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SOME RESULTS RELATED TO A CONJECTURE OF R. BRÜCK CONCERNING MEROMORPHIC FUNCTIONS SHARING ONE SMALL FUNCTION WITH THEIR DERIVATIVES

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Abstract. In this paper, we investigate uniqueness problems of meromorphic functions that share a small function with one of its derivatives, and give some results which are related to a conjecture of R. Brück, and also answer some questions of Kit-Wing Yu.

1. Introduction and results

In this paper a meromorphic function will mean meromorphic in the whole complex plane. We say that two meromorphic functions f and g share a finite value a IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities). It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [5] and [14]. For any non-constant meromorphic function f, we denote by $S(r,f)$ any quantity satisfying

$$\lim_{r\to\infty}\frac{S(r,f)}{T(r,f)}=0,$$

possibly outside of a set of finite linear measure in $\mathbf{R}$. Suppose that a is a meromorphic function, we say that $a(z)$ is a small function of f, if $T(r,a)=S(r,f)$.

Rubel and Yang [8], Mues and Steinmetz [7], Gundersen [3] and Yang [9], Zheng and Wang [16], and many other authors have obtained elegant results on the uniqueness problems of entire functions that share values CM or IM with their first or k-th derivatives. In the aspect of only one CM value, R. Brück [1] posed the following question.

What results can be obtained if one assumes that f and f' share only one value CM plus some growth condition?

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And he presented the following conjecture.

Conjecture. *Let f be a non-constant entire function. Suppose that $\rho_1(f)$ is not a positive integer or infinite, if f and f' share one finite value a CM, then*

$$\frac{f'-a}{f-a}=c$$

for some non-zero constant c, where $\rho_1(f)$ is the first iterated order of f which is defined by

$$\rho_1(f)=\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}.$$

Brück also showed in the same paper that the conjecture is true if $a=0$ or $N(r,1/f')=S(r,f)$ (no any growth condition in the later case). Furthermore in 1998, Gundersen and Yang [4] proved that the conjecture is true if f is of finite order, and in 1999, Yang [10] generalized their result to the k-th derivatives. In 2004, Chen and Shon [2] proved that the conjecture is true for entire functions of first iterated order $\rho_1<1/2$. In 2003, Yu [15] considered the case that a is a small function, and obtained the following results.

Theorem A. *Let f be a non-constant entire function, let k be a positive integer, and let a be a small meromorphic function of f such that $a(z)\not\equiv 0,\infty$. If $f-a$ and $f^{(k)}-a$ share the value 0 CM and $\delta(0,f)>3/4$, then $f\equiv f^{(k)}$.*

Theorem B. *Let f be a non-constant, non-entire meromorphic function, let k be a positive integer, and let a be a small meromorphic function of f such that $a(z)\not\equiv 0,\infty$, f and a do not have any common pole. If $f-a$ and $f^{(k)}-a$ share the value 0 CM and $4\delta(0,f)+2(8+k)\Theta(\infty,f)>19+2k$, then $f\equiv f^{(k)}$.*

In the same paper, Yu [15] posed the following questions.

Question 1. *Can a CM shared value be replaced by an IM shared value in Theorem A?*

Question 2. *Is the condition $\delta(0,f)>3/4$ sharp in Theorem A?*

Question 3. *Is the condition $4\delta(0,f)+2(8+k)\Theta(\infty,f)>19+2k$ sharp in Theorem B?*

Question 4. *Can the condition "f and a do not have any common pole" be deleted in Theorem B?*

In 2004, Liu and Gu [6] obtainted the following results.

Theorem C. *Let $k\geq 1$ and let f be a non-constant meromorphic function, and let a be a small meromorphic function of f such that $a(z)\not\equiv 0,\infty$. If $f-a$ and $f^{(k)}-a$ share the value 0 CM and $f^{(k)}$ and a do not have any common poles of same multiplicity and*

$$2\delta(0,f)+4\Theta(\infty,f)>5,$$

then $f\equiv f^{(k)}$.

Theorem D. *Let $k \geq 1$ and let f be a non-constant entire function, and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0, f) > 1/2$, then $f \equiv f^{(k)}$.*

It is natural to ask what happens if $f^{(k)}$ is replaced by $L(f)$ in Theorem C and D? where

$$L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_0 f, \tag{1.1}$$

a_j $(j = 0, 1, \cdots, k-1)$ are polynomials. Corresponding to this question, we obtain the following results which improve Theorem A $\sim$ D and answer the four questions mentioned above.

Theorem 1. *Let $k \geq 1$, f be a non-constant meromorphic function, and let a be a small meromorphic function such that $a(z) \not\equiv 0, \infty$. Suppose that $L(f)$ is defined by (1.1). If $f - a$ and $L(f) - a$ share the value 0 IM and*

$$5\delta(0, f) + (2k+6)\Theta(\infty, f) > 2k + 10, \tag{1.2}$$

then $f \equiv L(f)$.

Theorem 2. *Let $k \geq 1$, f be a non-constant meromorphic function, and let a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. Suppose that $L(f)$ is defined by (1.1). If $f - a$ and $L(f) - a$ share the value 0 CM and $2\delta(0, f) + 3\Theta(\infty, f) > 4$, then $f \equiv L(f)$.*

Corollary 1. *Let $k \geq 1$, and let f be a non-constant meromorphic function, a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 IM and $5\delta(0, f) + (2k+6)\Theta(\infty, f) > 2k + 10$, then $f \equiv f^{(k)}$.*

Corollary 2. *Let $k \geq 1$, and let f be a non-constant meromorphic function, a be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $2\delta(0, f) + 3\Theta(\infty, f) > 4$, then $f \equiv f^{(k)}$.*

Corollary 3. *Let $k \geq 1$, and let f be a non-constant meromorphic function, $L(f)$ be defined by (1.1). Suppose that f and $L(f)$ have the same fixed points (counting multiplicities) and that $2\delta(0, f) + 3\Theta(\infty, f) > 4$, then $f \equiv L(f)$.*

Corollary 4. *Let $k \geq 1$, and let f be a non-constant meromorphic function, $L(f)$ be be given by (1.1). Suppose that f and $L(f)$ share the value 1 CM and that $2\delta(0, f) + 3\Theta(\infty, f) > 4$, then $f \equiv L(f)$.*

2. Some lemmas

Lemma 2.1. ([11]) *Let f be a non-constant meromorphic function, then*

$$N\left(r, \frac{1}{f^{(n)}}\right) \leq T(r, f^{(n)}) - T(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f), \tag{2.1}$$

$$N\left(r, \frac{1}{f^{(n)}}\right) \leq N\left(r, \frac{1}{f}\right) + n\overline{N}(r, f) + S(r, f). \tag{2.2}$$

Now let h be a non-constant meromorphic function. We denote by $N_{1)}(r,1/h)$ the counting function of simple zeros of h, and by $N_{(2}(r,1/h)$ the counting function of multiple zeros of h, where each zero in these counting functions is counted only once(see [14]). By the above definitions, we have

$$\overline{N}\left(r,\frac{1}{h}\right)+N_{(2}\left(r,\frac{1}{h}\right)\le N\left(r,\frac{1}{h}\right). \tag{2.3}$$

Let F and G be two non-constant meromorphic functions such that F and G share the value 1 IM. Let z_0 be a 1-point of F of order p, a 1-point of G of order q. We denote by $N_L(r,\frac{1}{F-1})$ the counting function of those 1-points of F where $p>q$; by $N_E^{1)}(r,\frac{1}{F-1})$ the counting function of those 1-points of F where $p=q=1$; by $N_E^{(2}(r,\frac{1}{F-1})$ the counting function of those 1-points of F where $p=q\ge 2$; each point in these counting functions is counted only once. In the same way, we can define $N_L(r,\frac{1}{G-1})$, $N_E^{1)}(r,\frac{1}{G-1})$, and $N_E^{(2}(r,\frac{1}{G-1})$ (see [13]). Particularly, if F and G share 1 CM, then

$$\overline{N}_L\left(r,\frac{1}{F-1}\right)=\overline{N}_L\left(r,\frac{1}{G-1}\right)=0. \tag{2.4}$$

With these notations, if F and G share 1 IM, it is easy to see that

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)&=N_E^{1)}\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{F-1}\right)\\&\quad+N_L\left(r,\frac{1}{G-1}\right)+N_E^{(2}\left(r,\frac{1}{G-1}\right)\\&=\overline{N}\left(r,\frac{1}{G-1}\right).\end{aligned} \tag{2.5}$$

Lemma 2.2. ([12]) *Let*

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right), \tag{2.6}$$

where F and G are two nonconstant meromorphic functions. If F and G share 1 *IM and $H\not\equiv 0$, then*

$$N_E^{1)}\left(r,\frac{1}{F-1}\right)\le N(r,H)+S(r,F)+S(r,G). \tag{2.7}$$

Lemma 2.3. *Let f be a transcendental meromorphic function, $L(f)$ be defined by* (1.1)*. If $L(f)\not\equiv 0$, we have*

$$N\left(r,\frac{1}{L}\right)\le T(r,L)-T(r,f)+N\left(r,\frac{1}{f}\right)+S(r,f), \tag{2.8}$$

$$N\left(r,\frac{1}{L}\right)\le k\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+S(r,f). \tag{2.9}$$

Proof. By the first fundamental theorem and the lemma of logarithmic derivatives, we get:

$$\begin{aligned} N\left(r,\frac{1}{L}\right) &= T(r,L) - m\left(r,\frac{1}{L}\right) + O(1) \\ &\le T(r,L) - \big(m\,(r,1/f) - m\,(r,L/f)\big) + O(1) \\ &\le T(r,L) - \big(T(r,f) - N(r,1/f)\big) + S(r,f) \\ &\le T(r,L) - T(r,f) + N\left(r,\frac{1}{f}\right) + S(r,f). \end{aligned}$$

This proves (2.8). Since

$$\begin{aligned} T(r,L) &= m(r,L) + N(r,L) \\ &\le m(r,f) + m\left(r,\frac{L}{f}\right) + N(r,f) + k\overline{N}(r,f) \\ &= T(r,f) + k\overline{N}(r,f) + S(r,f), \end{aligned}$$

from this and (2.8), we obtain (2.9), Lemma 2.3 is thus proved. □

3. Proof of Theorem 1

Let

$$F = \frac{L(f)}{a}, \quad G = \frac{f}{a}. \tag{3.1}$$

From the conditions of Theorem 1, we know that F and G share 1 IM. From (3.1), we have

$$T(r,F) = O\big(T(r,f)\big) + S(r,f), \;\; T(r,G) \le T(r,f) + S(r,f), \tag{3.2}$$

$$\overline{N}(r,F) = \overline{N}(r,G) + S(r,f). \tag{3.3}$$

Obviously f is a transcendental meromorphic function, then $T(r,a_j) = S(r,f)$, for $0 \le j \le k-1$. Let H be defined by (2.6). Suppose that $H \not\equiv 0$, by Lemma 2.2 we know that (2.7) holds. From (2.6) and (3.3), we have

$$\begin{aligned} N(r,H) \le\; & N_{(2}\left(r,\frac{1}{F}\right) + N_{(2}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + N_L\left(r,\frac{1}{F-1}\right) \\ & + N_L\left(r,\frac{1}{G-1}\right) + N_0\left(r,\frac{1}{F'}\right) + N_0\left(r,\frac{1}{G'}\right), \end{aligned} \tag{3.4}$$

where $N_0(r,\frac{1}{F'})$ denotes the counting function corresponding to the zeros of F' which are not the zeros of F and $F-1$, $N_0(r,\frac{1}{G'})$ denotes the counting function corresponding to the zeros of G' which are not the zeros of G and $G-1$. From The

Second Fundamental Theorem in Nevanlinna's Theory, we have

$$\begin{aligned} T(r,F)+T(r,G) \leq{}& \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G}\right) \\ &+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,f). \end{aligned} \tag{3.5}$$

Noting that F and G share 1 IM, we get from (2.5),

$$\begin{aligned} &\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right) \\ &= 2N_E^{1)}\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{G-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right). \end{aligned}$$

Combining with (2.7) and (3.4), we obtain

$$\begin{aligned} \overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right) \leq{}& N_{(2}\left(r,\frac{1}{F}\right)+N_{(2}\left(r,\frac{1}{G}\right) \\ &+\overline{N}(r,G)+3N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right) \\ &+2N_E^{(2}\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f). \end{aligned} \tag{3.6}$$

It is easy to see that

$$\begin{aligned} &N_L\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{G-1}\right)+2N_E^{(2}\left(r,\frac{1}{G-1}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right) \\ &\leq N\left(r,\frac{1}{G-1}\right) \leq T(r,G)+O(1). \end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we have

$$\begin{aligned} \overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right) \leq{}& N_{(2}\left(r,\frac{1}{F}\right)+N_{(2}\left(r,\frac{1}{G}\right) \\ &+\overline{N}(r,G)+2N_L\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{G-1}\right)+T(r,G) \\ &+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f). \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.5) and by using (2.3) and (3.3), we have

$$\begin{aligned} T(r,F) \leq{}& 3\overline{N}(r,G)+N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)+2N_L\left(r,\frac{1}{F-1}\right) \\ &+N_L\left(r,\frac{1}{G-1}\right)+S(r,f). \end{aligned} \tag{3.9}$$

Noting that

$$N\left(r,\frac{1}{F}\right)=N\left(r,\frac{a}{L}\right)\le N\left(r,\frac{1}{L}\right)+S(r,f),$$

we obtain from (2.8), (3.1) and (3.9) that

$$\begin{aligned}T(r,f)&\le 3\overline{N}(r,f)+2N\left(r,\frac{1}{f}\right)+2N_L\left(r,\frac{1}{F-1}\right)\\&\quad+N_L\left(r,\frac{1}{G-1}\right)+S(r,f).\end{aligned}\tag{3.10}$$

From (2.2), (2.9) and (3.1), we have

$$\begin{aligned}&2N_L\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{G-1}\right)\le 2N\left(r,\frac{1}{F'}\right)+N\left(r,\frac{1}{G'}\right)\\&\le 2\big(N(r,1/F)+\overline{N}(r,F)\big)+N(r,1/f)+\overline{N}(r,f)+S(r,f)\\&\le 2\big(N(r,1/f)+k\overline{N}(r,f)\big)+N(r,1/f)+3\overline{N}(r,f)+S(r,f)\\&\le 3N(r,1/f)+(2k+3)\overline{N}(r,f)+S(r,f).\end{aligned}\tag{3.11}$$

From (3.10) and (3.11), we have

$$T(r,f)\le 5N(r,1/f)+(2k+6)\overline{N}(r,f)+S(r,f),\tag{3.12}$$

which contradicts the assumption (1.2) of Theorem 1. Thus, $H\equiv 0$. By integration, we get from (2.6) that

$$\frac{1}{G-1}=\frac{A}{F-1}+B,$$

where $A(\ne 0)$ and B are constants. Thus

$$G=\frac{(B+1)F+(A-B-1)}{BF+(A-B)},\quad F=\frac{(B-A)G+(A-B-1)}{BG-(B+1)}.\tag{3.13}$$

We discuss the following three cases.

Case 1. Suppose that $B\ne 0,-1$. From (3.13) we have $\overline{N}\left(r,1/\left(G-\frac{B+1}{B}\right)\right)=\overline{N}(r,F)$.From this and the second fundamental theorem, we have

$$\begin{aligned}T(r,f)&\le T(r,G)+S(r,f)\\&\le \overline{N}(r,G)+\overline{N}(r,1/G)+\overline{N}\left(r,\frac{1}{G-\frac{B+1}{B}}\right)+S(r,f)\\&\le N(r,1/G)+\overline{N}(r,F)+\overline{N}(r,G)+S(r,f)\\&\le N(r,1/f)+2\overline{N}(r,f)+S(r,f),\end{aligned}$$

which contradicts the assumption (1.2).

Case 2. Suppose that $B=0$. From (3.13) we have

$$G=\frac{F+(A-1)}{A},\quad F=AG-(A-1).\tag{3.14}$$

If $A \neq 1$, from (3.14) we can obtain $N\left(r, 1/\left(G - \frac{A-1}{A}\right)\right) = N(r, 1/F)$, by (2.9) and the same arguments as in case 1, we have a contradiction. Thus $A = 1$. From (3.14) we have $F \equiv G$, then $f \equiv L$.

Case 3. Suppose that $B = -1$, from (3.13) we have

$$G = \frac{A}{-F + (A+1)}, \quad F = \frac{(A+1)G - A}{G}. \tag{3.15}$$

If $A \neq -1$, we obtain from (3.15) that $N\left(r, 1/\left(G - \frac{A}{A+1}\right)\right) = N(r, 1/F)$. By the same reasoning discussed in the case 2, we obtain a contradiction. Hence $A = -1$. From (3.15), we get $F \cdot G \equiv 1$, that is

$$f \cdot L \equiv a^2. \tag{3.16}$$

From (3.16), we have

$$N\left(r, \frac{1}{f}\right) + N(r, f) = S(r, f), \tag{3.17}$$

and so $T(r, f^{(k)}/f) = S(r, f)$. From (3.17), we obtain

$$2T\left(r, \frac{f}{a}\right) = T\left(r, \frac{f^2}{a^2}\right) = T\left(r, \frac{a^2}{f^2}\right) + O(1) = T\left(r, \frac{L}{f}\right) + O(1) = S(r, f),$$

and so $T(r, f) = S(r, f)$, this is impossible. This completes the proof of Theorem 1. □

4. Proof of Theorem 2

Let F and G be given by (3.1), from the assumption of Theorem 2, we know that F and G share 1 CM. Similar to the proof of Theorem 1, we obtain (3.10). Notice that (2.4) holds in this case, and so (3.10) gives

$$T(r, f) \leq 3\overline{N}(r, f) + 2N\left(r, \frac{1}{f}\right) + S(r, f),$$

which contradicts the assumption of Theorem 2. Thus, $H \equiv 0$. By the same reasoning as in the proof of Theorem 1, we obtain the result of Theorem 2, and we complete the proof of Theorem 2. □

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