion $i : TM \subset T^{(r)}M$, we have $A : TT^{(r)}M \to TM \subset T^{(r)}M$. Then, by Proposition 1, $A = \lambda\pi^T + \mu(i \circ T\pi)$. Since $r \ge 2$, A is not surjective. Then $\lambda = 0$. □

3. Let $\underline{A} : TT^{(r)}M \to T^{(r)}M$ be a natural transformation over n-manifolds.

We say that a natural transformation $A : TT^{(r)}M \to TT^{(r)}M$ over n-manifolds is *over* $\underline{A}$ if $\pi^T \circ A = \underline{A}$.

If $B : TT^{(r)}M \to T^{(r)}M$ is another natural transformation over n-manifolds, we define a natural transformation

$$\underline{A}^B := (\underline{A}, B) : TT^{(r)}M \to T^{(r)}M \times_M T^{(r)}M \tilde{=} VT^{(r)}M \subset TT^{(r)}M \ .$$

Clearly, $\underline{A}^B$ is over $\underline{A}$. We call $\underline{A}^B$ *the B-vertical lift of $\underline{A}$*.

In particular, considering the natural transformations $\pi^T : TT^{(r)}M \to T^{(r)}M$ and $i \circ T\pi : TT^{(r)}M \to T^{(r)}M$, we produce natural transformations $\underline{A}^{\pi^T} : TT^{(r)}M \to TT^{(r)}M$ and $\underline{A}^{i\circ T\pi} : TT^{(r)}M \to TT^{(r)}M$ over $\underline{A}$.

The above natural transformations $\underline{A}^B$ are of vertical type, i.e. they have values in $VT^{(r)}M$.

If $A : TT^{(r)}M \to VT^{(r)}M \tilde{=} T^{(r)}M \times_M T^{(r)}M$ is a natural transformation of vertical type over $\underline{A}$, then $A = (\underline{A}, B)$ for natural transformation $B = pr_2 \circ A : TT^{(r)}M \to T^{(r)}M$, i.e. $A = \underline{A}^B$ for some B.

Then applying Proposition 1 we obtain the following proposition.

Proposition 3. *Let $\underline{A} : TT^{(r)}M \to T^{(r)}M$ be a natural transformation over n-manifolds. Any natural transformation $A : TT^{(r)}M \to VT^{(r)}M$ over n-manifolds of vertical type over $\underline{A}$ is a linear combination (with real coefficients) of $\underline{A}^{\pi^T}$ and $\underline{A}^{i\circ T\pi}$.*

In the next item it will be presented an example of a natural transformation $A : TT^{(r)}M \to TT^{(r)}M$ over n-manifolds over $\underline{A}$ which is not of vertical type.

4. Assume $r \geq 2$. Let $\lambda, \mu \in \mathbf{R}$. If $\underline{A} = \lambda\pi^T + \mu(i \circ T\pi) : TT^{(r)}M \to T^{(r)}M$, then we define a natural transformation $A^{(\lambda,\mu)} : TT^{(r)}M \to TT^{(r)}M$ over n-manifolds over $\underline{A}$ as follows.

Let $u_o \in T_{\omega_o}T^{(r)}M$, $\omega_o \in T^{(r)}_{x_o}M$, $x_o \in M$. There exists a vector field X and an element $\eta \in T^{(r)}_{x_o}M$ such that $u_o = X^C_{|\omega_o} + (\omega_o, \eta)$ under $VT^{(r)}M \tilde{=} T^{(r)}M \times_M T^{(r)}M$, where $(\ )^C$ is the complete lifting of vector fields to $T^{(r)}M$. We put

$$A^{(\lambda,\mu)}(u_o) := X^C_{|\lambda\omega_o + \mu i(X_{|x_o})} + (\lambda\omega_o + \mu i(X_{|x_o}), \lambda\eta - \mu\sigma^X) \ ,$$

where $i : TM \to T^{(r)}M$ is the inclusion and $\sigma^X \in T^{(r)}_{x_o}M$ is given by $\langle \sigma^X, j^r_{x_o}\gamma \rangle := X(X\gamma)(x_o)$ for any $\gamma : M \to \mathbf{R}$ with $\gamma(x_o) = 0$. (σ^X is defined as $r \geq 2$.)

The definition of $A^{(\lambda,\mu)}$ is correct. For proving this, we consider another $\tilde{X} = X + X'$ with $X'_{|x_o} = 0$ and $\tilde{\eta} \in T^{(r)}_{x_o}M$ such that $u_o = \tilde{X}^C_{|\omega_o} + (\omega_o, \tilde{\eta})$. Then $(X')^C_{|\omega_o} = (\omega_o, \eta - \tilde{\eta})$. We have to show that $X^C_{|\lambda\omega_o+\mu i(X_{|x_o})} + (\lambda\omega_o + \mu i(X_{|x_o}), \lambda\eta - \mu\sigma^X) = \tilde{X}^C_{|\lambda\omega_o+\mu i(X_{|x_o})} + (\lambda\omega_o + \mu i(X_{|x_o}), \lambda\tilde{\eta} - \mu\sigma^{\tilde{X}})$.

It is sufficient to show that

$$(X')^C_{|\lambda\omega_o+\mu i(X_{|x_o})} = (\lambda\omega_o + \mu i(X_{|x_o}), \lambda(\eta - \tilde{\eta}) - \mu(\sigma^X - \sigma^{\tilde{X}})) \ .$$

Let φ_t be the flow of X'. Denote $(X')^C_{|\lambda\omega_o+\mu i(X_{|x_o})} = (\lambda\omega_o + \mu i(X_{|x_o}), \theta)$, $\theta \in T^{(r)}_{x_o}M$. Then for any $\gamma : M \to \mathbf{R}$ with $\gamma(x_o) = 0$ we have

$$\begin{aligned}\langle \theta, j^r_{x_o}\gamma &= \langle \frac{d}{dt}_{|0} T^{(r)}_{x_o}\varphi_t(\lambda\omega_o + \mu i(X_{|x_o})), j^r_{x_o}\gamma\rangle \\ &= \langle \lambda\omega_o + \mu i(X_{|x_o}), j^r_{x_o}(\frac{d}{dt}_{|0}\gamma \circ \varphi_t)\rangle \\ &= \lambda\langle \omega_o, j^r_{x_o}(X'\gamma)\rangle + \mu\langle i(X_{|x_o}), j^r_{x_o}(X'\gamma)\rangle\,.\end{aligned}$$

From $(X')^C_{|\omega_o} = (\omega_o, \eta - \tilde{\eta})$ we have

$$\langle \eta - \tilde{\eta}, j^r_{x_o}\gamma\rangle = \langle \frac{d}{dt}_{|0} T^{(r)}_{x_o}\varphi_t(\omega_o), j^r_{x_o}\gamma\rangle = \langle \omega_o, j^r_{x_o}(X'\gamma)\rangle\,.$$

On the other hand we have

$$\begin{aligned}\langle \sigma^X - \sigma^{\tilde{X}}, j^r_{x_o}\gamma\rangle &= X(X\gamma)(x_o) - \tilde{X}(\tilde{X}\gamma)(x_o) = -X(X'\gamma)(x_o) \\ &= -\langle X_{|x_o}, j^1_{x_o}(X'\gamma)\rangle = -\langle i(X_{|x_o}), j^r_{x_o}(X'\gamma)\rangle\end{aligned}$$

modulo the isomorphism $TM = T^{(1)}M$.

Then $\theta = \lambda(\eta - \tilde{\eta}) - \mu(\sigma^X - \sigma^{\tilde{X}})$. That is why $A^{(\lambda,\mu)}$ is well-defined.

5. We end the paper by the following proposition.

Proposition 4. *Let $\lambda, \mu \in \mathbf{R}$. Put $\underline{A} := \lambda\pi^T + \mu(i \circ T\pi) : TT^{(r)}M \to T^{(r)}M$. If $r \geq 2$, then any natural transformation $A : TT^{(r)}M \to TT^{(r)}M$ over n-manifolds over $\underline{A}$ is a linear combination (with real coefficients) of $\underline{A}^{\pi^T}$, $\underline{A}^{i\circ T\pi}$ and $A^{(\lambda,\mu)}$.*

Proof. Let $A : TT^{(r)}M \to TT^{(r)}M$ be a natural transformation over n-manifolds over $\underline{A}$. The composition $T\pi \circ A : TT^{(r)}M \to TM$ is a natural transformation. By Proposition 2, there exists the real number ρ such that $T\pi \circ A = \rho T\pi$. Clearly, $T\pi \circ A^{(\lambda,\mu)} = T\pi$. Then $A - \rho A^{(\lambda,\mu)} : TT^{(r)}M \to TT^{(r)}M$ is of vertical type. Then Proposition 3 ends the proof. □

Remark. Clearly, any natural transformation $TT^{(r)}M \to TT^{(r)}M$ is over $\underline{A} = \pi^T \circ A$. Then Proposition 4 together with Proposition 1 gives a complete description of all natural transformations $TT^{(r)}M \to TT^{(r)}M$ over n-manifolds in the case where $r \geq 2$.

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