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BORDERLINE SHARP ESTIMATES FOR SOLUTIONS TO NEUMANN PROBLEMS

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Abstract. A priori estimates for solutions to homogeneous Neumann problems for uniformly elliptic equations in open subsets Ω of $\mathbf{R}^n$ are established, with data in the limiting space $L^{n/2}(\Omega)$, or, more generally, in the Lorentz spaces $L^{n/2,q}(\Omega)$. These estimates are optimal as far as either constants or norms are concerned.

1. Introduction and main results

We are concerned with optimal a priori estimates for solutions to homogeneous Neumann problems for linear elliptic equations in divergence form. Precisely, weak solutions are taken into account to problems having the form

$$\begin{cases} -\operatorname{div}\big(A(x)\nabla u\big) + B(x)\cdot\nabla u = f(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, Ω is a domain, namely a connected open set, in $\mathbf{R}^n$, $n \geq 3$, which is bounded and has a sufficiently regular boundary $\partial\Omega$; $A(x)$ is an $n\times n$ matrix with essentially bounded coefficients, uniformly positive definite for $x \in \Omega$, and normalized in such a way that

$$A(x)\xi\cdot\xi \geq |\xi|^2 \quad \text{for } \xi \in \mathbf{R}^n; \tag{1.2}$$

∇u denotes the gradient of u; $\boldsymbol{\nu}$ is the co-normal on $\partial\Omega$, namely $\boldsymbol{\nu} = A(x)^T\mathbf{n}$, where $\mathbf{n}$ is the normal unit vector on $\partial\Omega$; f and B are a given real-valued and a given vector-valued function in Ω, respectively; the dot $\cdot$ stands for scalar product in $\mathbf{R}^n$.

Weak solutions u to (1.1) are well-defined if B and f satisfy appropriate integrability conditions. Assume, for instance, that $|B| \in L^n(\Omega)$ and $f \in L^{\frac{2n}{n+2}}(\Omega)$. Then a function u from the Sobolev space $W^{1,2}(\Omega)$ is said to be a weak solution to (1.1) if

$$\int_\Omega A(x)\nabla u\cdot\nabla\phi\,dx + \int_\Omega B(x)\cdot\nabla u\,\phi\,dx = \int_\Omega f\,\phi\,dx \tag{1.3}$$

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for every $\phi \in W^{1,2}(\Omega)$.

By the very definition of $W^{1,2}(\Omega)$, any weak solution u to (1.1) is in $L^2(\Omega)$ and, by the Sobolev embedding theorem, it also belongs to $L^{\frac{2n}{n-2}}(\Omega)$. However, classical results tell us that, if f belongs to a smaller space than just $L^{\frac{2n}{n+2}}(\Omega)$, then u enjoys stronger summability properties. Consider, for instance, the case where $f \in L^p(\Omega)$ for some $p \geq 2n/(n+2)$, and $B \equiv 0$. If $p < n/2$, then a constant $C = C(p, \Omega)$ exists such that

$$\|u - \mathrm{m}(u)\|_{L^{\frac{np}{n-2p}}(\Omega)} \leq C\|f\|_{L^p(\Omega)} \tag{1.4}$$

for every weak solution to (1.1), where

$$\mathrm{m}(u) = \sup\big\{t \in \mathbf{R} : |\{u > t\}| \geq \ |\Omega|/2\big\}, \tag{1.5}$$

the median of u, and $|\cdot|$ denotes Lebesgue measure. Notice that a normalization condition for u is indispensable in (1.4), since being a weak solution to (1.1) is not affected by adding real constants. Of course, other choices would be possible—for example, $\mathrm{m}(u)$ could be replaced by the mean value of u over Ω; for convenience we shall work with $\mathrm{m}(u)$ throughout.

When $p > n/2$, u is in fact essentially bounded, and a constant $C = C(p, \Omega)$ exists such that

$$\|u - \mathrm{m}(u)\|_{L^\infty(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \tag{1.6}$$

We refer e.g. to [Ma1, MS1] for these results. Let us also mention that improvements and extensions of (1.4) in terms of Lorentz norms could be proved similarly by [Ta], where Dirichlet problems are taken into account.

In the present paper we focus the limiting case where f belongs to $L^{n/2}(\Omega)$, or to Lorentz spaces close to $L^{n/2}(\Omega)$. In this case, weak solutions to (1.1) not only belong to $L^q(\Omega)$ for every $q < \infty$, but they are also exponentially summable. More precisely, constants $C_1 = C_1(\Omega)$ and $C_2 = C_2(\Omega)$ exist such that

$$\int_\Omega \exp\Big(C_1\frac{|u - \mathrm{m}(u)|}{\|f\|_{L^{n/2}}}\Big)^{\frac{n}{n-2}}\,dx \leq C_2, \tag{1.7}$$

as a combination of the estimate of [MS1] with an argument of [GT] easily shows. Our purpose is to improve on estimate (1.7) in two directions.

As a first result, we find the best constant C_1 for inequality (1.7) to hold for every Ω in suitable classes of domains, for every $f \in L^{n/2}(\Omega)$ and for every weak solution u to (1.1). It turns out that, if $\partial\Omega$ is smooth enough, of class $\mathscr{C}^{1,\alpha}$ say, then such a best constant depends only on the dimension n, and equals $n(n-2)(\omega_n/2)^{2/n}$, where ω_n is the measure of the unit ball in $\mathbf{R}^n$. This is a special case, corresponding to the choice $q = n/2$, of the following theorem, where f is allowed to belong to any Lorentz space $L^{n/2,q}(\Omega)$, with $1 < q \leq \infty$, and B is not necessarily identically equal to 0.

Theorem 1.1. *Let Ω be a bounded domain from the class $\mathscr{C}^{1,\alpha}$, for some $\alpha \in (0,1]$. Let $f \in L^{n/2,q}(\Omega)$ for some $q \in (1,\infty]$ and let $|B| \in L^{\sigma,\tau}(\Omega)$, for some $\sigma > n$ and $\tau \in [2,\infty]$. Let u be a weak solution to (1.1).*

(i) Case $1 < q < \infty$. A constant $C = C(\Omega, q, \|B\|_{L^{\sigma,\tau}})$ exists such that

$$\int_\Omega \exp\left(n(n-2)\left(\frac{\omega_n}{2}\right)^{2/n} \frac{|u - \mathrm{m}(u)|}{\|f\|_{L^{n/2,q}}}\right)^{q'} dx \le C. \tag{1.8}$$

The constant $n(n-2)(\omega_n/2)^{2/n}$ in (1.8) is sharp. Indeed, domains $\Omega \in \mathscr{C}^{1,\alpha}$ exist such that the left-hand side of (1.8), with $n(n-2)(\omega_n/2)^{2/n}$ replaced by any larger constant, cannot be uniformly bounded as f ranges among all functions from $L^{n/2,q}(\Omega)$ and u is a weak solution to (1.1) with $B(x) \equiv 0$ and $A(x) \equiv I$, the $n \times n$ unit matrix.

(ii) Case $q = \infty$. For every $\gamma < n(n-2)(\omega_n/2)^{2/n}$, a constant $C = C(\Omega, \gamma, \|B\|_{L^{\sigma,\tau}})$ exists such that

$$\int_\Omega \exp\left(\gamma \frac{|u - \mathrm{m}(u)|}{\|f\|_{L^{n/2,\infty}}}\right) dx \le C. \tag{1.9}$$

The result is sharp. Indeed, there exist domains $\Omega \in \mathscr{C}^{1,\alpha}$, functions $f \in L^{n/2,\infty}(\Omega)$ and weak solutions to (1.1) with $B(x) \equiv 0$ and $A(x) \equiv I$ such that the left-hand side of (1.9) diverges for every $\gamma \ge n(n-2)(\omega_n/2)^{2/n}$.

A more general version of Theorem 1.1, where irregular domains Ω with singularities on $\partial\Omega$ of conical type are admitted, is stated and proved in Section 3. Let us mention in advance that for these domains the best constant in (1.8)–(1.9) does depend on the geometry of $\partial\Omega$.

Notice that the case where $q = 1$ is not dealt with in Theorem 1.1, since weak solutions to (1.1) are in $L^\infty(\Omega)$ when $f \in L^{n/2,1}(\Omega)$ (see [Al] for the case of Dirichlet problems; the result for Neumann problems can be derived similarly via estimate (3.2), Section 3).

Results like those of Theorem 1.1 usually go under the name of Moser type inequalities, since they were first proved in [Mo] in the framework of Sobolev embeddings for the limiting space $W_0^{1,n}(\Omega)$. Estimates analogous to (1.8)–(1.9) for solutions to elliptic Dirichlet boundary value problems are the object of [AF, AFT, FFV1, FFV2, FFV3, L]. However, the discussion of the optimality of the constant in the case of equations seems to appear here for the first time, at least in this generality.

In order to illustrate the other improvement of inequality (1.7) that will be established, let us observe that (1.7) is equivalent to

$$\|u - \mathrm{m}(u)\|_{\exp L^{\frac{n}{n-2}}(\Omega)} \le C\, \|f\|_{L^{n/2}(\Omega)}, \tag{1.10}$$

for some positive constant $C = C(\Omega)$, where $\|\cdot\|_{\exp L^{\frac{n}{n-2}}(\Omega)}$ denotes the Luxemburg norm in the Orlicz space $\exp L^{\frac{n}{n-2}}(\Omega)$ associated with the Young function $e^{t^{\frac{n}{n-2}}} - 1$. Our second result ensures that if $f \in L^{n/2}(\Omega)$, then u is not just in $\exp L^{\frac{n}{n-2}}(\Omega)$, but belongs in fact to the strictly smaller Lorentz-Zygmund space $L^{\infty,n/2}(\log L)^{-1}(\Omega)$. Moreover, the result is sharp in the framework of all rearrangement invariant (briefly, r.i.) spaces, namely, those Banach function spaces where the norm of a function depends only on its decreasing rearrangement. Indeed, $L^{\infty,n/2}(\log L)^{-1}(\Omega)$ turns out to be the smallest possible space from this class to which any weak solution u to (1.1) belongs when $f \in L^{n/2}(\Omega)$. Recall that for $1 \le p, q \le \infty$ and $\alpha \in \mathbf{R}$, the space $L^{p,q}(\log L)^{\alpha}(\Omega)$ consists of those measurable functions g in Ω for which the quantity

$$\|g\|_{L^{p,q}(\log L)^{\alpha}(\Omega)} = \|s^{\frac{1}{p}-\frac{1}{q}}(1+\log(|\Omega|/s))^{\alpha} g^*(s)\|_{L^q(0,|\Omega|)} \tag{1.11}$$

Here, g^* is the decreasing rearrangement of g.

This result is part of Theorem 1.2 below, where data f and B in Lorentz spaces are considered. Observe that this theorem requires weaker regularity assumptions on Ω than Theorem 1.1; actually, any bounded domain satisfying a relative isoperimetric inequality with exponent $n' = \frac{n}{n-1}$ is allowed (see Section 2 for the definition). In particular, bounded domains with a Lipschitz boundary are admissible.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbf{R}^n$, $n \ge 3$, satisfying a relative isoperimetric inequality with exponent n'. Let $f \in L^{n/2,q}(\Omega)$ for some $q \in (1,\infty]$, and let $|B| \in L^{\sigma,\tau}(\Omega)$ for some $\sigma > n$ and $\tau \in [2,\infty]$. Let u be a weak solution to (1.1). Then a constant $C = C(\Omega, q, \|B\|_{L^{\sigma,\tau}})$ exists such that*

$$\|u - \mathrm{m}(u)\|_{L^{\infty,q}(\log L)^{-1}(\Omega)} \le C\|f\|_{L^{n/2,q}(\Omega)}. \tag{1.12}$$

The space $L^{\infty,q}(\log L)^{-1}(\Omega)$ is optimal among all r.i. spaces, in the sense that if Ω is any domain as above and $X(\Omega)$ is any r.i. space such that (1.12) holds with $\|\cdot\|_{L^{\infty,q}(\log L)^{-1}(\Omega)}$ replaced by $\|\cdot\|_{X(\Omega)}$ for every $f \in L^{n/2,q}(\Omega)$ and every weak solution to any problem having the form (1.1), then $L^{\infty,q}(\log L)^{-1}(\Omega) \subseteq X(\Omega)$.

As far as we know, estimates like (1.12), although appearing in the framework of Sobolev embeddings [BW, Han, CP, KP], are not known for solutions to elliptic equations, even subject to Dirichlet boundary conditions. Such estimates for solutions to Dirichlet problems can be derived by the methods of this paper.

We conclude the present section by a few considerations about these methods. Our approach to Theorems 1.1 and 1.2 rests upon a priori estimates for solutions to problem (1.1) in terms of rearrangements which go back to [MS1, MS2, Be, C2], and trace their origins in the work of Maz'ya [Ma1] and Talenti [Ta]. Similarly to analogous results for solutions to Dirichlet problems, these estimates rely upon isoperimetric inequalities. In the case where homogeneous Dirichlet boundary conditions are prescribed, the standard isoperimetric inequality in $\mathbf{R}^n$ is involved, solutions to properly spherically symmetrized problems enjoy suitable extremal properties, and thus any bound for these symmetric solutions translates into a corresponding bound for the solution to any problem in an appropriate class. The picture for Neumann problems is not so neat. Indeed, as elucidated in the fundamental paper [Ma1], the isoperimetric inequality in $\mathbf{R}^n$ has to be replaced by the relative isoperimetric inequality for open subsets of $\mathbf{R}^n$, and the latter is not explicitly known, except in very few cases. Furthermore, no extremal problem exists and, consequently, sharp bounds for solutions to problems like (1.1) are not automatically reduced to analogous bounds for solutions to some symmetrized problem. Fortunately, the piece of information which can be deduced from the relevant rearrangement estimate is sufficient, in fact, to derive optimal results at least as far as norms are concerned. This explains why Theorem 1.2 can be proven by these techniques. The situation for Theorem 1.1 is more delicate, since a sharp constant is involved. Actually, optimal constants in a priori bounds for solutions to Neumann problems are usually not derivable via rearrangement estimates. This approach is successful here because the constant in question turns out to depend only on an asymptotic form of the relative isoperimetric inequality for subsets of Ω whose measure approaches zero. Such an asymptotic inequality can actually be established, at least for sufficiently regular domains, as demonstrated by recent results of [C4], where Moser type inequalities for functions not necessarily vanishing on the boundary are discussed (see also [AH-S, CY, Ch, EH-S] for related exponential inequalities).

2. Prerequisites

We collect in this section some miscellaneous definitions and results known in the mathematical literature and coming into play in the proofs of Theorems 1.1 and 1.2.

2.1. Isoperimetric inequalities. The isoperimetric function $h_\Omega\colon (0,|\Omega|) \to [0,+\infty)$ of an open set Ω in $\mathbf{R}^n$ is defined as

$$h_\Omega(s) = \inf\{\mathscr{P}(E;\Omega) : E \subset \Omega, |E| = s\} \quad \text{for } s \in (0,|\Omega|), \tag{2.1}$$

where $\mathscr{P}(E;\Omega)$ is the perimeter of E relative to Ω (see e.g. [AFP, Definition 3.35]). Notice that $\mathscr{P}(E;\Omega) = \mathscr{H}^{n-1}(\partial E \cap \Omega)$, the $(n-1)$-dimensional Hausdorff measure of $\partial E \cap \Omega$, if E is sufficiently smooth. Equation (2.1) immediately implies that

$$\mathscr{P}(E;\Omega) \geq h_\Omega(|E|) \tag{2.2}$$

for every measurable subset E of Ω. Inequality (2.2) is called the relative isoperimetric inequality in Ω. The isoperimetric function of any open set Ω having finite measure is symmetric about $|\Omega|/2$; namely,

$$h_\Omega(s) = h_\Omega(|\Omega| - s) \quad \text{for } s \in (0,|\Omega|). \tag{2.3}$$

Unfortunately, h_Ω is explicitly known only for very special sets Ω, such as balls, hyperplanes and convex cones. Nevertheless, many applications just involve the behavior of h_Ω at 0, and information on this point is much easier to derive. For instance, if Ω is connected, has finite measure and satisfies the cone property, there exists a positive constant $C = C(\Omega)$ such that

$$h_\Omega(s) \geq C \min{}^{1/n'}\{s, |\Omega| - s\} \quad \text{for } s \in (0,|\Omega|) \tag{2.4}$$

([Ma2, Corollary 3.2.1/3]). A domain Ω fulfilling (2.4) is usually said to satisfy a relative isoperimetric inequality with exponent $1/n'$.

A precise asymptotic estimate for the isoperimetric function h_Ω is available for domains from the class $\Sigma^{1,\alpha}$ defined as follows.

Definition 2.1. Let $\alpha \in (0,1]$. An open subset Ω of $\mathbf{R}^n$ is said to be a domain of class $\Sigma^{1,\alpha}$ if a finite family $\{U_k\}_{k\in K}$, $K \subset \mathbf{N}$, of open subsets of $\mathbf{R}^n$ exists satisfying the following properties:

(i) $\overline{\Omega} \subset \cup_{k\in K} U_k$;

(ii) for each $k \in K$ there exists an open subset V_k of $\mathbf{R}^n$, a diffeomorphism $\Phi_k\colon U_k \to V_k$, a point $x_k \in U_k$ and an open convex cone Λ_k (possibly the whole of $\mathbf{R}^n$) with vertex at $\Phi_k(x_k)$ and smooth boundary, such that

$$\Phi_k\colon \overline{\Omega} \cap U_k \to \overline{\Lambda}_k \cap V_k$$

is a homeomorphism;

(iii) the Jacobian matrix $\mathrm{J}\,\Phi_k(x_k) = \mathrm{I}$;

(iv)

$$|\mathrm{J}\,\Phi_k(x) - \mathrm{J}\,\Phi_k(y)| \le L|x-y|^\alpha$$

for some constant L and for every $x, y \in U_k$.

In particular, we have

Definition 2.2. Let $\alpha \in (0,1]$. An open subset Ω of $\mathbf{R}^n$ is said to be a domain of class $\mathscr{C}^{1,\alpha}$ if it satisfies the definition of domain of class $\Sigma^{1,\alpha}$ with Λ_k either equal to a half-space or to $\mathbf{R}^n$ for every $k \in K$.

In our applications, the minimum of the solid apertures

$$\theta_k = |\Lambda_k \cap B_1(\Phi_k(x_k))|$$

of the cones Λ_k in Definition 2.1 will play a role, where $B_r(x)$ denotes the ball centered at x and having radius r. We call such aperture θ_Ω; namely, we set

$$\theta_\Omega = \min_{k\in K} \theta_k.$$

Note that, with this notation in force, the class $\mathscr{C}^{1,\alpha}$ can be identified with the subclass of those domains Ω in $\Sigma^{1,\alpha}$ satisfying $\theta_\Omega = \omega_n/2$.

An asymptotically sharp relative isoperimetric inequality for domains in $\Sigma^{1,\alpha}$ is proved in [C4, Theorem1.3]. We shall make use of a consequence of that inequality, which tells us that if Ω is a bounded domain from the class $\Sigma^{1,\alpha}$ for some $\alpha \in (0,1]$, there exist constants $\beta = \beta(\Omega) > 0$, $C = C(\Omega) > 0$ and $s_1 = s_1(\Omega) \in \big(0, |\Omega|/2\big]$ such that if $h\colon (0,|\Omega|) \to (0,+\infty)$ is the function defined as

$$h(s) = \begin{cases} n\,\theta_\Omega^{1/n}\, s^{1/n'}\,(1 - Cs^\beta) & \text{if } s \in (0, s_1], \\ h(s_1) & \text{if } s \in (s_1, |\Omega|/2], \end{cases} \tag{2.5}$$

and symmetric about $|\Omega|/2$, then $h(s)$ and $\dfrac{s}{h(s)}$ are nondecreasing in $\big(0, |\Omega|/2\big)$, and

$$h_\Omega(s) \ge h(s) \quad \text{for } s \in (0, |\Omega|) \tag{2.6}$$

(see [C4, Corollary 2.1]).

For a detailed study and for applications of isoperimetric functions and inequalities, we refer to [C1, G, Ma1, MP].

2.2. Rearrangements and rearrangement invariant spaces. Let Ω be a measurable subset of $\mathbf{R}^n$ having finite measure and let u be a real-valued measurable function in Ω. The *decreasing rearrangement* u^* of u is the unique non-increasing right-continuous function from $[0,+\infty)$ into $[0,+\infty]$ which is equidistributed with u. In formulas,

$$u^*(s) = \sup\{t \geq 0 : |\{x \in \Omega : |u(x)| > t\}| > s\} \quad \text{for } s \geq 0.$$

The function $u^{**}\colon (0,+\infty) \to [0,+\infty]$, defined as $u^{**}(s) = \frac{1}{s}\int_0^s u^*(r)\,dr$ for $s > 0$, is also non-increasing and satisfies $u^* \leq u^{**}$.

A rearrangement invariant space $X(\Omega)$ is a Banach function space equipped with a norm $\|\cdot\|_{X(\Omega)}$ satisfying

$$\|v\|_{X(\Omega)} = \|u\|_{X(\Omega)} \quad \text{if } u^* = v^*. \tag{2.7}$$

The *associate space* $X'(\Omega)$ of $X(\Omega)$ is the r.i. space of those measurable functions v in Ω for which the r.i. norm

$$\|v\|_{X'(\Omega)} = \sup_{u \neq 0} \frac{\int_\Omega |uv|\,dx}{\|u\|_{X(\Omega)}} \tag{2.8}$$

is finite. As a consequence, the Hölder type inequality

$$\int_\Omega |uv|\,dx \leq \|u\|_{X(\Omega)}\|v\|_{X'(\Omega)} \tag{2.9}$$

holds for every $u \in X(\Omega)$ and $v \in X'(\Omega)$.

The *representation space* $\overline{X}(0,|\Omega|)$ of an r.i. space $X(\Omega)$ is the unique r.i. space on $(0,|\Omega|)$ satisfying

$$\|u\|_{X(\Omega)} = \|u^*\|_{\overline{X}(0,|\Omega|)} \tag{2.10}$$

for every $u \in X(\Omega)$. In most instances, an expression for the norm $\|\cdot\|_{\overline{X}(0,|\Omega|)}$ is immediately derived from that of $\|\cdot\|_{X(\Omega)}$. In general, one has

$$\|\varphi\|_{\overline{X}(0,|\Omega|)} = \sup_{\|v\|_{X'(\Omega)} \leq 1} \int_0^{|\Omega|} \varphi^*(r)\,v^*(r)\,dr. \tag{2.11}$$

Hardy's lemma ensures that if $X(\Omega)$ is any r.i. space and u and v are measurable functions in Ω such that $v \in X(\Omega)$, then

$$u^{**} \leq v^{**} \quad \text{implies} \quad \|u\|_{X(\Omega)} \leq \|v\|_{X(\Omega)}. \tag{2.12}$$

Let $l > 0$ and let $X(0,l)$ be any r.i. space on $(0,l)$. Then the linear operator $T\colon X(0,l) \to X(0,l)$, defined by

$$(T\varphi)(s) = \varphi(s/2) \quad \text{for } s \in (0,l)\,, \tag{2.13}$$

is bounded, and

$$\|T\| \le 2. \tag{2.14}$$

Lebesgue, Orlicz, Lorentz and Lorentz-Zygmund spaces are customary examples of r.i. spaces. If $1 < p < \infty$ and $1 \le q \le \infty$, or $p = q = \infty$, the Lorentz space $L^{p,q}(\Omega)$ is a special case of the Lorentz-Zygmund spaces $L^{p,q}(\log L)^{\alpha}(\Omega)$ defined in Section 1, corresponding to $\alpha = 0$. Notice that the quantities $\|\cdot\|_{L^{p,q}(\Omega)}$ and $\|\cdot\|_{L^{p,q}(\log L)^{\alpha}(\Omega)}$ need not be norms, but they are always equivalent, up to multiplicative constants, to the r.i. norms $\|\cdot\|_{L^{(p,q)}(\Omega)}$ and $\|\cdot\|_{L^{(p,q)}(\log L)^{\alpha}(\Omega)}$ obtained on replacing u^* by u^{**} in the definition. Thus, in particular, positive constants $C_1 = C_1(p,q)$ and $C_2 = C_2(p,q)$ exist such that

$$C_1 \|u\|_{L^{p,q}(\Omega)} \le \|u\|_{L^{(p,q)}(\Omega)} \le C_2 \|u\|_{L^{p,q}(\Omega)} \tag{2.15}$$

for every $u \in L^{p,q}(\Omega)$. The associate space to $L^{(p,q)}(\Omega)$ is, up to equivalent norms, $L^{(p',q')}(\Omega)$ for all admissible values of p and q. Thus, owing to (2.8), (2.9) and (2.15), positive constants $C_1 = C_1(p,q)$ and $C_2 = C_2(p,q)$ exist such that

$$C_1 \|v\|_{L^{p',q'}(\Omega)} \le \sup_{u \in L^{p,q}(\Omega)} \frac{\int_\Omega |uv|\,dx}{\|u\|_{L^{p,q}(\Omega)}} \le C_2 \|v\|_{L^{p',q'}(\Omega)} \tag{2.16}$$

for every $v \in L^{p',q'}(\Omega)$.

A thorough treatment of r.i. spaces can be found in [BS].

2.3. One-dimensional inequalities. A weighted version of the Hardy inequality states the following. Let $l > 0$, let $q \in [1,\infty]$ and let μ and ν be nonnegative locally integrable functions in $[0,l]$. Define

$$K_1 = \sup_{s \in (0,l)} \|\mu\|_{L^q(s,l)} \|1/\nu\|_{L^{q'}(0,s)}. \tag{2.17}$$

If $K_1 < \infty$, then

$$\left\| \mu(s) \int_0^s \varphi(r)\,dr \right\|_{L^q(0,l)} \le (q')^{1/q'} q^{1/q} K_1 \|\nu(s)\,\varphi(s)\|_{L^q(0,l)} \tag{2.18}$$

for every nonnegative measurable function φ in $[0,l]$. Define

$$K_2 = \sup_{s \in (0,l)} \|\mu\|_{L^q(0,s)} \|1/\nu\|_{L^{q'}(s,l)}. \tag{2.19}$$

If $K_2 < \infty$, then

$$\left\| \mu(s) \int_s^l \varphi(r)\,dr \right\|_{L^q(0,l)} \le (q')^{1/q'} q^{1/q} K_2 \|\nu(s)\,\varphi(s)\|_{L^q(0,l)} \tag{2.20}$$

for every nonnegative measurable function φ in $[0,l]$ (see e.g. [Ma2, Section 1.3.1] or [OK]).

A Hölder type inequality for non-increasing function ensures that if $l > 0$, $q \in (1, \infty)$ and μ is as above, then there exists a constant $C = C(q)$ such that

$$\begin{aligned}\int_0^l \varphi(s)\,\psi(s)\,ds \le C\Big(\int_0^l \varphi(s)^q\,\mu(s)\,ds\Big)^{1/q} \\ \cdot\Bigg[\Bigg(\int_0^l\Big(\int_0^s \psi(r)\,dr\Big)^{q'}\frac{\mu(s)}{\Big(\int_0^s \mu(r)\,dr\Big)^{q'}}\,ds\Bigg)^{1/q'} \\ + \frac{\int_0^l \psi(s)\,ds}{\Big(\int_0^l \mu(s)\,ds\Big)^{1/q}}\Bigg]\end{aligned} \tag{2.21}$$

for any measurable $\psi\colon [0, l] \to [0, +\infty)$ and any non-increasing $\varphi\colon [0, l] \to [0, +\infty)$ (see [Sa, Theorem 1]).

An extension of Moser's one-dimensional lemma, appearing in [FFV1], tells us the following. Let $l \in \mathbf{R}$, $q \in (1, \infty)$ and let $k\colon (l, +\infty) \times (l, +\infty) \to [0, \infty)$ be a measurable kernel. Set

$$S = \sup_{t>l} \int_t^\infty k(t, \zeta)^{q'}\,d\zeta.$$

Assume that

$$S < \infty \tag{2.22}$$

and that there exists $g\colon (l, +\infty) \to [0, +\infty)$ satisfying

$$\begin{cases} k(t, \zeta) \le 1 + g(\zeta) & \text{if } l < \zeta < t, \\ g \in L^1(l, \infty) \cap L^{q'}(l, \infty). \end{cases} \tag{2.23}$$

Then a constant $C = C\left(q, l, \|g\|_{L^1}, \|g\|_{L^{q'}}, S\right)$ exists such that

$$\int_l^\infty \exp\left[\left(\frac{1}{\|\varphi\|_{L^q(l,\infty)}}\int_l^\infty k(t, \zeta)\,\varphi(\zeta)\,d\zeta\right)^{q'} - t\right] dt \le C \tag{2.24}$$

for every $\varphi \in L^q(l, \infty)$.

3. Moser type estimates

The point of departure in our proofs is the following rearrangement estimate for solutions to problem (1.1). Let Ω be a bounded domain in $\mathbf{R}^n$. Assume that $f \in L^{n/2,q}(\Omega)$ for some $q \in [1, \infty]$ and that $|B| \in L^{\sigma,\tau}(\Omega)$ for some $\sigma > n$ and $\tau \in [2, \infty]$. Let u be a weak solution to problem (1.1). Then there exists a measurable function

$b\colon (0,|\Omega|) \to [0,+\infty)$, fulfilling

$$\begin{aligned} &\int_0^s (b^*(r))^2\,dr \le \int_0^s (|B|^*(r))^2\,dr \quad \text{for } s \in (0,|\Omega|), \\ &\int_0^{|\Omega|} (b^*(r))^2\,dr = \int_0^{|\Omega|} (|B|^*(r))^2\,dr, \end{aligned} \tag{3.1}$$

such that

$$\big(u - \mathrm{m}(u)\big)_i^*(s) \le \int_s^{|\Omega|/2} \frac{1}{h_\Omega^2(\rho)} \left(\int_0^\rho \exp\Big(\int_\zeta^\rho \frac{b(\eta)}{h_\Omega(\eta)}\,d\eta \Big) f_i^*(\zeta)\,d\zeta \right) d\rho, \tag{3.2}$$

$i = 1, 2$, for $s \in \big(0, |\Omega|/2\big)$, where $f_1 = \max\{f, 0\}$, $f_2 = \max\{-f, 0\}$, the positive and the negative part of f, respectively, and $\big(u - \mathrm{m}(u)\big)_i$, $i = 1, 2$, are defined analogously. Notice that, as a consequence of (2.15), (3.1) and (2.12), a constant $C = C\,(\sigma, \tau)$ exists such that for every $\sigma > 2$ and $\tau \ge 2$

$$\|b\|_{L^{\sigma,\tau}(\Omega)} \le C\,\|B\|_{L^{\sigma,\tau}(\Omega)}. \tag{3.3}$$

A version of inequality (3.2) is established in [Be] for $|B| \in L^\infty(\Omega)$ and for domains Ω satisfying a relative isoperimetric inequality with exponent $1/n'$, with $h_\Omega(s)$ replaced by the right-hand side of (2.4). A proof of the slightly more general estimate (3.2) can be accomplished similarly; the necessary modifications can be patterned on the arguments of [AFT], where an analogous estimate for solutions to Dirichlet problems is given.

As announced in Section 1, we prove a generalized version of Theorem 1.1 for domains from the class $\Sigma^{1,\alpha}$.

Theorem 3.1. *Let Ω be a bounded domain from the class $\Sigma^{1,\alpha}$ for some $\alpha \in (0,1]$. Let $f \in L^{n/2,q}(\Omega)$ for some $q \in (1,\infty]$ and let $|B| \in L^{\sigma,\tau}(\Omega)$ for some $\sigma > n$ and $\tau \in [2,\infty]$. Let u be a weak solution to (1.1).*

(i) Case $1 < q < \infty$. A constant $C = C\,(\Omega, q, \|B\|_{L^{\sigma,\tau}})$ exists such that

$$\int_\Omega \exp\Big(n(n-2)\,\theta_\Omega^{2/n} \frac{|u - \mathrm{m}(u)|}{\|f\|_{L^{n/2,q}}} \Big)^{q'} dx \le C. \tag{3.4}$$

The constant $n(n-2)\,\theta_\Omega^{2/n}$ in (3.4) is sharp in the same sense as in Theorem 1.1, part (i).

(ii) Case $q = \infty$. For every $\gamma < n(n-2)\,\theta_\Omega^{2/n}$, a constant $C = C\,(\Omega, \gamma, \|B\|_{L^{\sigma,\tau}})$ exists such that

$$\int_\Omega \exp\Big(\gamma \frac{|u - \mathrm{m}(u)|}{\|f\|_{L^{n/2,\infty}}} \Big)\,dx \le C. \tag{3.5}$$

The result is sharp in the same sense as in Theorem 1.1, part (ii).

Remark 3.2. The same conclusions as in Theorem 3.1 hold when Ω is any bounded convex polytope in $\mathbf{R}^n$, provided that θ_Ω is replaced by the minimum of the solid apertures of the support cones to Ω. The proof is completely analogous to that given below, on making use of a version of estimate (2.6) which follows from [C4, Prop. 2.1].

Our approach to Theorem 3.1 is related to that of [Mo] and of [Ad, AFT, FFV1, Fo]. We split the proof in two parts. In Part I inequalities (3.4)-(3.5) are established. Their optimality is proved in Part II.

Proof of Theorem 3.1, Part I. Consider first the case where $q < \infty$. From estimates (3.2) and (2.6) we get

(3.6)
$$\big(u - \mathrm{m}(u)\big)_i^*(s) \le \int_s^{|\Omega|/2} \frac{1}{h^2(\rho)} \left(\int_0^\rho \exp\Big(\int_\zeta^\rho \frac{b(\eta)}{h(\eta)}\, d\eta \Big) f_i^*(\zeta)\, d\zeta \right) d\rho, \; i = 1, 2,$$

for $s \in (0, |\Omega|/2)$. An application of Fubini's theorem to the integral on right-hand side of (3.6) yields

$$\int_s^{|\Omega|/2} \frac{1}{h^2(\rho)} \left(\int_0^\rho \exp\Big(\int_\zeta^\rho \frac{b(\eta)}{h(\eta)}\, d\eta \Big) f_i^*(\zeta)\, d\zeta \right) d\rho = \int_0^{|\Omega|/2} f_i^*(r)\, a(s,r)\, dr, \tag{3.7}$$

$i = 1, 2$, for $s \in (0, |\Omega|/2)$, where $a\colon (0, |\Omega|/2) \times (0, |\Omega|/2) \to (0, +\infty)$ is defined as

$$a(s,r) = \begin{cases} \displaystyle\int_s^{|\Omega|/2} \frac{1}{h^2(\rho)} \exp\Big(\int_r^\rho \frac{b(\eta)}{h(\eta)}\, d\eta \Big)\, d\rho & \text{if } 0 < r \le s < |\Omega|/2, \\ \displaystyle\int_r^{|\Omega|/2} \frac{1}{h^2(\rho)} \exp\Big(\int_r^\rho \frac{b(\eta)}{h(\eta)}\, d\eta \Big)\, d\rho & \text{if } 0 < s < r < |\Omega|/2. \end{cases} \tag{3.8}$$

From (3.6)–(3.7), via a change of variable, one obtains

$$\big(u - \mathrm{m}(u)\big)_i^* (|\Omega|\, e^{-t}) \le |\Omega| \int_{\log 2}^\infty f_i^*(|\Omega|\, e^{-\zeta})\, a(|\Omega|\, e^{-t}, |\Omega|\, e^{-\zeta})\, d\zeta, \quad i = 1, 2, \tag{3.9}$$

for $t > \log 2$. Another change of variables in the integrals defining the function a tells us that

(3.10)
$$\begin{aligned} & a(|\Omega|\, e^{-t}, |\Omega|\, e^{-\zeta}) \\ &= \begin{cases} \displaystyle |\Omega| \int_{\log 2}^t \frac{e^{-\rho}}{h^2(|\Omega|\, e^{-\rho})} \exp\Big(|\Omega| \int_\rho^\zeta \frac{b(|\Omega|\, e^{-\lambda})}{h(|\Omega|\, e^{-\lambda})}\, e^{-\lambda}\, d\lambda \Big)\, d\rho & \text{if } \log 2 < t \le \zeta, \\ \displaystyle |\Omega| \int_{\log 2}^\zeta \frac{e^{-\rho}}{h^2(|\Omega|\, e^{-\rho})} \exp\Big(|\Omega| \int_\rho^\zeta \frac{b(|\Omega|\, e^{-\lambda})}{h(|\Omega|\, e^{-\lambda})}\, e^{-\lambda}\, d\lambda \Big)\, d\rho & \text{if } \log 2 < \zeta < t, \end{cases} \\ &\le \begin{cases} \displaystyle |\Omega| \int_{\log 2}^t \frac{e^{-\rho}}{h^2(|\Omega|\, e^{-\rho})} \exp\Big(|\Omega| \int_\rho^\infty \frac{b(|\Omega|\, e^{-\lambda})}{h(|\Omega|\, e^{-\lambda})}\, e^{-\lambda}\, d\lambda \Big)\, d\rho & \text{if } \log 2 < t \le \zeta, \\ \displaystyle |\Omega| \int_{\log 2}^\zeta \frac{e^{-\rho}}{h^2(|\Omega|\, e^{-\rho})} \exp\Big(|\Omega| \int_\rho^\infty \frac{b(|\Omega|\, e^{-\lambda})}{h(|\Omega|\, e^{-\lambda})}\, e^{-\lambda}\, d\lambda \Big)\, d\rho & \text{if } \log 2 < \zeta < t. \end{cases} \end{aligned}$$

Hence, on setting

$$\phi_i(\zeta) = f_i^*(|\Omega|e^{-\zeta})e^{-2\zeta/n}\,|\Omega|^{2/n} \quad \text{for } \zeta > \log 2$$

and

$$K(t,\zeta) = \begin{cases} |\Omega|^{2/n'} e^{(-1+2/n)\zeta} \displaystyle\int_{\log 2}^{t} \frac{e^{-\rho}}{h^2(|\Omega|\,e^{-\rho})} \exp\Big(|\Omega| \int_\rho^\infty \frac{b(|\Omega|\,e^{-\lambda})}{h(|\Omega|\,e^{-\lambda})}\,e^{-\lambda}\,d\lambda\Big)\,d\rho \\ \text{if } \log 2 < t \le \zeta, \\ |\Omega|^{2/n'} e^{(-1+2/n)\zeta} \displaystyle\int_{\log 2}^{\zeta} \frac{e^{-\rho}}{h^2(|\Omega|\,e^{-\rho})} \exp\Big(|\Omega| \int_\rho^\infty \frac{b(|\Omega|\,e^{-\lambda})}{h(|\Omega|\,e^{-\lambda})}\,e^{-\lambda}\,d\lambda\Big)\,d\rho \\ \text{if } \log 2 < \zeta < t, \end{cases}$$

we have

$$\big(u - \mathrm{m}(u)\big)_i^*\,(|\Omega|\,e^{-t}) \le \int_{\log 2}^\infty K(t,\zeta)\,\phi_i(\zeta)\,d\zeta, \quad i = 1, 2, \tag{3.11}$$

for $t > \log 2$. We claim that the kernel $k(t,\zeta) = n(n-2)\theta_\Omega^{2/n} K(t,\zeta)$ satisfies assumptions (2.22)–(2.23) with $p = q$ and

$$g(\zeta) = \max\{\overline{g}(\zeta) - 1, 0\}, \tag{3.12}$$

where

$$\begin{aligned} \overline{g}(\zeta) &= n(n-2)\theta_\Omega^{2/n}|\Omega|^{2/n'} e^{(-1+2/n)\zeta} \\ &\quad \cdot \int_{\log 2}^{\zeta} \frac{e^{-\rho}}{h^2(|\Omega|\,e^{-\rho})}\, \exp\Big(|\Omega| \int_\rho^\infty \frac{b(|\Omega|\,e^{-\lambda})}{h(|\Omega|\,e^{-\lambda})}\,e^{-\lambda}\,d\lambda\Big)\,d\rho \end{aligned} \tag{3.13}$$

for $\zeta > \log 2$. Indeed, we have

$$\begin{aligned} &\sup_{t>\log 2} \int_t^\infty K(t,\zeta)^{q'}\,d\zeta \\ &= \sup_{t>\log 2} |\Omega|^{\frac{2}{n'}q'} \int_t^\infty e^{-\frac{n-2}{n}\,q'\,\zeta} \Big(\int_{\log 2}^{t} \frac{e^{-\rho}}{h^2(|\Omega|e^{-\rho})} \\ &\quad \cdot \exp\Big(|\Omega| \int_\rho^\infty \frac{b(|\Omega|\,e^{-\lambda})}{h(|\Omega|\,e^{-\lambda})}\,e^{-\lambda}\,d\lambda\Big)\,d\rho\Big)^{q'}\,d\zeta \\ &= \sup_{t>\log 2} |\Omega|^{\frac{2}{n'}q'} \Big(\int_{\log 2}^{t} \frac{e^{-\rho}}{h^2(|\Omega|e^{-\rho})} \\ &\quad \cdot \exp\Big(|\Omega| \int_\rho^\infty \frac{b(|\Omega|\,e^{-\lambda})}{h(|\Omega|\,e^{-\lambda})}\,e^{-\lambda}\,d\lambda\Big)\,d\rho\Big)^{q'} \int_t^\infty e^{-\frac{n-2}{n}\,q'\,\zeta}\,d\zeta. \end{aligned} \tag{3.14}$$

Now, by (2.16) and (3.3)

$$
\begin{aligned}
|\Omega| \int_{\log 2}^{\infty} \frac{b(|\Omega|\, e^{-\lambda})}{h(|\Omega|\, e^{-\lambda})}\, e^{-\lambda}\, d\lambda &= \int_{0}^{|\Omega|/2} \frac{b(\eta)}{h(\eta)}\, d\eta \\
&\le C \|b\|_{L^{\sigma,\tau}(0,|\Omega|/2)} \big\|1/h\big\|_{L^{\sigma',\tau'}(0,|\Omega|/2)} \\
&\le C_1 \|B\|_{L^{\sigma,\tau}(0,|\Omega|/2)} \big\|1/h\big\|_{L^{\sigma',\tau'}(0,|\Omega|/2)}
\end{aligned}
\tag{3.15}
$$

for some constants $C = C(\sigma,\tau)$ and $C_1 = C_1(\sigma,\tau)$. Owing to (2.5), the last norm is finite. Consequently, by (3.14)–(3.15), a constant $C = C\left(\Omega, \sigma, \tau, q, \|B\|_{L^{\sigma,\tau}}\right)$ exists such that

$$
\sup_{t>\log 2} \int_t^{\infty} K(t,\zeta)^{\frac{n}{n-2}}\, d\zeta \le \sup_{t>\log 2} C\, e^{(-\frac{n-2}{n})\, q'\, t} \Big(\int_{\log 2}^{t} \frac{e^{-\rho}}{h^2(|\Omega| e^{-\rho})}\, d\rho \Big)^{q'}. \tag{3.16}
$$

By (2.5), a constant $C = C(\Omega)$ exists such that the last integral does not exceed $C\big(1 + \int_{\log(|\Omega|/s_1)}^{t} e^{\frac{n-2}{n}\rho}\, d\rho\big)$. Hence, (2.22) follows. As far as (2.23) is concerned, the inequality is trivial. As for the second condition, an application of De L'Hopital rule shows that the function $\overline{g}$ given by (3.13) satisfies $\lim\limits_{\zeta\to+\infty} \overline{g}(\zeta) = 0$. Moreover, if $\alpha < \min\left\{1 - \frac{2}{n}, \beta, \frac{1}{n} - \frac{1}{\sigma}\right\}$,

$$
\begin{aligned}
&\lim_{\zeta\to+\infty} \frac{\overline{g}(\zeta)}{e^{-\alpha\zeta}} \\
&= \lim_{\zeta\to+\infty} \frac{1}{e^{(1-2/n-\alpha)\zeta}} \Big[n(n-2)\theta_\Omega^{2/n} |\Omega|^{2/n'} \Big(\int_{\log 2}^{\zeta} \frac{e^{-\rho}}{h^2(|\Omega| e^{-\rho})} \\
&\quad \cdot \exp\Big(|\Omega| \int_{\rho}^{\infty} \frac{b(|\Omega|\, e^{-\lambda})}{h(|\Omega|\, e^{-\lambda})}\, e^{-\lambda}\, d\lambda \Big)\, d\rho \Big) - e^{(1-2/n)\zeta} \Big] \\
&= \lim_{\zeta\to+\infty} \frac{1}{\big(1-\frac{2}{n}-\alpha\big) e^{(1-2/n-\alpha)\zeta}} \Big[n(n-2)\theta_\Omega^{2/n} |\Omega|^{2/n'} \frac{e^{-\zeta}}{h^2(|\Omega| e^{-\zeta})} \\
&\quad \cdot \exp\Big(|\Omega| \int_{\zeta}^{\infty} \frac{b(|\Omega|\, e^{-\lambda})}{h(|\Omega|\, e^{-\lambda})}\, e^{-\lambda}\, d\lambda \Big) - \Big(1 - \frac{2}{n}\Big) e^{(1-2/n)\zeta} \Big] \\
&= \lim_{\zeta\to+\infty} \frac{1}{\big(1-\frac{2}{n}-\alpha\big) e^{(1-2/n-\alpha)\zeta}} \Big[\frac{n-2}{n} \frac{e^{-\zeta}}{e^{-\frac{2}{n'}\zeta}\big(1 - C\,|\Omega|^{\beta}\, e^{-\beta\zeta}\big)^2} \\
&\quad \cdot \exp\Big(|\Omega| \int_{\zeta}^{\infty} \frac{b(|\Omega|\, e^{-\lambda})}{h(|\Omega|\, e^{-\lambda})}\, e^{-\lambda}\, d\lambda \Big) - \Big(1 - \frac{2}{n}\Big) e^{(1-2/n)\zeta} \Big] \\
&= \lim_{\zeta\to+\infty} \frac{n-2}{n} \frac{e^{(1-2/n)\,\zeta}}{\big(1-\frac{2}{n}-\alpha\big)\, e^{(1-2/n-\alpha)\,\zeta}} \Big[\Big(1 + 2\,C\,|\Omega|^{\beta}\, e^{-\beta\zeta} + o\,(e^{-\zeta\beta})\Big) \\
&\quad \cdot \Big(1 + |\Omega| \int_{\zeta}^{\infty} \frac{b(|\Omega|\, e^{-\lambda})}{h(|\Omega|\, e^{-\lambda})}\, e^{-\lambda}\, d\lambda + o\Big(\int_{\zeta}^{\infty} \frac{b(|\Omega|\, e^{-\lambda})}{h(|\Omega|\, e^{-\lambda})}\, e^{-\lambda}\, d\lambda \Big)\Big) - 1 \Big],
\end{aligned}
\tag{3.17}
$$

where the second equality follows from an application of De L'Hopital rule, and the third one is due to (2.5). Here, the notation $o(\phi(\zeta))$ means that $\lim_{\zeta\to+\infty} o(\phi(\zeta))/\phi(\zeta) = 0$. Equation (2.5) also ensures that a constant $C = C(\Omega, \sigma, \tau)$ exists such that

$$\big\|1/h\big\|_{L^{\sigma',\tau'}(0,|\Omega|e^{-t})} \le C\, e^{-(\frac{1}{n}-\frac{1}{\sigma})\,t}$$

if t is sufficiently large. Hence, similarly as in (3.15), we have

$$|\Omega| \int_t^{\infty} \frac{b(|\Omega|\, e^{-\lambda})}{h(|\Omega|\, e^{-\lambda})}\, e^{-\lambda}\, d\lambda = \int_0^{|\Omega|e^{-t}} \frac{b(\eta)}{h(\eta)}\, d\eta \le C\, \|B\|_{L^{\sigma,\tau}(0,|\Omega|/2)}\, e^{-(\frac{1}{n}-\frac{1}{\sigma})\,t} \tag{3.18}$$

for some constant $C = C(\Omega, \sigma, \tau)$ and for large t. From (3.17)–(3.18) one easily infers that

$$\lim_{t\to+\infty} \frac{\bar g(t)}{e^{-\alpha t}} = 0. \tag{3.19}$$

Therefore, $g \in L^1(\log 2, \infty) \cap L^{q'}(\log 2, \infty)$, and (2.23) holds.

Now, one has

$$\begin{aligned} &\int_\Omega \exp\Big(n(n-2)\theta_\Omega^{2/n} \frac{|u - \mathrm{m}(u)|}{\|f\|_{L^{n/2,q}}}\Big)^{q'} dx \\ &\le \sum_{i=1}^{2} \int_\Omega \exp\Big(n(n-2)\theta_\Omega^{2/n} \frac{(u - \mathrm{m}(u))_i}{\|f_i\|_{L^{n/2,q}}}\Big)^{q'} dx \\ &= \sum_{i=1}^{2} \int_0^{\frac{|\Omega|}{2}} \exp\Big(n(n-2)\theta_\Omega^{2/n} \frac{(u - \mathrm{m}(u))_i^*(s)}{\|f_i\|_{L^{n/2,q}}}\Big)^{q'} ds \\ &= |\Omega| \sum_{i=1}^{2} \int_{\log 2}^{\infty} \exp\Big[\Big(n(n-2)\theta_\Omega^{2/n} \frac{(u - \mathrm{m}(u))_i^*(|\Omega|e^{-t})}{\|f_i\|_{L^{n/2,q}}}\Big)^{q'} - t\Big]\, dt \\ &\le |\Omega| \sum_{i=1}^{2} \int_{\log 2}^{\infty} \exp\Big[\Big(n(n-2)\theta_\Omega^{2/n} \frac{1}{\|f_i\|_{L^{n/2,q}}} \int_{\log 2}^{\infty} K(t,\zeta)\phi_i(\zeta)\, d\zeta\Big)^{q'} - t\Big]\, dt, \end{aligned} \tag{3.20}$$

where the last inequality is a consequence of (3.11). On the other hand,

$$\begin{aligned} \|\phi_i\|_{L^q(\log 2,\infty)} &= \Big(\int_{\log 2}^{\infty} \big(f_i^*(|\Omega|e^{-\zeta})\, e^{-2\zeta/n}|\Omega|^{2/n}\big)^q\, d\zeta\Big)^{1/q} \\ &= \Big(\int_0^{|\Omega|/2} \big(f_i^*(s)s^{2/n}\big)^q \frac{ds}{s}\Big)^{1/q} = \|f_i\|_{L^{n/2,q}(0,|\Omega|/2)}. \end{aligned} \tag{3.21}$$

By (3.20)–(3.21) and by (2.24) inequality (3.4) follows.

Assume now that $q = \infty$. The very definition of Lorentz norm entails that

$$\frac{f_i^*(s)}{\|f\|_{L^{n/2,\infty}}} \le \frac{f_i^*(s)}{\|f_i\|_{L^{n/2,\infty}}} \le s^{-2/n} \quad \text{for } s > 0. \tag{3.22}$$

Given $\gamma > 0$, inequalities (3.6) and (3.22) yield

$$
\begin{aligned}
\int_\Omega \exp\Big(\gamma \frac{|u-\mathrm{m}(u)|}{\|f\|_{L^{n/2,\infty}}}\Big)\,dx &\le \sum_{i=1}^{2}\int_0^{\frac{|\Omega|}{2}} \exp\Big(\gamma\,\frac{(u-\mathrm{m}(u))_i^*(s)}{\|f_i\|_{L^{n/2,\infty}}}\Big)\,ds \\
&\le 2\int_0^{\frac{|\Omega|}{2}} \exp\Big[\gamma \int_s^{\frac{|\Omega|}{2}} \frac{1}{h^2(\rho)}\Big(\int_0^\rho \exp\Big(\int_\zeta^\rho \frac{b(\eta)}{h(\eta)}d\eta\Big)\,\zeta^{-2/n}\,d\zeta\Big)\,d\rho\Big]\,ds \\
&\le 2\int_0^{\frac{|\Omega|}{2}} \exp\Big[\gamma\frac{n}{n-2}\int_s^{\frac{|\Omega|}{2}} \frac{1}{h^2(\rho)}\exp\Big(\int_0^\rho \frac{b(\eta)}{h(\eta)}\,d\eta\Big)\,\rho^{1-2/n}\,d\rho\Big]\,ds.
\end{aligned} \tag{3.23}
$$

From equation (2.5) and inequality (3.18) one deduces that, for every $\varepsilon > 0$, a constant $C = C\,(\Omega, \varepsilon, \sigma, \tau, \|B\|_{L^{\sigma,\tau}})$ exists such that

$$
\int_s^{\frac{|\Omega|}{2}} \frac{\rho^{1-2/n}}{h^2(\rho)}\,\exp\Big(\int_0^\rho \frac{b(\eta)}{h(\eta)}\,d\eta\Big)\,d\rho \le C + \frac{1+\varepsilon}{n^2\theta_\Omega^{2/n}}\,\log\Big(\frac{1}{s}\Big) \tag{3.24}
$$

for $s \in \big(0, |\Omega|/2\big)$. Owing to the arbitrariness of ε, inequalities (3.23)-(3.24) yield (3.5) for every $\gamma \in (0, n(n-2)\theta_\Omega^{2/n})$. □

Proof of Theorem 3.1, Part II. Assume that $q < \infty$. In order to prove the optimality of (3.4), for every $\beta > n(n-2)\theta_\Omega^{2/n}$ we exhibit a domain $\Omega \in \Sigma^{1,\alpha}$ and a sequence $\{f_k\}_{k\in\mathbf{N}}$ of functions in $L^{n/2,q}(\Omega)$ such that the corresponding sequence $\{u_k\}_{k\in\mathbf{N}}$ of solutions to the problems

$$
\begin{cases} -\Delta u_k = f_k & \text{in } \Omega, \\ \dfrac{\partial u_k}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.25}
$$

satisfies

$$
\lim_{k\to+\infty} \int_\Omega \exp\Big(\beta \frac{|u_k-\mathrm{m}(u_k)|}{\|f_k\|_{L^{n/2,q}}}\Big)^{q'}\,dx = +\infty. \tag{3.26}
$$

Let $\varphi\colon \mathbf{R} \to [0,+\infty)$ be an increasing smooth function (of class $\mathscr{C}^2$, say) such that

$$
\varphi(t) \equiv 0 \quad \text{if } t \le 0 \quad \text{and} \quad \varphi(t) = t \quad \text{if } t \ge 1.
$$

Given $\varepsilon \in (0,1)$, let $\psi_\varepsilon\colon \mathbf{R} \to [0,1]$ be defined as

$$
\psi_\varepsilon(t) = \begin{cases} 0 & \text{if } t < 0, \\ \varepsilon\,\varphi(t/\varepsilon) & \text{if } 0 \le t \le \varepsilon, \\ t & \text{if } \varepsilon < t < 1-\varepsilon, \\ 1-\varepsilon\,\varphi\big(\frac{1-t}{\varepsilon}\big) & \text{if } 1-\varepsilon < t \le 1, \\ 1 & \text{if } t > 1. \end{cases} \tag{3.27}
$$

Furthermore, for $k \in \mathbf{N}$, let $\xi_{\varepsilon,k}\colon (0,+\infty) \to [0,1]$ be given by

$$
\xi_{\varepsilon,k}(r) = \psi_\varepsilon\left(\frac{\log\frac{1}{r}}{\log k}\right) \quad \text{for } r > 0. \tag{3.28}
$$

Fixed $\theta \in (0, \frac{\omega_n}{2}]$, choose as a ground domain any bounded domain $\Omega \in \Sigma^{1,\alpha}$ satisfying $B_2(0) \cap \Omega = B_2(0) \cap \Lambda$ for some cone Λ having aperture θ. Finally, define $u_{\varepsilon,k} \colon \Omega \to \mathbf{R}$ as

$$u_{\varepsilon,k}(x) = \begin{cases} \xi_{\varepsilon,k}(|x|) & \text{if } x \in B_1(0) \cap \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{3.29}$$

Thanks to our assumptions on φ, for every $\varepsilon \in (0,1)$ and $k \in \mathbf{N}$, $u_{\varepsilon,k} \in \mathscr{C}^2(\Omega)$. Since $|\Omega| \geq |B_2(0) \cap \Lambda| > 2|B_1(0) \cap \Lambda| = 2|B_1(0) \cap \Omega|$, we have

$$\mathrm{m}(u_{\varepsilon,k}) = 0. \tag{3.30}$$

Moreover, $\dfrac{\partial\, u_{\varepsilon,k}}{\partial \mathbf{n}} = 0$ on $\partial\Omega$. Set

$$f_{\varepsilon,k} = \Delta\, u_{\varepsilon,k}.$$

Hence,

$$f_{\varepsilon,k}(x) = \psi_\varepsilon''\bigg(\frac{\log \frac{1}{|x|}}{\log k}\bigg) \frac{1}{|x|^2\, (\log k)^2} - (n-2)\psi_\varepsilon'\bigg(\frac{\log \frac{1}{|x|}}{\log k}\bigg) \frac{1}{|x|^2 \log k} \tag{3.31}$$

if $x \in B_1(0) \cap \Omega$, and $f_{\varepsilon,k}(x) \equiv 0$ otherwise. Thus, a constant $C = C\,(\varphi)$, which we choose larger than $n-2$, exists such that, on setting

$$\Upsilon(\varepsilon, k) = C\Big(1 + \frac{1}{\varepsilon \log k}\Big), \tag{3.32}$$

we have

$$|f_{\varepsilon,k}(x)| \begin{cases} = 0 & \text{if } 0 < |x| < \dfrac{1}{k}, \\ \leq \dfrac{\Upsilon(\varepsilon,k)}{|x|^2\, \log k} & \text{if } \dfrac{1}{k} \leq |x| < \dfrac{1}{k^{1-\varepsilon}}, \\ = \dfrac{n-2}{|x|^2\, \log k} & \text{if } \dfrac{1}{k^{1-\varepsilon}} \leq |x| < \dfrac{1}{k^{\varepsilon}}, \\ \leq \dfrac{\Upsilon(\varepsilon,k)}{|x|^2\, \log k} & \text{if } \dfrac{1}{k^{\varepsilon}} \leq |x| < 1, \end{cases} \tag{3.33}$$

for $x \in B_1(0) \cap \Omega$. Consequently, if $g_{\varepsilon,k} \colon [0, +\infty) \to [0, +\infty)$ is given by

$$g_{\varepsilon,k}(s) = \begin{cases} \dfrac{\Upsilon(\varepsilon,k)\theta^{2/n}}{s^{2/n}\, \log k} & \text{if } \dfrac{\theta}{k^n} < s < \dfrac{\theta}{k^{(1-\varepsilon)n}}, \\ \dfrac{(n-2)\theta^{2/n}}{s^{2/n}\, \log k} & \text{if } \dfrac{\theta}{k^{(1-\varepsilon)n}} \leq s < \dfrac{\theta}{k^{\varepsilon\, n}}, \\ \dfrac{\Upsilon(\varepsilon,k)\theta^{2/n}}{s^{2/n}\, \log k} & \text{if } \dfrac{\theta}{k^{\varepsilon\, n}} \leq s < \theta, \\ 0 & \text{otherwise,} \end{cases}$$

then, by (3.33),

$$f_{\varepsilon,k}(x) \le g_{\varepsilon,k}(\theta\,|x|^n) \quad \text{for } x \in B_1(0)\cap\Omega, \tag{3.34}$$

and hence

$$f^*_{\varepsilon,k}(s) \le g^*_{\varepsilon,k}(s) \quad \text{for } s > 0. \tag{3.35}$$

Computations show that, if k is sufficiently large, then

(3.36)

$$g^*_{\varepsilon,k}(s) = \begin{cases} \dfrac{\Upsilon(\varepsilon,k)\,\theta^{2/n}}{(\log k)(s+\frac{\theta}{k^n})^{2/n}} & \text{if } 0 < s < \dfrac{\theta}{k^{(1-\varepsilon)n}} - \dfrac{\theta}{k^n}, \\ \dfrac{(n-2)\,\theta^{2/n}}{(\log k)(s+\frac{\theta}{k^n})^{2/n}} & \text{if } \dfrac{\theta}{k^{(1-\varepsilon)n}} - \dfrac{\theta}{k^n} \le s \\ & \quad < \dfrac{\theta}{k^{\varepsilon n}}\Big(\dfrac{n-2}{\Upsilon(\varepsilon,k)}\Big)^{n/2} - \dfrac{\theta}{k^n}, \\ \left(\dfrac{(n-2)^{n/2} + \Upsilon(\varepsilon,k)^{n/2}}{\frac{\theta}{k^n} + \frac{\theta}{k^{\varepsilon n}} + s}\right)^{2/n} \dfrac{\theta^{2/n}}{\log k} & \text{if } \dfrac{\theta}{k^{\varepsilon n}}\Big(\dfrac{n-2}{\Upsilon(\varepsilon,k)}\Big)^{n/2} - \dfrac{\theta}{k^n} \\ & \quad \le s < \dfrac{\theta}{k^{\varepsilon n}}\Big(\dfrac{\Upsilon(\varepsilon,k)}{n-2}\Big)^{n/2} - \dfrac{\theta}{k^n}, \\ \dfrac{\Upsilon(\varepsilon,k)\,\theta^{2/n}}{(\log k)(s+\frac{\theta}{k^n})^{2/n}} & \text{if } \dfrac{\theta}{k^{\varepsilon n}}\Big(\dfrac{\Upsilon(\varepsilon,k)}{n-2}\Big)^{n/2} - \dfrac{\theta}{k^n} \\ & \quad \le s < \theta - \dfrac{\theta}{k^n}, \\ 0 & \text{if } \theta - \dfrac{\theta}{k^n} \le s. \end{cases}$$

We have

$$\begin{aligned} &\int_\Omega \exp\Big(\beta\,\frac{|u_{\varepsilon,k} - \mathrm{m}(u_{\varepsilon,k})|}{\|f_{\varepsilon,k}\|_{L^{n/2,q}}}\Big)^{q'} dx \\ &\quad \ge \int_{\{x\in B_1(0)\cap\Omega:\,|x|\le 1/k\}} \exp\Big(\beta\,\frac{u_{\varepsilon,k}}{\|f_{\varepsilon,k}\|_{L^{n/2,q}}}\Big)^{q'} dx \\ &\quad \ge \int_{\{x\in B_1(0)\cap\Omega:\,|x|\le 1/k\}} \exp\Big(\beta\,\frac{u_{\varepsilon,k}}{\|g_{\varepsilon,k}\|_{L^{n/2,q}}}\Big)^{q'} dx \\ &\quad = \frac{\theta}{k^n}\ \exp\Big(\frac{\beta}{\|g_{\varepsilon,k}\|_{L^{n/2,q}}}\Big)^{q'} \\ &\quad = \theta\ \exp\Big[\Big(\frac{\beta}{\|g_{\varepsilon,k}\|_{L^{n/2,q}}}\Big)^{q'} - n\,\log k\Big] \\ &\quad = \theta\exp\Big[\log k\,\Big(-n + \frac{1}{\log k}\Big(\frac{\beta}{\|g_{\varepsilon,k}\|_{L^{n/2,q}}}\Big)^{q'}\Big)\Big], \end{aligned} \tag{3.37}$$

where the first inequality is due to (3.30) and to the fact that $u_{\varepsilon,k}(x) \equiv 1$ if $0 \leq |x| \leq 1/k$, and the second one holds owing to (3.35). From equation (3.37) and the definition of Lorentz norm one can deduce, after some computations that,

$$\lim_{k\to+\infty} \frac{1}{\log k \, \|g_{\varepsilon,k}\|^{q'}_{L^{n/2,q}}} = \Big[n(1-2\varepsilon)((n-2)\theta^{2/n})^q + 2n\varepsilon(C\theta^{2/n})^q\Big]^{\frac{1}{1-q}}$$

for every $\varepsilon \in (0,1)$, where C is the constant appearing in (3.32). Hence, the right-most side of (3.37) diverges to $+\infty$ as $k \to +\infty$ if

$$-n + \beta^{q'}\big[n(1-2\varepsilon)((n-2)\theta^{2/n})^q + 2n\varepsilon(C\theta^{2/n})^q\big]^{\frac{1}{1-q}} > 0. \tag{3.38}$$

Thus, given any $\beta > n(n-2)\theta_\Omega^{2/n}$ and chosen any $\varepsilon \in (0,1)$ for which (3.38) is fulfilled, equation (3.26) holds with $u_k = u_{\varepsilon,k}$.

Consider now the case where $q = \infty$. Let Ω be as above. We exhibit a function $f \in L^{n/2,\infty}(\Omega)$ and a solution u to the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

fulfilling

$$\int_\Omega \exp\Big(\gamma \frac{|u - \mathrm{m}(u)|}{\|f\|_{L^{n/2,\infty}}}\Big)\, dx = +\infty \tag{3.39}$$

for every $\gamma \geq n(n-2)\theta^{2/n}$. Let $\Psi\colon \mathbf{R} \to [0,+\infty)$ be a smooth convex function, $\mathscr{C}^2$ say, such that

$$\Psi(t) = 0 \quad \text{if } t \leq 0 \quad \text{and} \quad \Psi(t) = t - \frac{1}{2} \quad \text{if } t \geq 1.$$

Given $a > 1$, define $\xi_a\colon (0,+\infty) \to \mathbf{R}$ as

$$\xi_a(r) = \Psi\left(\frac{\log\frac{1}{r}}{\log a}\right) \quad \text{for } r > 0,$$

$u_a\colon \Omega \to \mathbf{R}$ as

$$u_a(x) = \begin{cases} \xi_a(|x|) & \text{if } x \in B_1(0) \cap \Omega, \\ 0 & \text{otherwise}, \end{cases} \tag{3.40}$$

and

$$f_a(x) = \Delta\, u_a(x).$$

Then

$$f_a(x) = \Psi''\left(\frac{\log\frac{1}{|x|}}{\log a}\right)\frac{1}{|x|^2\,(\log a)^2} - (n-2)\Psi'\left(\frac{\log\frac{1}{|x|}}{\log a}\right)\frac{1}{|x|^2\,\log a}.$$

Observe that Ψ'' is bounded and nonnegative, $0 \le \Psi'(t) \le 1$ for every $t \in \mathbf{R}$, and $\Psi'(t) \equiv 1$, $\Psi''(t) \equiv 0$ if $t \ge 1$. Thus, for $x \in B_1(0) \cap \Omega$,

$$|f_a(x)| \begin{cases} = \dfrac{n-2}{|x|^2 \log a} & \text{if } |x| \le \dfrac{1}{a}, \\ \le \dfrac{n-2}{|x|^2 \log a} & \text{if } \dfrac{1}{a} < |x| \le 1, \\ = 0 & \text{if } |x| > 1, \end{cases}$$

provided that a is sufficiently large. Hence,

$$\|f_a\|_{L^{n/2,\infty}(\Omega)} \le \frac{(n-2)\, \theta^{2/n}}{\log a}. \tag{3.41}$$

By (3.40)–(3.41),

$$\begin{aligned} &\int_\Omega \exp\Big(\gamma \frac{|u_a - \mathrm{m}(u_a)|}{\|f_a\|_{L^{n/2,\infty}}}\Big)\, dx \ge \int_{B_1(0)\cap\Omega} \exp\left[\gamma \frac{\Psi\Big(\frac{\log \frac{1}{|x|}}{\log a}\Big) \log a}{(n-2)\theta^{2/n}}\right] dx \\ &\ge \int_{\{x \in B_1(0)\cap\Omega : |x| \le 1/a\}} \exp\left[\gamma \frac{\Big(\frac{\log \frac{1}{|x|}}{\log a} - \frac{1}{2}\Big) \log a}{(n-2)\theta^{2/n}}\right] dx \\ &= \theta \int_0^{\frac{1}{a}} r^{n-1} \exp\Big(\gamma \frac{\log \frac{1}{r}}{(n-2)\theta^{2/n}}\Big) \exp\Big(-\frac{\gamma \log a}{2\,(n-2)\theta^{2/n}}\Big)\, dr \\ &= \theta \exp\Big(-\frac{\gamma \log a}{2\,(n-2)\theta^{2/n}}\Big) \int_0^{\frac{1}{a}} r^{-\frac{\gamma}{(n-2)\theta^{2/n}} + n - 1}\, dr = +\infty \end{aligned}$$

if $\gamma \ge n(n-2)\theta^{2/n}$. Hence, (3.39) follows with $u = u_a$ and $f = f_a$. □

4. Optimal summability

The present section is devoted to the proof of Theorem 1.2.

Proof of Theorem 1.2. From inequality (3.2), via (2.4) and (3.15), we deduce that

$$\big(u - \mathrm{m}(u)\big)_i^*(s) \le C \int_s^{|\Omega|/2} r^{-2+2/n} \int_0^r f_i^*(\rho)\, d\rho\, dr, \quad i = 1, 2, \tag{4.1}$$

for some constant $C = C(\Omega, \sigma, \tau, \|B\|_{L^{\sigma,\tau}})$ and for every $s \in \big(0, |\Omega|/2\big)$. Hence,

$$
\begin{aligned}
\|u - \mathrm{m}(u)\|_{L^{\infty,q}(\log L)^{-1}(\Omega)} &\le \sum_{i=1}^{2} \|(u - \mathrm{m}(u))_i\|_{L^{\infty,q}(\log L)^{-1}(\Omega)} \\
&\le \sum_{i=1}^{2} \left\| \frac{(u - \mathrm{m}(u))_i^*(s)}{s^{1/q}\big(1 + \log \frac{|\Omega|}{2s}\big)} \right\|_{L^q(0,|\Omega|/2)} \\
&\le C \sum_{i=1}^{2} \left\| \frac{\int_s^{|\Omega|/2} r^{-2+2/n} \int_0^r f_i^*(\rho)\, d\rho\, dr}{s^{1/q}\big(1 + \log \frac{|\Omega|}{2s}\big)} \right\|_{L^q(0,|\Omega|/2)} .
\end{aligned} \tag{4.2}
$$

An application of Fubini's theorem and simple estimates yield

$$
\begin{aligned}
&\left\| \frac{\int_s^{|\Omega|/2} r^{-2+2/n} \int_0^r f_i^*(\rho)\, d\rho\, dr}{s^{1/q}\big(1 + \log \frac{|\Omega|}{2s}\big)} \right\|_{L^q(0,|\Omega|/2)} \\
&\le \frac{n}{n-2} \left(\left\| \frac{s^{-1+\frac{2}{n}-\frac{1}{q}}}{1 + \log \frac{|\Omega|}{2s}} \int_0^s f_i^*(\rho)\, d\rho \right\|_{L^q(0,|\Omega|/2)} \right. \\
&\quad \left. + \left\| \frac{s^{-\frac{1}{q}}}{1 + \log \frac{|\Omega|}{2s}} \int_s^{|\Omega|/2} f_i^*(\rho)\, \rho^{-1+2/n}\, d\rho \right\|_{L^q(0,|\Omega|/2)} \right),
\end{aligned} \tag{4.3}
$$

for $i = 1, 2$. By an application of (2.18) and (2.20) to the first norm and to the second norm, respectively, on the right hand side of (4.3), one deduces that

$$
\begin{aligned}
&\left\| \frac{\int_s^{|\Omega|/2} r^{-2+2/n} \int_0^r f_i^*(\rho)\, d\rho\, dr}{s^{1/q}\big(1 + \log \frac{|\Omega|}{2s}\big)} \right\|_{L^q(0,|\Omega|/2)} \\
&\le \frac{n}{n-2}\, C \left\| s^{\frac{2}{n}-\frac{1}{q}} f_i^*(s) \right\|_{L^q(0,|\Omega|/2)}, \quad i = 1, 2\,,
\end{aligned} \tag{4.4}
$$

for some constant $C = C(q)$ and for every $f \in L^{n/2,q}(\Omega)$. Inequality (1.12) then follows from (4.2) and (4.4).

We now prove the optimality of the space $L^{\infty,q}(\log L)^{-1}(\Omega)$. Let Ω be a bounded domain in $\mathbf{R}^n$, let $q \in (1, \infty]$, and let $X(\Omega)$ be any r.i. space on Ω having the following property: a constant C exists such that

$$
\|u - \mathrm{m}(u)\|_{X(\Omega)} \le C \|f\|_{L^{n/2,q}(\Omega)} \tag{4.5}
$$

for every $f \in L^{n/2,q}(\Omega)$ and for every solution u to the Neumann problem

$$
\begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}
$$

We may assume, without loss of generality, that $0 \in \Omega$. Let $R > 0$ be such that $B_R(0) \subset\subset \Omega$ and $|B_R(0)| < |\Omega|/2$. Set $|B_R(0)| = V$. Let ϕ be any function in

$L^{n/2,q}(0,\infty)$ fulfilling

$$\phi(s) = -\phi(V-s) \quad \text{if } s \in (0,V), \quad \phi(s) = 0 \quad \text{if } s > V, \tag{4.7}$$

and

$$\phi(s) \geq 0 \quad \text{if } s \in (0, V/2). \tag{4.8}$$

Hence, in particular,

$$\int_0^s \phi(r)\,dr \geq 0 \quad \text{if } s \in [0,V] \quad \text{and} \quad \int_0^s \phi(r)\,dr = 0 \quad \text{if } s \geq V. \tag{4.9}$$

Define $f\colon \Omega \to \mathbf{R}$ as

$$f(x) = \phi(\omega_n |x|^n) \quad \text{for } x \in \Omega.$$

Thus, by (4.9),

$$\int_\Omega f(x)\,dx = \int_0^V \phi(s)\,ds = 0. \tag{4.10}$$

Moreover, define $u\colon \Omega \to \mathbf{R}$ as

$$u(x) = \frac{1}{n^2\,\omega_n^{2/n}} \int_{\omega_n|x|^n}^{V/2} r^{-2+2/n} \int_0^r \phi(\rho)\,d\rho\,dr \quad \text{for } x \in \Omega. \tag{4.11}$$

Therefore,

$$-\Delta\, u = f \quad \text{in } \Omega,$$

and since

$$u(x) = -\frac{1}{n^2\,\omega_n^{2/n}} \int_{V/2}^{V} r^{-2+2/n} \int_0^r \phi(\rho)\,d\rho\,dr \quad \text{if } x \in \Omega \setminus B_R(0), \tag{4.12}$$

then $\frac{\partial u}{\partial \mathbf{n}} = 0$ on $\partial\Omega$. In other words, u is a solution to problem (4.6). Observe that

$$\mathrm{m}(u) = -\frac{1}{n^2\,\omega_n^{2/n}} \int_{V/2}^{V} r^{-2+2/n} \int_0^r \phi(\rho)\,d\rho\,dr < 0, \tag{4.13}$$

where the equality holds owing to (4.12) and to the fact that $V < |\Omega|/2$, and the inequality in (4.13) is a consequence of (4.9). Thus,

$$\begin{aligned} |u(x) - \mathrm{m}(u)| &= \frac{1}{n^2\,\omega_n^{2/n}} \bigg| \int_{\omega_n|x|^n}^{V/2} r^{-2+2/n} \int_0^r \phi(\rho)\,d\rho\,dr \\ &\qquad + \int_{V/2}^{V} r^{-2+2/n} \int_0^r \phi(\rho)\,d\rho\,dr \bigg| \\ &= \frac{1}{n^2\,\omega_n^{2/n}} \bigg| \int_{\omega_n|x|^n}^{V} r^{-2+2/n} \int_0^r \phi(\rho)\,d\rho\,dr \bigg| \\ &\geq \frac{\chi_{[0,V/2]}(\omega_n|x|^n)}{n^2\,\omega_n^{2/n}} \int_{\omega_n|x|^n}^{V/2} r^{-2+2/n} \int_0^r \phi(\rho)\,d\rho\,dr \geq 0 \end{aligned} \tag{4.14}$$

for $x \in \Omega$. Consequently,

$$\|u - \mathrm{m}(u)\|_{X(\Omega)} \geq \left\| \frac{\chi_{[0,V/2]}(s)}{n^2\, \omega_n^{2/n}} \int_s^{V/2} r^{-2+2/n} \int_0^r \phi(\rho)\, d\rho\, dr \right\|_{\overline{X}(0,|\Omega|)}. \tag{4.15}$$

The following chain holds

$$\begin{aligned}
&\left\| \frac{\chi_{[0,V/2]}(s)}{n^2\, \omega_n^{2/n}} \int_s^{V/2} r^{-2+\frac{2}{n}} \int_0^r \phi(\rho)\, d\rho\, dr \right\|_{\overline{X}(0,|\Omega|)} \\
&= \left\| \frac{\chi_{[0,V/2]}(s)}{2^{2/n}\, n^2\, \omega_n^{2/n}} \int_{2s}^{V} r^{-2+\frac{2}{n}} \int_0^r \phi\Big(\frac{\rho}{2}\Big)\, d\rho\, dr \right\|_{\overline{X}(0,|\Omega|)} \\
&\geq \frac{1}{2^{1+2/n}\, n^2\, \omega_n^{2/n}} \left\| \chi_{[0,V]}(s) \int_s^{V} r^{-2+\frac{2}{n}} \int_0^r \phi\Big(\frac{\rho}{2}\Big)\, d\rho\, dr \right\|_{\overline{X}(0,|\Omega|)} \\
&= \frac{1}{2^{1+2/n}\, n^2\, \omega_n^{2/n}} \left\| \frac{n\, \chi_{[0,V]}(s)}{n-2} \Big(\big(s^{-1+\frac{2}{n}} - V^{-1+\frac{2}{n}}\big) \int_0^s \phi(r/2)\, dr \right. \\
&\qquad \left. + \int_s^V \phi(r/2) \big(r^{-1+\frac{2}{n}} - V^{-1+\frac{2}{n}}\big)\, dr \Big) \right\|_{\overline{X}(0,|\Omega|)} \\
&\geq \frac{1}{n\,(n-2) 2^{1+2/n}\, \omega_n^{2/n}} \left\| \chi_{[0,V]}(s) \int_s^V \phi(r/2) \big(r^{-1+\frac{2}{n}} - V^{-1+\frac{2}{n}}\big)\, dr \right\|_{\overline{X}(0,|\Omega|)} \\
&\geq \frac{1}{n\,(n-2) 2^{1+2/n}\, \omega_n^{2/n}} \left\| \chi_{[0,V]}(2s) \int_{2s}^V \phi(r/2) \big(r^{-1+\frac{2}{n}} - V^{-1+\frac{2}{n}}\big)\, dr \right\|_{\overline{X}(0,|\Omega|)} \\
&\geq \frac{(1 - 2^{-1+2/n})}{n\,(n-2) 2^{1+2/n}\, \omega_n^{2/n}} \left\| \chi_{[0,V]}(2s) \int_{2s}^V \phi(r/2)\, r^{-1+\frac{2}{n}}\, dr \right\|_{\overline{X}(0,|\Omega|)} \\
&\geq \frac{(1 - 2^{-1+2/n})}{2n\,(n-2) 2^{1+2/n}\, \omega_n^{2/n}} \left\| \chi_{[0,V]}(s) \int_s^V \phi(r/2)\, r^{-1+\frac{2}{n}}\, dr \right\|_{\overline{X}(0,|\Omega|)},
\end{aligned} \tag{4.16}$$

where the first and the fifth inequality are consequences of (2.14), and the second inequality holds since $\phi(r/2) \geq 0$ if $r \in (0, V)$. On the other hand,

$$\|f\|_{L^{n/2,q}(\Omega)} = \left\| \phi(s/2) \chi_{[0,V]}(s) \right\|_{L^{n/2,q}(0,|\Omega|)}. \tag{4.17}$$

From (4.5), (4.15), (4.16) and (4.17) we deduce that

$$\left\| \chi_{[0,V]}(s) \int_s^V \phi(r/2)\, r^{-1+\frac{2}{n}}\, dr \right\|_{\overline{X}(0,|\Omega|)} \leq C \left\| \phi(s/2) \chi_{[0,V]}(s) \right\|_{L^{n/2,q}(0,|\Omega|)}, \tag{4.18}$$

for some constant $C = C(X(\Omega), q)$. Hence, in particular,

$$\left\| \int_s^{|\Omega|} \psi(r)\, r^{-1+\frac{2}{n}}\, dr \right\|_{\overline{X}(0,|\Omega|)} \leq C \,\| \psi\|_{L^{n/2,q}(0,|\Omega|)} \tag{4.19}$$

for every $\psi \in L^{n/2,q}(0,|\Omega|)$ vanishing outside $[0,V]$. If ψ need not vanish outside $[0,V]$, then

$$\begin{aligned} \left\| \int_s^{|\Omega|} \psi(r)\, r^{-1+\frac{2}{n}}\, dr \right\|_{\overline{X}(0,|\Omega|)} \leq & \left\| \int_s^{|\Omega|} \chi_{[0,V]}(r)\, \psi(r)\, r^{-1+\frac{2}{n}}\, dr \right\|_{\overline{X}(0,|\Omega|)} \\ & + \left\| \int_s^{|\Omega|} \chi_{[V,|\Omega|]}(r)\, \psi(r)\, r^{-1+\frac{2}{n}}\, dr \right\|_{\overline{X}(0,|\Omega|)}. \end{aligned} \tag{4.20}$$

We have

$$\begin{aligned} & \left\| \int_s^{|\Omega|} \chi_{[V,|\Omega|]}(r)\, \psi(r)\, r^{-1+\frac{2}{n}}\, dr \right\|_{\overline{X}(0,|\Omega|)} \\ & \leq V^{-1+\frac{2}{n}} \left\| \int_s^{|\Omega|} |\psi(r)|\, dr \right\|_{\overline{X}(0,|\Omega|)} \\ & \leq V^{-1+\frac{2}{n}}\, \|1\|_{\overline{X}(0,|\Omega|)} \int_0^{|\Omega|} |\psi(r)|\, dr \\ & \leq C\, V^{-1+\frac{2}{n}}\, \|1\|_{\overline{X}(0,|\Omega|)}\ \|1\|_{L^{\frac{n}{n-2},q'}(0,|\Omega|)}\ \|\psi\|_{L^{n/2,q}(0,|\Omega|)} \end{aligned} \tag{4.21}$$

for some constant $C = C(X(\Omega), q)$. From (4.20)–(4.21) and from (4.19) applied with ψ replaced by $\chi_{[0,V]}\, \psi$, one infers that

$$\left\| \int_s^{|\Omega|} \psi(r)\, r^{-1+\frac{2}{n}}\, dr \right\|_{\overline{X}(0,|\Omega|)} \leq C\, \|\psi\|_{L^{n/2,q}(0,|\Omega|)} \tag{4.22}$$

for some constant $C = C(X(\Omega), q)$ and for every $\psi \in L^{n/2,q}(0,|\Omega|)$. We now use an argument involving associate spaces analogous to that of [C3, Proof of Lemma 1] and [EKP, Proof of Theorem 4.6]. Inequality (4.22) entails that

$$
\begin{aligned}
C &\geq \sup_{\psi\in L^{n/2,q}(0,|\Omega|)} \frac{\left\| \int_s^{|\Omega|} |\psi(r)|\, r^{-1+2/n}\, dr \right\|_{\overline{X}(0,|\Omega|)}}{\|\psi\|_{L^{n/2,q}(0,|\Omega|)}} \\
&= \sup_{\substack{\psi\in L^{n/2,q}(0,|\Omega|)\\ \xi\in \overline{X}'(0,|\Omega|)}} \frac{\int_0^{|\Omega|} \xi^*(s) \int_s^{|\Omega|} |\psi(r)|\, r^{-1+\frac{2}{n}}\, dr\, ds}{\|\xi\|_{\overline{X}'(0,|\Omega|)}\, \|\psi\|_{L^{n/2,q}(0,|\Omega|)}} \\
&= \sup_{\substack{\psi\in L^{n/2,q}(0,|\Omega|)\\ \xi\in \overline{X}'(0,|\Omega|)}} \frac{\int_0^{|\Omega|} |\psi(r)|\, r^{-1+\frac{2}{n}} \int_0^{r} \xi^*(s)\, ds\, dr}{\|\xi\|_{\overline{X}'(0,|\Omega|)}\, \|\psi\|_{L^{n/2,q}(0,|\Omega|)}} \\
&\geq C_1 \sup_{\xi\in \overline{X}'(0,|\Omega|)} \frac{\left\| r^{2/n}\, \xi^{**}(r) \right\|_{L^{\frac{n}{n-2},q'}(0,|\Omega|)}}{\|\xi\|_{\overline{X}'(0,|\Omega|)}},
\end{aligned}
\tag{4.23}
$$

for some constant $C_1 = C_1(X(\Omega), q)$, where the first equality is due to (2.11). Consequently,

$$
\begin{aligned}
\|\phi\|_{\overline{X}(0,|\Omega|)} &= \sup_{\xi\in \overline{X}'(0,|\Omega|)} \frac{\int_0^{|\Omega|} \phi^*(s)\, \xi^*(s)\, ds}{\|\xi\|_{\overline{X}'(0,|\Omega|)}} \\
&\leq C \sup_{\xi\in L^{\frac{n}{n-2},q'}(0,|\Omega|)} \frac{\int_0^{|\Omega|} \phi^*(s)\, \xi^*(s)\, ds}{\|s^{2/n}\, \xi^{**}(s)\|_{L^{\frac{n}{n-2},q'}(0,|\Omega|)}}
\end{aligned}
\tag{4.24}
$$

for some constant $C = C(X(\Omega), q)$ and for every $\phi \in \overline{X}(0, |\Omega|)$.

In the case where $q < \infty$, by [KP, Theorem 3.9] a constant $C = C(n, q, |\Omega|)$ exists such that

$$
\begin{aligned}
C\, \|s^{2/n}\, \xi^{**}(s)\|_{L^{\frac{n}{n-2},q'}(0,|\Omega|)} &\geq \sup_{s\leq r\leq |\Omega|} \|r^{2/n}\, \xi^{**}(r)\|_{L^{\frac{n}{n-2},q'}(0,|\Omega|)} \\
&= \left(\int_0^{|\Omega|} \left[\left(\sup_{s\leq r\leq |\Omega|} r^{2/n}\, \xi^{**}(r) \right) r^{\frac{n-2}{n}} \right]^{q'} \frac{ds}{s} \right)^{1/q'} \\
&\geq \left(\int_0^{|\Omega|} \xi^{**}(s)^{q'}\, s^{q'-1}\, ds \right)^{1/q'}
\end{aligned}
\tag{4.25}
$$

for every $\xi \in L^{\frac{n}{n-2},q'}(0, |\Omega|)$. From (4.24)–(4.25) we get

$$
\|\phi\|_{\overline{X}(0,|\Omega|)} \leq C \sup_{\xi\in L^{\frac{n}{n-2},q'}(0,|\Omega|)} \frac{\int_0^{|\Omega|} \phi^*(s)\, \xi^*(s)\, ds}{\left(\int_0^{|\Omega|} \xi^{**}(s)^{q'}\, s^{q'-1}\, ds \right)^{1/q'}},
\tag{4.26}
$$

for some constant $C = C(X(\Omega), q)$. An application of (2.21) with

$$\mu(s) = \frac{1}{s\,(1 + \log\frac{|\Omega|}{s})^q}$$

tells us that the supremum in (4.26) does not exceed

$$C\left(\int_0^{|\Omega|}\left(\frac{\phi^*(s)}{1+\log\frac{|\Omega|}{s}}\right)^q\frac{ds}{s}\right)^{1/q}$$

for some constant $C = C\,(q, |\Omega|)$. Notice that here we have made use of the fact that

$$\int_0^{|\Omega|}\xi^*(s)\,ds \le C\left(\int_0^{|\Omega|}\xi^{**}(s)^{q'}\,s^{q'-1}\,ds\right)^{1/q'}$$

for some constant $C = C(q, |\Omega|)$ and for every measurable function ξ in $(0, |\Omega|)$, see e.g. [GP, Theorem 4.1]. Hence, by (2.10), $L^{\infty,q}(\log L)^{-1}(\Omega)$ is continuously embedded into $X(\Omega)$.

Finally, assume that $q = \infty$, and hence $q' = 1$. One has

$$\|s^{2/n}\,\xi^{**}(s)\|_{L^{\frac{n}{n-2},1}(0,|\Omega|)} = \int_0^{|\Omega|}\xi^{**}(s)\,ds = \int_0^{|\Omega|}\xi^*(s)\log\big(|\Omega|/s\big)\,ds \tag{4.27}$$

for $\xi \in L^{\frac{n}{n-2},1}(0, |\Omega|)$. Since

$$\int_0^s\Big(1+\log\big(|\Omega|/r\big)\Big)\,dr \le (1+\mathrm{e}\,)\int_0^s\log\big(|\Omega|/r\big)\,dr \quad \text{for } s\in(0,|\Omega|), \tag{4.28}$$

one has by Hardy's Lemma (see e.g. [BS, Proposition 3.6, Chap. 2])

$$\int_0^{|\Omega|}\xi^*(s)\Big(1+\log\big(|\Omega|/s\big)\Big)\,ds \le (1+\mathrm{e})\int_0^{|\Omega|}\xi^*(s)\log\big(|\Omega|/s\big)\,ds \tag{4.29}$$

for every ξ as above. Combining (4.24), (4.27) and (4.29) tells us that

$$\begin{aligned}\|\phi\|_{\overline{X}(0,|\Omega|)} &\le C\,(1+e)\sup_{\xi\in L^{1,1}(\log L)(\Omega)}\frac{\int_0^{|\Omega|}\phi^*(s)\,\xi^*(s)\,ds}{\int_0^{|\Omega|}\xi^*(s)\big(1+\log\big(|\Omega|/s\big)\big)\,ds}\\ &\le C\,(1+\mathrm{e})\Big\|\frac{\phi^*(s)}{1+\log\big(|\Omega|/s\big)}\Big\|_{L^\infty(0,|\Omega|)} = C\,(1+e)\,\|\phi\|_{L^{\infty,\infty}(\log L)^{-1}(0,|\Omega|)}\end{aligned} \tag{4.30}$$

for some constant $C = C(X(\Omega), q)$. Hence, $L^{\infty,\infty}(\log L)^{-1}(\Omega)$ is continuously embedded into $X(\Omega)$. The proof is complete. □

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References

[Ad] Adams, D. R.: A sharp inequality of J. Moser for higher order derivatives. - Ann. of Math. 128, 1988, 385–398.

[AH-S] Adams, D. R., and R. Hurri-Syrjänen: Vanishing exponential integrability for functions whose gradients belong to $L^n(\log(e+L))^\alpha$. - J. Funct. Anal. 197, 2003, 162–178.

[AF] Alberico, A., and V. Ferone: Regularity properties of solutions of elliptic equations in $\mathbf{R}^2$ in limit cases. - Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 6, 1995, 237–250.

[Al] Alvino, A.: Formule di maggiorazione e regolarizzazione per soluzioni di equazioni ellittiche del secondo ordine in un caso limite. - Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 62, 1977, 335–340 .

[AFT] Alvino, A., V. Ferone, and G. Trombetti: Moser-type inequalities in Lorentz spaces. - Potential Anal. 5:3, 1996, 273–299.

[AFP] Ambrosio, L., N. Fusco, and D. Pallara: Functions of bounded variation and free discontinuity problems. - Clarendon Press, Oxford, 2000.

[BS] Bennett, C., and R. Sharpley: Interpolation of Operators. - Academic Press, Boston, 1988.

[Be] Betta, M. F.: Neumann problem: comparison results. - Rend. Accad. Sci. Fis. Mat. Napoli (4) 57, 1990, 41–58.

[BW] Brezis, H., and S. Wainger: A note on limiting cases of Sobolev embeddings and convolution inequalities. - Comm. Partial Differential Equations 5, 1980, 773–789.

[CY] Chang, S.-Y. A., and P. C. Yang: Conformal deformation of metrics on S^2. - J. Diff. Geom. 27, 1988, 259–296.

[Ch] Cherrier, P.: Cas d'exception du theoréme d'inclusion de Sobolev sur les variétés riemanniennes et applications. - Bull. Sci. Math. (2) 105, 1981, 235–288.

[C1] Cianchi, A.: On relative isoperimetric inequalities in the plane. - Boll. Un. Mat. Ital. B (7) 3, 1989, 289–325.

[C2] Cianchi, A.: Elliptic equations on manifolds and isoperimetric inequalities. - Proc. Roy. Soc. Edinburgh Sect. A 114, 1990, 213–227.

[C3] Cianchi, A.: A sharp embedding theorem for Orlicz-Sobolev spaces. - Indiana Univ. Math. J. 45, 1996, 39–65.

[C4] Cianchi, A.: Moser-Trudinger inequalities without boundary conditions and isoperimetric problems. - Indiana Univ. Math. J. 54, 2005, 669–705.

[CP] Cwikel, M., and E. Pustylnik: Sobolev type embeddings in the limiting case. - J. Fourier Anal. Appl. 4, 1998, 433–446.

[EH-S] Edmunds, D. E., and R. Hurri-Syrjänen: Sobolev inequalities of exponential type. - Israel Math. J. 123, 2001, 61–92.

[EKP] Edmunds, D. E., R. A. Kerman, and L. Pick: Optimal Sobolev imbeddings involving rearrangement invariant quasi-norms. - J. Funct. Anal. 170, 2000, 307–355.

[FFV1] Ferone, A., V. Ferone, and R. Volpicelli: Moser-type inequalities for solutions of linear elliptic equations with lower-order terms. - Differential Integral Equations 10, 1997, 1031–1048.

[FFV2] Ferone, A., V. Ferone, and R. Volpicelli: Moser-type inequalities for solutions of nonlinear elliptic equations. - Atti Sem. Mat. Fis. Univ. Modena 46, suppl., 1998, 727–743.

[FFV3] Ferone, A., V. Ferone, and R. Volpicelli: On the summability of solutions of nonlinear elliptic equations in limit cases. - Ricerche Mat. 50, 2001, 145–158.

[Fe] Ferone, V.: Symmetrization in a Neumann problem. - Le Matematiche 41, 1986, 67–78.

[Fo] Fontana, L.: Sharp borderline Sobolev inequalities on compact Riemannian manifolds. - Comment. Math. Helv. 68, 1993, 415–454.

[G] Gallot, S.: Inégalités isopérimétriques et analitiques sur les variétés riemanniennes. - Astérisque 163, 1988, 31–91.

[GP] Gogatishvili, A., and L. Pick: Discretization and antidiscretization of rearrangement-invariant norms. - Publ. Mat. 47, 2003, 311–358.

[GT] Giarrusso, E., and G. Trombetti: Estimates for solutions of elliptic equations in a limit case. - Bull. Austral. Math. Soc. 36, 1987, 425–434.

[Han] Hansson, K.: Imbedding theorems of Sobolev type in potential theory. - Math. Scand. 45, 1979, 77–102.

[KP] Kerman, R. A., and L. Pick: Optimal Sobolev imbeddings. - Forum Math. 18, 2006, 535–570.

[L] Liskevich, V. A.: Some limit cases in estimates for solutions of second order elliptic equations. - Houston J. Math. 19, 1993, 661–673.

[MS1] Maderna, C., and S. Salsa: Symmetrization in Neumann problems. - Appl. Anal. 9, 1979, 247–256.

[MS2] Maderna, C., and S. Salsa: A priori bounds in non-linear Neumann problems. - Boll. Un. Mat. Ital. B (5) 16, 1979, 1144–1153.

[Ma1] Maz'ya, V. G.: On weak solutions of the Dirichlet and Neumann problems. - Trans. Moscow Math. Soc. 20, 1969, 135–172.

[Ma2] Maz'ya, V. G.: Sobolev Spaces. - Springer-Verlag, Berlin, 1985.

[MP] Maz'ya, V. G., and S. V. Poborchi: Differentiable functions on bad domains. - World Scientific, Singapore, 1997.

[Mo] Moser, J.: A sharp form of an inequality by Trudinger. - Indiana Univ. Math. J. 20, 1971, 1077–1092.

[OK] Opic, B., and A. Kufner: Hardy-type inequalities. - Longman Scientific & Technical, Harlow, 1990.

[Sa] Sawyer, E.: Boundedness of classical operators on classical Lorentz spaces. - Studia Math. 96, 1990, 145–158.

[Ta] Talenti, G.: Elliptic equations and rearrangements. - Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 3, 1976, 697–718.

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