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MAY THE CAUCHY TRANSFORM OF A NON-TRIVIAL FINITE MEASURE VANISH ON THE SUPPORT OF THE MEASURE?

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Abstract. Consider a finite complex Radon measure μ in the plane whose Cauchy transform vanishes μ-almost everywhere on the support of μ. It looks like, excluding some trivial cases, μ should be the zero measure. We show that this is the case if certain additional conditions hold.

1. Introduction

Let μ be a finite complex Radon measure in the plane and let $\mathscr{C}(\mu) = (1/z)\star\mu$ be its Cauchy transform. Being the convolution of the locally integrable function $1/z$ with a finite measure, $\mathscr{C}(\mu)$ is a locally integrable function (with respect to planar Lebesgue measure dA) and thus it is defined almost everywhere with respect to dA. Let S be the closed support of the measure μ. For $z \in S$, the value $\mathscr{C}(\mu)(z)$ does not need to be defined μ-almost everywhere, because μ may be singular with respect to dA. We set

$$\mathscr{C}_\varepsilon(\mu)(z) = \int_{|z-w|>\varepsilon} \frac{d\mu(w)}{z-w}$$

and

$$\mathscr{C}(\mu)(z) = \lim_{\varepsilon\to 0} \mathscr{C}_\varepsilon(\mu)(z) = \mathrm{P.V.}\int \frac{d\mu(w)}{z-w},$$

whenever the principal value integral exists. Notice that for $z \notin S$ the above limit exists everywhere and coincides with the value of the locally integrable function $\mathscr{C}(\mu)$.

Melnikov and Volberg raised the following question. Assume that $\mathscr{C}(\mu)$ exists and vanishes μ-almost everywhere on S. Does it then follow that $\mu = 0$?

A first remark is that the answer is obviously no for a point mass. On the other hand, we do not know any example without point masses that provides a negative answer.

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A second remark is that if for any reason $\mathscr{C}(\mu)$ turns out to be continuous everywhere and S is compact, then the answer is yes, just by the maximum principle. This happens, for instance, if the Newtonian Potential $(1/|z|) \star \mu$ is continuous and S is compact; in particular, if $\mu = f\,dA$ with f a compactly supported function in $\in L^p(\mathbf{C})$, $p > 2$.

In this paper we provide two sufficient conditions that ensure that the answer to our problem is positive. In fact we strongly believe that the answer should be yes except for the case of measures with point masses.

Theorem 1. *Let $\mu = f dA$ with f a complex valued function in $L^1(dA)$ and assume that $\mathscr{C}(f\,dA)$ vanishes dA-almost everywhere on the support of f. Then $f = 0\,dA$-almost everywhere.*

To state the next result we need to introduce some notation and recall some well-known facts.

Given a positive Radon measure μ in the plane the total Menger curvature (see [Mel]) of μ is

$$c^2(\mu) = \iiint \frac{1}{R(z,w,\zeta)^2}\, d\mu(z)\, d\mu(w)\, d\mu(\zeta),$$

where $R(z,w,\zeta)$ is the radius of the disc through z, w and ζ (the inverse of $R(z,w,\zeta)$ is called the Menger curvature of the triple (z,w,ζ)).

The one-dimensional fractional maximal function of μ is defined by

$$M_1\mu(z) = \sup_{r>0} \frac{\mu D(z,r)}{r},$$

where $D(z,r)$ is the open disc with center z and radius r.

We will also use the standard Hardy–Littlewood maximal operator associated to area measure, namely

$$M_2\mu(z) = \sup_{r>0} \frac{\mu D(z,r)}{\pi r^2}$$

and the Hardy–Littlewood maximal operator associated to a positive Radon measure μ

$$M_\mu f(z) = \sup_{r>0} \frac{1}{\mu\big(D(z,r)\big)} \int_{D(z,r)} |f(w)|\, d\mu(w), \quad f \in L^1_{\mathrm{loc}}(\mu),$$

where z is in the support of μ.

The upper one-dimensional density of μ is $\theta^*_\mu(x) = \limsup_{r\to 0} \mu\big(D(x,r)\big)/r$.

It is a deep theorem of Léger [Lé] that if $c^2(\mu) < \infty$ and $0 < \theta^*_\mu(z) < \infty$, μ-almost everywhere, then μ vanishes out of a rectifiable set, that is, out of a countable union of rectifiable curves.

Using this, it was proved in [To1] that if $c^2(\mu) < \infty$ and $M_1\mu(z) < \infty$ μ-almost everywhere, then the principal value integral $\mathscr{C}(\mu)(z)$ exists μ-almost everywhere. To get readily a more complete and symmetric statement, set $\theta_\mu(z) = \lim_{r\to 0} \mu\bigl(D(x,r)\bigr)/r$, whenever the limit exists.

We now obtain the following: if μ is a finite positive Radon measure on $\mathbf{C}$ such that $c^2(\mu) < \infty$ and $\int (M_1\mu)^2\, d\mu < \infty$, then $\theta_\mu(z)$ and the principal value integral $\mathscr{C}(\mu)(z)$ exist μ-almost everywhere. Indeed, let $F = \{z \in \mathbf{C} : \theta^*_\mu(z) > 0\}$. From the results in [Lé] it follows that F is rectifiable. Since $M_1\mu(x) < \infty$, μ-a.e., there exists a non negative function f such that $\mu_{|F}$ is absolutely continuous with respect to length measure on F and has density f. By the rectifiability of F and Lebesgue's differentiation theorem, $\theta_\mu(z)$ exists and coincides with $f(z)$ at μ-almost all $z \in F$. On the other hand, for $z \notin F$ we have $\theta_\mu(z) = 0$. The existence of $\mathscr{C}(\mu)$ in the principal value sense μ almost everywhere follows now from the results in [MaMe] and [To1].

The surprising new relevant fact is that we can moreover obtain the following precise identity.

Theorem 2. *Assume that μ is a positive finite Radon measure with $c^2(\mu) < \infty$ and $\int (M_1\mu)^2\, d\mu < \infty$. Then we have*

$$(1)\qquad \|\mathscr{C}(\mu)\|^2_{L^2(\mu)} = \lim_{\varepsilon\to 0} \|\mathscr{C}_\varepsilon\mu\|^2_{L^2(\mu)} = \frac{\pi^2}{3}\int \theta_\mu(z)^2\, d\mu(z) + \frac{1}{6}\, c^2(\mu),$$

where $\mathscr{C}(\mu)$ is understood in the principal value sense.

In particular, if $\mathscr{C}(\mu) = 0$ μ-almost everywhere, then $\mu = 0$.

Notice that the latter statement follows readily from the former, because if $\mathscr{C}(\mu)$ vanishes μ-almost everywhere, then the total Menger curvature of μ is zero, which means that μ is supported on a straight line. Since the density $\theta_\mu(z)$ also vanishes μ-almost everywhere and μ is absolutely continuous with respect to length on that line, μ must be the zero measure.

Let us remark that the identity (1) has already been proved in [To4, Lemma 6] in the particular case where μ is the arc length measure on a finite collection of pairwise disjoint compact segments.

For some recent results concerning principal values of Cauchy integrals, see [Ma1], [MaP], [To2] and [JP]. See also [Ma2, Section 6] for other related open problems and [To3] for another connected question.

Section 1 contains the proof of Theorem 1 and Section 2 the proof of Theorem 2. We use standard notation and terminology. In particular, the letter C will denote a constant that does not depend on the relevant parameters involved and that may vary from one occurrence to another. The notation $A \lesssim B$ is equivalent to $A \le CB$.

2. Proof of Theorem 1

The next lemma is folklore. We include a proof for the reader's convenience.

Lemma 3. *Let ν be a positive finite Radon measure. Then*

$$\left|\left\{w \in \mathbf{C} : \left(\frac{1}{|z|} \star \nu\right)(w) > t\right\}\right| \le C \frac{\|\nu\|^2}{t^2},$$

where $|E|$ denotes the area of the set E.

Proof. Set

$$E = \left\{w \in \mathbf{C} : \left(\frac{1}{|z|} \star \nu\right)(w) > t\right\}.$$

Then

$$|E| \le \int_E \frac{1}{t}\left(\frac{1}{|z|} \star \nu\right)(w)\, dA(w) = \frac{1}{t}\int \left(\frac{1}{|z|} \star \chi_E\, dA\right)(\zeta)\, d\nu(\zeta).$$

We now take $R = |E|^{1/2}$ and write:

$$\left(\frac{1}{|z|} \star \chi_E\, dA\right)(\zeta) = \int_{D(\zeta,R)} \frac{1}{|\zeta - z|}\, \chi_E(z)\, dA(z) + \int_{\mathbf{C}\setminus D(\zeta,R)} \frac{1}{|\zeta - z|}\, \chi_E(z)\, dA(z).$$

The first integral is estimated by $2\pi R$ and the second by $|E|/R = |E|^{1/2}$. Hence

$$|E| \le C\, \frac{1}{t}\, |E|^{1/2}\, \|\nu\|,$$

which proves the lemma. □

Let μ be a complex finite Radon measure and set

$$F(z) = \frac{1}{\pi}\, \mathscr{C}(\mu)(z) = \frac{1}{\pi}\int \frac{1}{z - w}\, d\mu(w),$$

$$\tilde{\mu}(z) = \lim_{\varepsilon \to 0} \frac{\mu D(z,\varepsilon)}{\pi \varepsilon^2},$$

and

$$B\mu(z) = -\frac{1}{\pi}\, \mathrm{P.V.} \int \frac{d\mu(w)}{(z-w)^2}.$$

Notice that the above expressions exist dA-almost everywhere. Indeed, $\tilde{\mu}$ coincides dA-almost everywhere with the absolutely continuous part of μ (and thus vanishes if μ is singular with respect to dA) and $B(\mu)$ is the Beurling transform of μ. We are interested in the differentiability properties of F. Recall that

$$\frac{\partial F}{\partial \bar{z}} = \mu \quad \text{and} \quad \frac{\partial F}{\partial z} = B\mu,$$

in the sense of distributions. Thus we set for $w \ne z$

$$Q(w,z) = Q_\mu(w,z) = \frac{|F(w) - F(z) - B\mu(z)(w-z) - \tilde{\mu}(z)(\overline{w} - \bar{z})|}{|w - z|}.$$

It is not true that F is differentiable at dA-almost all points z, but the following weaker substitute result is available (the method is inspired by [St, Chapter 8]).

Lemma 4. *For dA-almost all $z \in \mathbf{C}$ we have*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup_{t>0} t \bigl| \{ w \in D(z,\varepsilon) : Q_\mu(w,z) > t \} \bigr|^{1/2} = 0.$$

Proof. Notice that the conclusion of the lemma holds if μ is of the form $\varphi\, dA$, with φ a compactly supported infinitely differentiable function, or if μ is supported on a closed set of zero area. Therefore the set of measures for which the conclusion of the lemma holds is dense in the space of all finite complex Radon measures endowed with the total variation norm. Consider the sub-linear operator

$$T\mu(z) = \sup_{\varepsilon>0} \frac{1}{\varepsilon} \sup_{t>0} t \bigl| \{ w \in D(z,\varepsilon) : Q_\mu(w,z) > t \} \bigr|^{1/2}.$$

By the opening remark it is clearly enough to show that T satisfies the weak type inequality

$$\bigl| \{ z \in \mathbf{C} : T\mu(z) > t \} \bigr| \le \frac{C}{t} \, \|\mu\|,$$

which follows from

$$T\mu(z) \le C\, \{ M_2\mu(z) + B^*\mu(z) \}, \quad z \in \mathbf{C}, \tag{2}$$

where B^* stands for the maximal Beurling transform, that is,

$$B^*\mu(z) = \sup_{\varepsilon>0} |B_\varepsilon \mu(z)|$$

and

$$B_\varepsilon \mu(z) = \frac{1}{\pi} \int_{|w-z|>\varepsilon} \frac{d\mu(w)}{(z-w)^2}.$$

To prove (2) take $z=0$, $w \neq 0$ and set $\delta = |w|$. Then

$$\begin{aligned}
&|F(w) - F(0) - B\mu(0)w - \tilde{\mu}(0)\overline{w}| \\
&\qquad \le |F(w) - F(0) - B_{2\delta}\mu(0)w| + |w|\, |B_{2\delta}\mu(0) - B\mu(0)| + |w| M_2\mu(0) \\
&\qquad \le |F(w) - F(0) - B_{2\delta}\mu(0)w| + 2|w| B^*\mu(0) + |w| M_2\mu(0)
\end{aligned}$$

and, since we can assume without loss of generality that μ is a positive measure,

$$\begin{aligned}
|F(w) - F(0) - B_{2\delta}\mu(0)w| \le \frac{1}{\pi} \int_{|\zeta|<2\delta} &\left| \frac{1}{w-\zeta} + \frac{1}{\zeta} \right| d\mu(\zeta) \\
&+ \frac{1}{\pi} \int_{|\zeta|>2\delta} \left| \frac{1}{w-\zeta} + \frac{1}{\zeta} + \frac{w}{\zeta^2} \right| d\mu(\zeta) = I + II.
\end{aligned}$$

We first estimate the term II. Observe that if $|\zeta| > 2\delta$ then

$$\left|\frac{1}{w-\zeta} + \frac{1}{\zeta} + \frac{w}{\zeta^2}\right| \le \frac{|w|^2}{|w-\zeta||\zeta|^2} \le 2\frac{|w|^2}{|\zeta|^3}.$$

Thus

$$\pi\, II \le \int_{|\zeta|>2\delta} 2\frac{|w|^2}{|\zeta|^3}\, d\mu(\zeta) \le C|w| M_2\mu(0),$$

where in the last inequality we use the standard idea of decomposing the domain of integration in annuli centered at 0 whose distance to 0 is of the order of $\delta\, 2^n$.

The term I is π times the integral

$$\int_{|\zeta|<2\delta} \frac{|w|}{|w-\zeta|\,|\zeta|}\, d\mu(\zeta) = |w| \left(\frac{1}{|\zeta|} \star \chi_{D(0,2\delta)} \frac{d\mu}{|\zeta|}\right)(w).$$

Since $\delta = |w|$ we obtain using Lemma 3

$$\begin{aligned}
&\frac{t}{\varepsilon}\left|\left\{w \in D(0,\varepsilon) : \left(\frac{1}{|\zeta|} \star \chi_{D(0,2\delta)} \frac{d\mu}{|\zeta|}\right)(w) > t\right\}\right|^{1/2} \\
&\quad \le \frac{t}{\varepsilon}\left|\left\{w \in D(0,\varepsilon) : \left(\frac{1}{|\zeta|} \star \chi_{D(0,2\varepsilon)} \frac{d\mu}{|\zeta|}\right)(w) > t\right\}\right|^{1/2} \\
&\quad \le \frac{C}{\varepsilon} \int_{D(0,\,2\varepsilon)} \frac{d\mu(\zeta)}{|\zeta|} \le C M_2\mu(0),
\end{aligned}$$

where in the last inequality we decompose again the domain of integration in annuli centered at 0 whose distance to 0 is of the order of $\varepsilon\, 2^{-n}$. □

Proof of Theorem 1. Let S be the closed support of the function f and set $E = \{z \in S : F(z) = 0 \text{ and } f(z) \ne 0\}$, so that we have to show that $|E| = 0$. We will prove that almost all points of E are not points of density of S. Take a point $z \in E$ which is a Lebesgue point of f and at which the conclusion of Lemma 4 holds. Assume that $z = 0$ and consider the $\mathbf{R}$-linear mapping $L\colon \mathbf{C} \to \mathbf{C}$ defined by $L(w) = aw + b\overline{w}$, where $a = B\mu(0), \mu = f\,dA$ and $b = f(0)$. Since $b \ne 0$ L is not identically 0. We distinguish two cases.

Case 1: the kernel of L is $\{0\}$. Then for some $\delta > 0$ we have

$$\frac{|aw + b\overline{w}|}{|w|} \ge \delta, \quad |w| = 1. \tag{3}$$

Write, as in the proof of Lemma 4,

$$F(z) = \frac{1}{\pi}\mathscr{C}(\mu)(z),$$

where $\mu = f\,dA$, and

$$Q(w) = \frac{|F(w) - aw - b\overline{w}|}{|w|}, \quad w \neq 0.$$

Take $t = \frac{1}{2}\delta$ (in fact, any other any positive number t less than δ would work). Taking into account that F vanishes on S dA-almost everywhere and (3), we obtain

$$\begin{aligned}\frac{t^2\big|\{w \in D(0,\varepsilon) : Q(w) > t\}\big|}{\varepsilon^2} &= \frac{t^2\big|\{w \in D(0,\varepsilon) \cap S : Q(w) > t\}\big|}{\varepsilon^2} \\ &\quad + \frac{t^2\big|\{w \in D(0,\varepsilon) \setminus S : Q(w) > t\}\big|}{\varepsilon^2} \\ &\geq \frac{t^2|D(0,\varepsilon) \cap S|}{\varepsilon^2}.\end{aligned}$$

Thus Lemma 4 tells us that $|D(0,\varepsilon) \cap S|/\varepsilon^2$ tends to 0 with ε, which means that 0 is not a point of density of S.

Case 2: the kernel of L is one-dimensional. Assume without loss of generality that $L(1) = 0$. Then $L(w) = aw - a\overline{w} = a2i\,\mathrm{Im}(w)$. Thus, on the cone $K = \big\{w \in \mathbf{C} : |\,\mathrm{Im}(w)| \geq |w|/\sqrt{2}\,\big\}$, we have the inequality

$$\frac{|aw + b\overline{w}|}{|w|} = 2|a|\frac{|\,\mathrm{Im}(w)|}{|w|} \geq \sqrt{2}\,|a| = \delta > 0,$$

where the last identity is a definition of δ. Take now any positive number t less than δ. As before we have

$$\frac{t^2\big|\{w \in D(0,\varepsilon) : Q(w) > t\}\big|}{\varepsilon^2} \geq \frac{t^2|D(0,\varepsilon) \cap S \cap K|}{\varepsilon^2}$$

and therefore by Lemma 4

$$\lim_{\varepsilon \to 0} \frac{|D(0,\varepsilon) \cap S \cap K|}{\varepsilon^2} = 0.$$

On the other hand

$$\frac{|D(0,\varepsilon) \cap S \setminus K|}{\pi\varepsilon^2} \leq \frac{|D(0,\varepsilon) \setminus K|}{\pi\varepsilon^2} \leq \frac{1}{2},$$

and hence 0 is not a point of density of S. □

3. Proof of Theorem 2

To prove Theorem 2 we will need some preliminary lemmas. The first one is the following well-known estimate.

Lemma 5. *Let μ be a positive Radon measure on $\mathbf{C}$, and $z\in\mathbf{C}$. For $0<r_1<r_2\le\infty$, we have*

$$\int_{r_1\le|z-w|\le r_2}\frac{1}{|z-w|^2}\,d\mu(w)\le\frac{C}{r_1}\sup_{r_1\le R\le r_2}\frac{\mu\big(D(z,R)\big)}{R}\le C\frac{M_1\mu(z)}{r_1}.$$

The next lemma is also easy.

Lemma 6. *Let μ be a positive Radon measure on $\mathbf{C}$, and $z\in\mathbf{C}$ such that $\theta_\mu(z)=0$. Then,*

$$\lim_{r\to0} r\int_{|z-w|\ge r}\frac{1}{|z-w|^2}\,d\mu(w)=0.$$

Proof. Let $K>1$ be some big constant. For any $r>0$, by the preceding lemma we have

$$\begin{aligned}\int_{|z-w|\ge r}\frac{r}{|z-w|^2}\,d\mu(w)&=\int_{r\le|z-w|\le Kr}\frac{r}{|z-w|^2}\,d\mu(w)\\&\quad+\int_{|z-w|>Kr}\frac{r}{|z-w|^2}\,d\mu(w)\\&\lesssim\sup_{r\le R\le Kr}\frac{\mu\big(D(z,R)\big)}{R}+\frac{1}{K}\,M_1\mu(z).\end{aligned}$$

As a consequence,

$$\limsup_{r\to0}\int_{|z-w|\ge r}\frac{r}{|z-w|^2}\,d\mu(w)\lesssim\frac{1}{K}\,M_1\mu(z).$$

Since $M_1\mu(z)<\infty$ and K is arbitrary, the lemma follows. □

The following version of Cotlar's inequality has been proved in [Vo, Lemma 5.1] (to be precise, there are some little differences between [Vo, Lemma 5.1] and the following result; however the arguments are almost the same):

Lemma 7. *Let μ be a positive Radon measure on $\mathbf{C}$. Then, at μ-a.e. $z\in\mathbf{C}$ we have*

$$\sup_{\varepsilon\ge\delta}|\mathscr{C}_\varepsilon\mu(z)|\le C\,\{M_\mu(\mathscr{C}_\delta\mu)(z)+M_1\mu(z)\},$$

where the constant C does not depend on δ.

Notice that in this lemma one does not need to assume the $L^2(\mu)$ boundedness of the Cauchy integral operator.

Recall the identity proved in [MeV]:

$$\begin{aligned}\|\mathscr{C}_\varepsilon\mu\|^2_{L^2(\mu)} &= \frac{1}{6}\iiint_{\substack{|z-w|>\varepsilon\\|z-\zeta|>\varepsilon\\|w-\zeta|>\varepsilon}} c(z,w,\zeta)^2\,d\mu(z)\,d\mu(w)\,d\mu(\zeta)\\ &\quad+\iiint_{\substack{|z-w|\le\varepsilon\\|z-\zeta|>\varepsilon\\|w-\zeta|>\varepsilon}} \frac{1}{(\zeta-z)\overline{(\zeta-w)}}\,d\mu(z)\,d\mu(w)\,d\mu(\zeta).\end{aligned}\tag{4}$$

It is well known that the last integral in (4) is bounded above by $C\int(M_1\mu)^2\,d\mu$. Indeed, for z, w, ζ such that $|z-w|\le\varepsilon$, $|z-\zeta|>\varepsilon$ and $|w-\zeta|>\varepsilon$ we have $|z-\zeta|\approx|w-\zeta|$, and so by Lemma 5, for any given z we get

$$\begin{aligned}\iint_{\substack{|z-w|\le\varepsilon\\|z-\zeta|>\varepsilon\\|w-\zeta|>\varepsilon}} \frac{1}{|\zeta-z|\,|\zeta-w|}\,d\mu(w)d\mu(\zeta) &\lesssim \iint_{\substack{|z-w|\le\varepsilon\\|z-\zeta|>\varepsilon\\|w-\zeta|>\varepsilon}} \frac{1}{|\zeta-z|^2}\,d\mu(w)d\mu(\zeta)\\ &\le \mu\big(D(z,\varepsilon)\big)\int_{|z-\zeta|>\varepsilon}\frac{1}{|\zeta-z|^2}\,d\mu(\zeta)\\ &\lesssim \mu\big(D(z,\varepsilon)\big)\frac{M_1\mu(z)}{\varepsilon}\le M_1\mu(z)^2,\end{aligned}\tag{5}$$

and our claim follows. So by the assumptions in the theorem, it turns out that $\|\mathscr{C}_\varepsilon\mu\|_{L^2(\mu)}$ is bounded uniformly on $\varepsilon>0$. As a consequence, by the preceding version of Cotlar's inequality and monotone convergence, we have $\|\mathscr{C}^*\mu\|_{L^2(\mu)}<\infty$, where $\mathscr{C}^*\mu(z)=\sup_{\varepsilon>0}|\mathscr{C}_\varepsilon\mu(z)|$. By dominated convergence this implies

$$\|\mathscr{C}\mu\|_{L^2(\mu)}=\lim_{\varepsilon\to0}\|\mathscr{C}_\varepsilon\mu\|_{L^2(\mu)}<\infty.$$

The identity (4) also tells us that

$$\lim_{\varepsilon\to0}\|\mathscr{C}_\varepsilon\mu\|^2_{L^2(\mu)}=\frac{1}{6}\,c^2(\mu)+\lim_{\varepsilon\to0}\iiint_{\substack{|z-w|\le\varepsilon\\|z-\zeta|>\varepsilon\\|w-\zeta|>\varepsilon}} \frac{1}{(\zeta-z)\overline{(\zeta-w)}}\,d\mu(z)\,d\mu(w)\,d\mu(\zeta).$$

Therefore, in order to prove the theorem we only have to show that the triple integral on the right side above converges to $\frac{1}{3}\pi^2\int\theta_\mu(z)^2\,d\mu$. By the assumption $\int M_1\mu^2\,d\mu<\infty$, (5), and dominated convergence, to prove this statement it suffices to show that at μ-a.e. $z\in\mathbf{C}$ we have

$$\lim_{\varepsilon\to0}\iint_{\substack{|z-w|\le\varepsilon\\|z-\zeta|>\varepsilon\\|w-\zeta|>\varepsilon}} \frac{1}{(\zeta-z)\overline{(\zeta-w)}}\,d\mu(w)\,d\mu(\zeta)=\frac{\pi^2}{3}\theta_\mu(z)^2.\tag{6}$$

If $\theta_\mu(z) = 0$ this is easy: from (5) we get

$$\iint_{\substack{|z-w|\le\varepsilon\\ |z-\zeta|>\varepsilon\\ |w-\zeta|>\varepsilon}} \frac{1}{|\zeta - z|\,|\zeta - w|}\, d\mu(w)\, d\mu(\zeta) \lesssim \mu\big(D(z,\varepsilon)\big)\frac{M_1\mu(z)}{\varepsilon},$$

which tends to 0 because $M_1\mu(z) < \infty$.

Suppose now that $\theta_\mu(z) > 0$. In this case z belongs to the rectifiable set F introduced above. We want to reduce the problem to the case where μ is supported on a Lipschitz graph. To this end, recall that there is a countable family of Lipschitz graphs Γ_n and a set Z with $\mu(Z) = 0$ such that $F \subset \bigcup_n \Gamma_n \cup Z$. Let z be a density point of Γ_n (i.e. $\mu\big(D(z,r)\setminus\Gamma_n)/\mu(D(z,r)\big) \to 0$ as $r \to 0$). Let us see that

$$\lim_{\varepsilon\to 0} \iint_{\substack{|z-w|\le\varepsilon\\ |z-\zeta|>\varepsilon\\ |w-\zeta|>\varepsilon\\ w\notin\Gamma_n \text{ or } \zeta\notin\Gamma_n}} \frac{1}{(\zeta - z)\overline{(\zeta - w)}}\, d\mu(w)\, d\mu(\zeta) = 0. \tag{7}$$

By estimates analogous to the ones in (5) it is straightforward to check that

$$\lim_{\varepsilon\to 0} \iint_{\substack{|z-w|\le\varepsilon\\ |z-\zeta|>\varepsilon^{1/2}\\ |w-\zeta|>\varepsilon}} \frac{1}{|\zeta - z|\,|\zeta - w|}\, d\mu(w)\, d\mu(\zeta) = 0. \tag{8}$$

So we may assume that $|z-\zeta| \le \varepsilon^{1/2}$ in the integral in (7). We consider first the case $w \notin \Gamma_n$:

$$\begin{aligned}
&\iint_{\substack{|z-w|\le\varepsilon\\ \varepsilon<|z-\zeta|\le\varepsilon^{1/2}\\ |w-\zeta|>\varepsilon\\ w\notin\Gamma_n}} \frac{1}{|\zeta - z|\,|\zeta - w|}\, d\mu(w)\, d\mu(\zeta)\\
&\qquad \lesssim \mu\big(D(z,\varepsilon)\setminus\Gamma_n\big) \int_{|z-\zeta|>\varepsilon} \frac{1}{|z-\zeta|^2}\, d\mu(\zeta)\\
&\qquad \le \mu\big(D(z,\varepsilon)\setminus\Gamma_n\big)\frac{M_1\mu(z)}{\varepsilon}\\
&\qquad = \frac{\mu\big(D(z,\varepsilon)\setminus\Gamma_n\big)}{\mu\big(D(z,\varepsilon)\big)}\frac{\mu\big(D(z,\varepsilon)\big)}{\varepsilon}\, M_1\mu(z)\\
&\qquad \le \frac{\mu\big(D(z,\varepsilon)\setminus\Gamma_n\big)}{\mu\big(D(z,\varepsilon)\big)} M_1\mu(z)^2 \to 0 \quad \text{as } \varepsilon\to 0,
\end{aligned}$$

assuming $M_1\mu(z) < \infty$. The case $\zeta \notin \Gamma_n$ is similar. The details are left to the reader.

By (7), when proving (6) for $z \in F$, we may assume that μ is supported on a Lipschitz graph Γ_n. We show that (6) holds in this case in a separate lemma.

Lemma 8. *Let Γ be a Lipschitz graph and μ a Radon measure supported on it such that $\mu = g\, d\mathscr{H}^1_{|\Gamma}$ and $\mu(\Gamma) < \infty$. Then,*

$$\lim_{\varepsilon\to 0} \iint_{\substack{|z-w|\le\varepsilon\\ |z-\zeta|>\varepsilon\\ |w-\zeta|>\varepsilon}} \frac{1}{(\zeta - z)\overline{(\zeta - w)}}\, d\mu(w)\, d\mu(\zeta) = \frac{\pi^2}{3}\, g(z)^2 \tag{9}$$

at μ-a.e. $z \in \Gamma$.

Proof. First we consider the case where Γ coincides with the real line and $\mu = \mathscr{H}^1_{|\mathbf{R}}$. The proof of (9) in this situation is basically contained in [To4, Lemma 7]. However, for completeness we will also show the detailed calculations here. We may assume that $z = 0$. By (8), showing that (9) holds is equivalent to proving that

$$I^1_\varepsilon := \iint_{\substack{|y|\le\varepsilon\\ \varepsilon<|z|\le\varepsilon^{1/2}\\ |y-z|>\varepsilon}} \frac{1}{z(z-y)}\, dy\, dz \to \frac{\pi^2}{3} \quad \text{as } \varepsilon \to 0,$$

where dy and dz denote the usual integration with respect to Lebesgue measure in $\mathbf{R}$. On the other hand, by symmetry it is easy to check that

$$I^1_\varepsilon = 2 \iint_{\substack{0<y\le\varepsilon\\ \varepsilon<|z|\le\varepsilon^{1/2}\\ |y-z|>\varepsilon}} \frac{1}{z(z-y)}\, dy\, dz.$$

Thus

$$I^1_\varepsilon = 2\int_0^\varepsilon \left(\int_{-\varepsilon^{1/2}}^{-\varepsilon} \frac{1}{z(z-y)}\, dz + \int_{y+\varepsilon}^{\varepsilon^{1/2}} \frac{1}{z(z-y)}\, dz \right) dy.$$

Since a primitive of $1/\bigl(z(z-y)\bigr)$ (with respect to z) is $y^{-1}\log|1-y/z|$, it follows that

$$I^1_\varepsilon = 2\int_0^\varepsilon \left[\frac{2}{y}\log\Bigl|1+\frac{y}{\varepsilon}\Bigr| + \frac{1}{y}\log\Bigl|1-\frac{y}{\varepsilon^{1/2}}\Bigr| - \frac{1}{y}\log\Bigl|1+\frac{y}{\varepsilon^{1/2}}\Bigr| \right] dy.$$

If we split the integral into two parts and we change variables, we get

$$I^1_\varepsilon = 4\int_0^1 \frac{1}{t}\log|1+t|\, dt + 2\int_0^{\varepsilon^{1/2}} \frac{1}{t}\bigl(\log|1-t| - \log|1+t|\bigr)\, dt. \tag{10}$$

It is well known that

$$\int_0^1 \frac{1}{t}\log|1+t|\, dt = \frac{\pi^2}{12}.$$

On the other hand, the last integral in (10) tends to 0 as $\varepsilon \to 0$ because the function inside the integral is bounded in $\bigl(0,\frac{1}{2}\bigr]$. Thus $I^1_\varepsilon \to \frac{1}{3}\pi^2$ as $\varepsilon \to 0$, and so the lemma holds for $\Gamma = \mathbf{R}$ and $\mu = \mathscr{H}^1_{|\mathbf{R}}$.

Consider now the case of a general Lipschitz graph. Let A be a Lipschitz function such that $\Gamma = \{(s, A(s)) : s \in \mathbf{R}\}$. Assume that $z \in \Gamma$ is a Lebesgue point of g with respect to $\mathscr{H}^1_{|\Gamma}$, that $M_1\mu(z) < \infty$, that $(s, A(s)) = z$ and that A is differentiable at s. Moreover, without loss of generality, we may suppose also that $s = 0$ and $A(0) = A'(0) = 0$ and that 0 is a Lebesgue point of A', since the term $1/\bigl[(\zeta - z)\overline{(\zeta - w)}\bigr]$ is invariant by translations and rotations[1] of z, w, ζ. By (8), showing that (9) holds for this z is equivalent to proving that

$$\lim_{\varepsilon\to 0} \iint_{\substack{|w|\le\varepsilon \\ \varepsilon<|\zeta|\le\varepsilon^{1/2} \\ |w-\zeta|>\varepsilon}} \frac{1}{\zeta\overline{(\zeta - w)}}\, d\mu(w)\, d\mu(\zeta) = \frac{\pi^2}{3} g(0)^2.$$

To prove this we will compare the integral above with

$$g(0)^2 \iint_{\substack{|s|\le\varepsilon \\ \varepsilon<|t|\le\varepsilon^{1/2} \\ |s-t|>\varepsilon}} \frac{1}{t(t-s)}\, ds\, dt,$$

where ds and dt denote the usual Lebesgue measure on $\mathbf{R}$. The latter integral converges to $\frac{1}{3}\pi^2$, as shown above. Let us denote $\gamma(s) = s + i\,A(s)$. We have

$$\begin{aligned}
&\left| \iint_{\substack{|w|\le\varepsilon \\ \varepsilon<|\zeta|\le\varepsilon^{1/2} \\ |w-\zeta|>\varepsilon}} \frac{1}{\zeta\overline{(\zeta - w)}}\, d\mu(w)\, d\mu(\zeta) - g(0)^2 \iint_{\substack{|s|\le\varepsilon \\ \varepsilon<|t|\le\varepsilon^{1/2} \\ |s-t|>\varepsilon}} \frac{1}{t(t-s)}\, ds\, dt \right| \\
&\quad\le \left| \iint_{\substack{|w|\le\varepsilon \\ \varepsilon<|\zeta|\le\varepsilon^{1/2} \\ |w-\zeta|>\varepsilon}} \frac{1}{\zeta\overline{(\zeta - w)}}\, d\mu(w)\, d\mu(\zeta) \right. \\
&\qquad \left. - \iint_{\substack{|w|\le\varepsilon \\ \varepsilon<|\zeta|\le\varepsilon^{1/2} \\ |w-\zeta|>\varepsilon}} \frac{g(0)}{\zeta\overline{(\zeta - w)}}\, d\mathscr{H}^1_{|\Gamma}(w)\, d\mu(\zeta) \right| \\
&\qquad + \left| \iint_{\substack{|w|\le\varepsilon \\ \varepsilon<|\zeta|\le\varepsilon^{1/2} \\ |w-\zeta|>\varepsilon}} \frac{g(0)}{\zeta\overline{(\zeta - w)}}\, d\mathscr{H}^1_{|\Gamma}(w)\, d\mu(\zeta) \right. \\
&\qquad \left. - \iint_{\substack{|w|\le\varepsilon \\ \varepsilon<|\zeta|\le\varepsilon^{1/2} \\ |w-\zeta|>\varepsilon}} \frac{g(0)^2}{\zeta\overline{(\zeta - w)}}\, d\mathscr{H}^1_{|\Gamma}(w)\, d\mathscr{H}^1_{|\Gamma}(\zeta) \right|
\end{aligned}$$

[1] One needs a rotation in order to get $A'(0) = 0$. However a little problem (easy to fix) may appear: when one rotates a Lipschitz graph one obtains the Lipschitz graph of another function if the rotation angle is small enough. To ensure that the rotation angle is small, if necessary one can assume that the Lipschitz constant of the graph is small, or even one can suppose that A' is continuous and then argue locally, etc.

$$
\begin{aligned}
&+\Biggl|\iint_{\substack{|w|\le\varepsilon\\ \varepsilon<|\zeta|\le\varepsilon^{1/2}\\ |w-\zeta|>\varepsilon}} \frac{g(0)^2}{\zeta\overline{(\zeta-w)}}\,d\mathscr{H}^1_{|\Gamma}(w)\,d\mathscr{H}^1_{|\Gamma}(\zeta)\\
&-\iint_{\substack{|\gamma(s)|\le\varepsilon\\ \varepsilon<|\gamma(t)|\le\varepsilon^{1/2}\\ |\gamma(s)-\gamma(t)|>\varepsilon}} \frac{g(0)^2}{\gamma(t)\overline{\bigl(\gamma(t)-\gamma(s)\bigr)}}\,ds\,dt\Biggr|\\
&+g(0)^2\iint_{\substack{|\gamma(s)|\le\varepsilon\\ \varepsilon<|\gamma(t)|\le\varepsilon^{1/2}\\ |\gamma(s)-\gamma(t)|>\varepsilon}} \left|\frac{1}{\gamma(t)\overline{\bigl(\gamma(t)-\gamma(s)\bigr)}}-\frac{1}{t(t-s)}\right|ds\,dt\\
&+g(0)^2\Biggl|\iint_{\substack{|\gamma(s)|\le\varepsilon\\ \varepsilon<|\gamma(t)|\le\varepsilon^{1/2}\\ |\gamma(s)-\gamma(t)|>\varepsilon}} \frac{1}{t(t-s)}\,ds\,dt\\
&-\iint_{\substack{|s|\le\varepsilon\\ \varepsilon<|t|\le\varepsilon^{1/2}\\ |s-t|>\varepsilon}} \frac{1}{t(t-s)}\,ds\,dt\Biggr|\\
&=: D_1+\cdots+D_5.
\end{aligned}
$$

We will show that each term $D_1,\dots,D_5$ tends to 0 as $\varepsilon\to 0$, and we will be done.

Let us deal with D_1 first:

$$
\begin{aligned}
D_1 &\lesssim \iint_{\substack{|w|\le\varepsilon\\ \varepsilon<|\zeta|\le\varepsilon^{1/2}\\ |w-\zeta|>\varepsilon}} \frac{1}{|\zeta|^2}|g(w)-g(0)|\,d\mathscr{H}^1_{|\Gamma}(w)\,d\mu(\zeta)\\
&\le \left(\varepsilon\int_{|\zeta|>\varepsilon}\frac{1}{|\zeta|^2}\,d\mu(\zeta)\right)\left(\frac{1}{\varepsilon}\int_{|w|<\varepsilon}|g(w)-g(0)|\,d\mathscr{H}^1_{|\Gamma}(w)\right).
\end{aligned}
$$

The first factor on the right side is bounded above by $M_1\mu(0)$, and the last factor tends to 0 because 0 is a Lebesgue point of g. Thus $D_1\to 0$ as $\varepsilon\to 0$.

We now consider the term D_2. Using again the fact that 0 is a Lebesgue point of g, we get

$$
\begin{aligned}
D_2 &\lesssim \iint_{\substack{|w|\le\varepsilon\\ \varepsilon<|\zeta|\le\varepsilon^{1/2}\\ |w-\zeta|>\varepsilon}} \frac{g(0)}{|\zeta|^2}|g(\zeta)-g(0)|\,d\mathscr{H}^1_{|\Gamma}(w)\,d\mathscr{H}^1_{|\Gamma}(\zeta)\\
&\lesssim \varepsilon g(0)\int_{\varepsilon<|\zeta|\le\varepsilon^{1/2}}\frac{|g(\zeta)-g(0)|}{|\zeta|^2}\,d\mathscr{H}^1_{|\Gamma}(\zeta)\\
&\lesssim \varepsilon g(0)\frac{1}{\varepsilon}\sup_{0<r<\varepsilon^{1/2}}\frac{1}{r}\int_{|\zeta|\le r}|g(\zeta)-g(0)|\,d\mathscr{H}^1_{|\Gamma}(\zeta)\to 0\quad\text{as }\varepsilon\to 0,
\end{aligned}
$$

where we applied Lemma 5 to measure $|g(\zeta)-g(0)|\,d\mathscr{H}^1_{|\Gamma}(\zeta)$ in the last inequality.

Let us turn our attention to D_3. We have

$$
D_3\lesssim g(0)^2\iint_{\substack{|\gamma(s)|\le\varepsilon\\ \varepsilon<|\gamma(t)|\le\varepsilon^{1/2}\\ |\gamma(s)-\gamma(t)|>\varepsilon}} \frac{1}{\gamma(t)^2}\bigl|(1+A'(s)^2)^{1/2}(1+A'(t)^2)^{1/2}-1\bigr|\,ds\,dt. \tag{11}
$$

Since A' is bounded, we get

$$\begin{aligned}&\big|(1+A'(s)^2)^{1/2}(1+A'(t)^2)^{1/2}-1\big|\\&\qquad\le\big|(1+A'(s)^2)^{1/2}-1\big|(1+A'(t)^2)^{1/2}+\big|(1+A'(t)^2)^{1/2}-1\big|\\&\qquad\lesssim|A'(s)|+|A'(t)|.\end{aligned}$$

We wish to replace the conditions on $w=\gamma(s)$ and $\zeta=\gamma(t)$ in the domain of integration of D_3 by similar conditions on s and t. To this end, notice that $w=s\big(1+o(1)\big)$ and $\zeta=t\big(1+o(1)\big)$ as $\varepsilon\to0$ since $A'(0)=0$ and $|s|\le\varepsilon$, $|t|\le\varepsilon^{1/2}$. As a consequence,

$$\big||w-\zeta|-|s-t|\big|\le|w-s|+|\zeta-t|\le(|w|+|\zeta|)o(1)\le2|\zeta|o(1)\lesssim|w-\zeta|o(1)$$

and so $|w-\zeta|=|s-t|\big(1+o(1)\big)$. Therefore, for ε small enough we have

$$\begin{aligned}D_3&\lesssim g(0)^2\iint_{\substack{|s|\le2\varepsilon\\ \varepsilon/2<|t|\le2\varepsilon^{1/2}\\ |s-t|>\varepsilon/2}}\frac{1}{t^2}\big(|A'(s)|+|A'(t)|\big)\,ds\,dt\\&\lesssim\frac{g(0)^2}{\varepsilon}\int_{|s|\le2\varepsilon}|A'(s)|\,ds+\varepsilon g(0)^2\int_{\varepsilon/2<|t|\le2\varepsilon^{1/2}}\frac{|A'(t)|}{t^2}\,dt\\&\lesssim\frac{g(0)^2}{\varepsilon}\int_{|s|\le2\varepsilon}|A'(s)|\,ds+g(0)^2\sup_{0<r\le2\varepsilon^{1/2}}\frac1r\int_{|t|\le r}|A'(t)|\,dt.\end{aligned}$$

Since $A'(0)=0$ and 0 is a Lebesgue point of A', we deduce that $D_3\to0$ as $\varepsilon\to0$.

Let us consider D_4. For s, t and $w=\gamma(s)$, $\zeta=\gamma(t)$ as in the integral in D_4 we have

$$\begin{aligned}\left|\frac{1}{\overline{\zeta(\zeta-w)}}-\frac{1}{\overline{t(t-s)}}\right|&\le\left|\frac{1}{\overline{\zeta(\zeta-w)}}-\frac{1}{\overline{t(\zeta-w)}}\right|+\left|\frac{1}{\overline{t(\zeta-w)}}-\frac{1}{\overline{t(t-s)}}\right|\\&\le\frac{|\zeta-t|}{|\zeta-w|\,|\zeta|\,|t|}+\frac{|\zeta-t|+|w-s|}{|t|\,|\zeta-w|\,|t-s|}\\&\lesssim\frac{o(1)}{|t|^2}.\end{aligned}$$

Thus,

$$D_4\lesssim g(0)^2\iint_{\substack{|s|\le2\varepsilon\\ \varepsilon/2<|t|\le2\varepsilon^{1/2}\\ |s-t|>\varepsilon/2}}\frac{o(1)}{t^2}\,ds\,dt\lesssim g(0)^2\,o(1).$$

So D_4 also tends to 0 when $\varepsilon\to0$.

Finally we estimate D_5. Let S be the symmetric difference between

$$\big\{(s,t)\in\mathbf{R}^2 : |w|<\varepsilon,\ \varepsilon<|\zeta|\le\varepsilon^{1/2},\ |w-\zeta|>\varepsilon\big\}$$

and

$$\big\{(s,t)\in\mathbf{R}^2 : |s|<\varepsilon,\ \varepsilon<|t|\le\varepsilon^{1/2},\ |s-t|>\varepsilon\big\}.$$

We have $D_5\lesssim g(0)^2\iint_S(1/t^2)\,ds\,dt$. We split the integral as follows:

$$\begin{aligned}\iint_S\frac{1}{t^2}\,ds\,dt &\le \iint_{\substack{\{|w|\le\varepsilon\}\triangle\{|s|\le\varepsilon\}\\ \varepsilon/2<|t|\le 2\varepsilon^{1/2}\\ |s-t|>\varepsilon/2}}\cdots+\iint_{\substack{|s|\le 2\varepsilon\\ \{|\zeta|>\varepsilon\}\triangle\{|t|>\varepsilon\}\\ |s-t|>\varepsilon/2}}\cdots\\ &\quad+\iint_{\substack{|s|\le 2\varepsilon\\ |t|>\varepsilon/2\\ \{|\zeta|\le\varepsilon^{1/2}\}\triangle\{|t|\le\varepsilon^{1/2}\}\\ |s-t|>\varepsilon/2}}\cdots\\ &\quad+\iint_{\substack{|s|\le 2\varepsilon\\ |t|>\varepsilon/2\\ \{|s-t|>\varepsilon\}\triangle\{|w-\zeta|>\varepsilon\}}}\cdots\\ &= I_1+\cdots+I_4.\end{aligned}$$

Let us deal with I_1:

$$\begin{aligned}I_1&\lesssim\mathscr{H}^1\big(\{s:|\gamma(s)|\le\varepsilon\}\triangle\{s:|s|\le\varepsilon\}\big)\int_{|t|>\varepsilon/2}\frac{1}{t^2}\,dt\\ &\approx\frac{1}{\varepsilon}\mathscr{H}^1\big(\{s:|\gamma(s)|\le\varepsilon\}\triangle\{s:|s|\le\varepsilon\}\big).\end{aligned}$$

Since $|s-\gamma(s)|=|s|o(1)\le\varepsilon o(1)$, we have

$$\mathscr{H}^1\big(\{s:|\gamma(s)|\le\varepsilon\}\triangle\{s:|s|\le\varepsilon\}\big)\le\varepsilon o(1),$$

and $I_1\to 0$ as $\varepsilon\to 0$. Consider now I_2:

$$I_2\lesssim\varepsilon\int_{\{|\zeta|>\varepsilon\}\triangle\{|t|>\varepsilon\}}\frac{1}{t^2}\,dt.$$

Since $|\zeta|=|\gamma(t)|=|t|\big(1+o(1)\big)$, we have

$$\{t:|\gamma(t)|>\varepsilon\}\triangle\{t:|t|>\varepsilon\}\subset B\big(0,\varepsilon\big(1+o(1)\big)\big)\setminus B\big(0,\varepsilon(1-o(1))\big)=:A(\varepsilon).$$

In the annulus $A(\varepsilon)$ we have $|t|\approx\varepsilon$. Moreover $\mathscr{H}^1\big(\mathbf{R}\cap A(\varepsilon)\big)\approx\varepsilon o(1)$. So we infer that $I_2\lesssim o(1)$.

The estimates for the integrals I_3 and I_4 are analogous. □

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