Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 31, 2006, 39–46

CHARACTERIZATION OF REARRANGEMENT INVARIANT SPACES WITH FIXED POINTS FOR THE HARDY–LITTLEWOOD MAXIMAL OPERATOR

Joaquim Martín and Javier Soria

Universitat Autònoma de Barcelona, Department of Mathematics
ES-08193 Bellaterra (Barcelona), Spain; jmartin@mat.uab.es

University of Barcelona, Department of Applied Mathematics and Analysis
ES-08071 Barcelona, Spain; soria@ub.edu

Abstract. We characterize the rearrangement invariant spaces for which there exists a non-constant fixed point, for the Hardy–Littlewood maximal operator (the case for the spaces $L^p(\mathbf{R}^n)$ was first considered in [7]). The main result that we prove is that the space $L^{n/(n-2),\infty}(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$ is minimal among those having this property.

1. Introduction

The centered Hardy–Littlewood maximal operator $\mathscr{M}$ is defined on the Lebesgue space $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ by

$$\mathscr{M}f(x)=\sup_{r>0}\frac{1}{|B_r|}\int_{B_r}|f(x-y)|\,dy,$$

where $|B_r|$ denotes the measure of the Euclidean ball B_r centered at the origin of $\mathbf{R}^n$.

In this paper we study the existence of non-constant fixed points of the maximal operator $\mathscr{M}$ (i.e., $\mathscr{M}f=f$) in the framework of the rearrangement invariant (r.i.) functions spaces (see Section 2 below). We will use some of the estimates proved in [7], where the case $L^p(\mathbf{R}^n)$ was studied, and show that they can be sharpened to obtain all the rearrangement invariant norms with this property (in particular we extend Korry's result to the end point case $p=n/(n-2)$, where the weak-type spaces have to considered). The main argument behind this problem is the existence of a minimal space $L^{n/(n-2),\infty}(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$ contained in all the r.i. spaces with the fixed point property.

2000 Mathematics Subject Classification: Primary 42B25, 46E30.

The research of the first-named author was partially supported by Grants MTM2004-02299, 2001SGR00069 and by Programa Ramón y Cajal (MCYT) and that of the second author was partially supported by Grants MTM2004-02299, 2001SGR00069.

2. Background on rearrangement invariant spaces

Since we work in the context of rearrangement invariant spaces it will be convenient to start by reviewing some basic definitions about these spaces.

A rearrangement invariant space $X = X(\mathbf{R}^n)$ (r.i. space) is a Banach function space on $\mathbf{R}^n$ endowed with a norm $\|\cdot\|_{X(\mathbf{R}^n)}$ such that

$$\|f\|_{X(\mathbf{R}^n)} = \|g\|_{X(\mathbf{R}^n)}$$

whenever $f^* = g^*$. Here f^* stands for the non-increasing rearrangement of f, i.e., the non-increasing, right-continuous function on $[0,\infty)$ equimeasurable with f.

An r.i. space $X(\mathbf{R}^n)$ has a representation as a function space on $\overline{X}(0,\infty)$ such that

$$\|f\|_{X(\mathbf{R}^n)} = \|f^*\|_{\overline{X}(0,\infty)}.$$

Any r.i. space is characterized by its *fundamental function*

$$\phi_X(s) = \|\chi_E\|_{X(\mathbf{R}^n)}$$

(here E is any subset of $\mathbf{R}^n$ with $|E| = s$) and the *fundamental indices*

$$\bar{\beta}_X = \inf_{s>1} \frac{\log M_X(s)}{\log s} \quad \text{and} \quad \underline{\beta}_X = \sup_{s<1} \frac{\log M_X(s)}{\log s},$$

where

$$M_X(s) = \sup_{t>0} \frac{\phi_X(ts)}{\phi_X(t)}, \quad s > 0.$$

It is well known that

$$0 \le \underline{\beta}_X \le \bar{\beta}_X \le 1.$$

(We refer the reader to [2] for further information about r.i. spaces.)

3. Main result

Before formulating our main result, it will be convenient to start with the following remarks (see [7]):

Remark 3.1. By Lebesgue's differentiation theorem one easily obtains that

$$|f(x)| \le \mathscr{M} f(x) \quad \text{a.e. } x \in \mathbf{R}^n;$$

thus f is a fixed point of $\mathscr{M}$, if and only if f is positive and

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy \le f(x) \quad \text{a.e. } x \in \mathbf{R}^n,$$

or equivalently f is a positive *super-harmonic* function (i.e. $\triangle f \le 0$, where $\triangle$ is the Laplacian operator).

Remark 3.2. If f is a non-constant fixed point of $\mathscr{M}$, and $\varphi \geq 0$ belongs to the Schwartz class $\mathscr{S}(\mathbf{R}^n)$, with $\int_{\mathbf{R}^n} \varphi(x)\, dx = 1$, then the function $f_t(x) = (f * \varphi_t)(x)$, with $\varphi_t(x) = t^{-n}\varphi(x/t)$ is also a non-constant fixed point of $\mathscr{M}$ which belongs to $\mathscr{C}^\infty(\mathbf{R}^n)$ (notice that using the Lebesgue differentiation theorem, there exists some $t > 0$ such that f_t is non-constant, since f is non-constant). In particular if $X(\mathbf{R}^n)$ is an r.i. space and $f \in X(\mathbf{R}^n)$ is a non-constant fixed point of $\mathscr{M}$, since $\mathscr{S}(\mathbf{R}^n) \subset L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ we get that $f_t \in X(\mathbf{R}^n) \cap \mathscr{C}^\infty(\mathbf{R}^n)$ is a non-constant fixed point of $\mathscr{M}$.

Remark 3.3. Using the theory of weighted inequalities for $\mathscr{M}$ (see [5]), if $\mathscr{M}f = f$, in particular $f \in A_1$ (the Muckenhoupt weight class), and hence $f(x)\, dx$ defines a doubling measure. Hence, $f \notin L^1(\mathbf{R}^n)$. Also, using the previous remark we see that if $f \in L^p(\mathbf{R}^n)$ is a fixed point, then $f \in L^q(\mathbf{R}^n)$, for all $p \leq q \leq \infty$.

Definition 3.4. Given an r.i. space $X(\mathbf{R}^n)$, we define

$$D_{I_2}\big(X(\mathbf{R}^n)\big) = \big\{f \in L^0(\mathbf{R}^n) : \|I_2 f\|_{X(\mathbf{R}^n)} < \infty\big\},$$

where I_2 is the Riesz potential,

$$(I_2 f)(x) = \int_{\mathbf{R}^n} |x-y|^{2-n} f(y)\, dy.$$

It is not hard to see that the space $D_{I_2}\big(X(\mathbf{R}^n)\big)$ is either trivial or is the largest r.i. space which is mapped by I_2 into $X(\mathbf{R}^n)$, and is also related with the theory of the optimal Sobolev embeddings (see [4] and the references quoted therein).

Theorem 3.5. *Let $X(\mathbf{R}^n)$ be an r.i. space. The following statements are equivalent*:

(1) *There is a non-constant fixed point $f \in X(\mathbf{R}^n)$ of $\mathscr{M}$.*
(2) *$n \geq 3$ and $|x|^{2-n}\chi_{\{x:|x|>1\}}(x) \in X(\mathbf{R}^n)$.*
(3) *$n \geq 3$ and $\chi_{[0,1]}(t) + t^{2/n-1}\chi_{[1,\infty)}(t) \in \overline{X}(0,\infty)$.*
(4) *$n \geq 3$ and $\big(L^{n/(n-2),\infty}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)\big) \subset X(\mathbf{R}^n)$.*
(5) *$n \geq 3$ and $D_{I_2}\big(X(\mathbf{R}^n)\big) \neq \{0\}$.*

Proof. (1) $\Rightarrow$ (2) Since if $n = 1$ or $n = 2$, the only positive super-harmonic functions are the constant functions (see [8, Remark 1, p. 210]), necessarily $n \geq 3$. Moreover, it is proved in [7] that, if $f \in \mathscr{C}^\infty(\mathbf{R}^n)$ is a non-constant fixed point of $\mathscr{M}$, then

$$f(x) \geq c|x|^{2-n}\chi_{\{x:|x|>1\}}(x).$$

Since $f \in X(\mathbf{R}^n)$, then $|x|^{2-n}\chi_{\{x:|x|>1\}}(x) \in X(\mathbf{R}^n)$.

(2) $\Rightarrow$ (3) Since if $|x|^{2-n}\chi_{\{x:|x|>1\}}(x) \in X(\mathbf{R}^n)$, then

$$F(x) = \chi_{\{x:|x|\leq 1\}}(x) + |x|^{2-n}\chi_{\{x:|x|>1\}}(x) \in X(\mathbf{R}^n).$$

An easy computation shows that

$$F^*(t) \simeq \chi_{[0,1]}(t) + t^{2/n-1}\chi_{[1,\infty)}(t).$$

(3) $\Rightarrow$ (4) Since $f \in \big(L^{n/(n-2),\infty}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)\big)$ if and only if

$$\sup_{t>0} f^*(t)W(t) < \infty,$$

where $W(t) = \max(1, t^{1-2/n})$, we have that

$$f^*(t) \le \|f\|_{L^{n/(n-2),\infty}(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)} W^{-1}(t)$$

and since $W^{-1}(t) = \chi_{[0,1]}(t) + t^{2/n-1}\chi_{[1,\infty)} \in \overline{X}\,(0,\infty)$ we have that

$$\|f\|_{X(\mathbf{R}^n)} = \|f^*\|_{\overline{X}\,(0,\infty)} \le c\|f\|_{L^{n/(n-2),\infty}(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)}$$

with $c = \|W^{-1}\|_{\overline{X}\,(0,\infty)}$.

(4) $\Rightarrow$ (5) Since (see [9] and [1])

$$(I_2 f)^*(t) \le c_1\bigg(t^{2/n-1}\int_0^t f^*(s)\,ds + \int_t^\infty f^*(s)s^{2/n-1}\,ds\bigg) \le c_2(I_2 f^0)^*(t)$$

where $f^0(x) = f^*(c_n|x|^n)$, $c_n =$ measure of the unit ball in $\mathbf{R}^n$. (Observe that $(f^0)^* = f^*$). Rewriting the middle term in the above inequalities, using Fubini's theorem, we get

$$(I_2 f)^*(t) \le d_1\bigg(\frac{n}{n-2}\int_t^\infty f^{**}(s)s^{2/n-1}\,ds\bigg) \le d_2(I_2 f^0)^*(t),$$

where $f^{**}(t) = t^{-1}\int_0^t f^*(s)\,ds$. Thus, $f \in D_{I_2}\big(X(\mathbf{R}^n)\big)$ if and only if

$$\bigg\|\int_t^\infty f^{**}(s)s^{2/n-1}\,ds\bigg\|_{\overline{X}\,(0,\infty)} < \infty. \tag{1}$$

Since

$$F(t) = \int_t^\infty \chi^{**}_{[0,1]}(s)s^{2/n-1}\,ds = c\big(\chi_{[0,1]}(t) + t^{2/n-1}\chi_{[1,\infty)}(t)\big)$$

is a decreasing function, and

$$\begin{aligned} F^0(x) &= F(c_n|x|^n) \\ &\simeq \big(\chi_{\{x:|x|\le 1\}}(x) + |x|^{2-n}\chi_{\{x:|x|>1\}}(x)\big) \in L^{n/(n-2),\infty}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) \end{aligned}$$

we get that $\chi^\circ_{[0,1]} \in D_{I_2}\big(X(\mathbf{R}^n)\big)$.

Another argument to prove this part is the following:
Since, if $n \geq 3$ (see [2, Theorem 4.18, p. 228])

$$I_2\colon L^1(\mathbf{R}^n) \to L^{n/(n-2),\infty}(\mathbf{R}^n) \quad \text{and} \quad I_2\colon L^{n/2,1}(\mathbf{R}^n) \to L^\infty(\mathbf{R}^n)$$

is bounded, we have that

$$I_2\colon \big(L^1(\mathbf{R}^n) \cap L^{n/2,1}(\mathbf{R}^n)\big) \to \big(L^{n/(n-2),\infty}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)\big) \subset X(\mathbf{R}^n)$$

is bounded, and hence $L^1(\mathbf{R}^n) \cap L^{n/2,1}(\mathbf{R}^n) \subset D_{I_2}\big(X(\mathbf{R}^n)\big)$.

(5) $\Rightarrow$ (1) Since $n \geq 3$, we can use the classical formula of potential theory (see [10, p. 126])

$$-h = \triangle(I_2 h)$$

to conclude that there is a positive function $f = I_2\chi^\circ_{[0,1]} \in X(\mathbf{R}^n)$. Then $0 \leq f_t = I_2(\chi^\circ_{[0,1]} * \varphi_t) \in X(\mathbf{R}^n) \cap \mathscr{C}^\infty(\mathbf{R}^n)$ and $\triangle f_t \leq 0$. □

We now consider particular examples, like the Lorentz spaces:

Corollary 3.6. *Let $1 \leq p < \infty$, and assume $\Lambda^p(\mathbf{R}^n, w)$ is a Banach space (i.e., $w \in B_p$ if $1 < p < \infty$ or $p \in B_{1,\infty}$ if $p = 1$, see [3]). Then, there exists a non-constant function $f \in \Lambda^p(\mathbf{R}^n, w)$ such that $\mathscr{M}(f) = f$ if and only if $n \geq 3$ and*

$$\int_1^\infty \frac{w(t)}{t^{p(1-2/n)}}\, dt < \infty.$$

In particular, this condition always holds, for $p > 1$ and n large enough.

Proof. The integrability condition follows by using the previous theorem. Now, if $w \in B_p$, then there exists an $\varepsilon > 0$ such that $w \in B_{p-\varepsilon}$, and hence it suffices to take $n > 2/\varepsilon$. Observe that if $w = 1$ and $p = 1$, then $\Lambda^1(\mathbf{R}^n, w) = L^1(\mathbf{R}^n)$, which does not have the fixed point property for any dimension n. □

Corollary 3.7. *Let $1 \leq p, q \leq \infty$ (if $p = 1$ we only consider $q = 1$). Then, there exists a non-constant function $f \in L^{p,q}(\mathbf{R}^n)$ such that $\mathscr{M}(f) = f$ if and only if $n \geq 3$ and*

$$\begin{cases} n/(n-2) < p \leq \infty \text{ or} \\ p = n/(n-2) \text{ and } q = \infty. \end{cases}$$

Corollary 3.8. (See [7]) *Let $1 \leq p \leq \infty$. There exists a non-constant function $f \in L^p(\mathbf{R}^n)$ such that $\mathscr{M}(f) = f$ if and only if $n \geq 3$ and $n/(n-2) < p \leq \infty$.*

It is interesting to know when given an r.i. space $X(\mathbf{R}^n)$, the space $D_{I_2}\big(X(\mathbf{R}^n)\big)$ is not trivial, or equivalently

$$(2) \qquad \overline{D_{I_2}\big(X(\mathbf{R}^n)\big)} := \left\{ f \in L^0([0,\infty)) : \left\| \int_t^\infty f^{**}(s) s^{2/n-1}\, ds \right\|_{\overline{X}\,(0,\infty)} < \infty \right\}$$

is not trivial. This will be done in terms of the fundamental indices of X. We start by computing the fundamental function of $D_{I_2}\big(X(\mathbf{R}^n)\big)$.

Lemma 3.9. *Let X be an r.i. space on $\mathbf{R}^n$, $n \geq 3$. Let Y be given by (2). Then*

$$\phi_Y(s) \simeq s^{n/2} \|P_{1-2/n}\chi_{[0,s]}\|_X$$

where $P_{1-2/n}f(t) = t^{2/n-1}\int_0^t f(s)s^{-2/n}\,ds$.

Proof.

$$\begin{aligned} s^{n/2}P_{1-2/n}\chi_{[0,s]}(t) &\simeq s^{n/2}\left(\chi_{[0,s]}(t) + \left(\frac{s}{t}\right)^{1-2/n}\chi_{[s,\infty)}(t)\right) \\ &\simeq \int_t^\infty \chi_{[0,s]}^{**}(r) r^{2/n-1}\,dr. \ \square \end{aligned}$$

Theorem 3.10. *Let X be an r.i. space on $\mathbf{R}^n$, $n \geq 3$. Let Y be given by (2). Then*

(1) *If $\bar{\beta}_X < 1 - 2/n$, then $Y \neq \{0\}$.*
(2) *If $Y \neq \{0\}$ then $\underline{\beta}_X \leq 1 - 2/n$.*

Proof. (1) Let $\chi_r = \chi_{[0,r]}$. Then

$$P_{1-2/n}\chi_r(t) = \int_0^1 \chi_r(\xi t)\frac{d\xi}{\xi^{n/2}} \leq c\sum_{k=0}^\infty 2^{-k(1-n/2)}\chi_{2^k r}(t).$$

Thus

$$\|P_{1-2/n}\chi_r\|_X \leq c\sum_{k=0}^\infty 2^{-k(1-n/2)}\phi_X(2^k r) \leq c\phi_X(r)\sum_{k=0}^\infty 2^{-k(1-n/2)}M_X(2^k).$$

Let $\varepsilon > 0$ be such that $\bar{\beta}_X + \varepsilon < 1 - 2/n$. Then by the definition of $\bar{\beta}_X$ it follows readily that there is a constant $c > 0$ such that

$$M_X(2^k) \leq c2^{k(\bar{\beta}_X + \varepsilon)},$$

and hence

$$\sum_{k=0}^\infty 2^{-k(1-n/2)}M_X(2^k) \leq \sum_{k=0}^\infty 2^{-k(1-n/2-\bar{\beta}_X-\varepsilon)} < \infty,$$

which implies that $\chi_r \in Y$.

(2) Since $Y \neq \{0\}$ if and only if $\|P_{1-2/n}\chi_{[0,1]}\|_X < \infty$ and

$$\sup_{t>0}(P_{1-2/n}\chi_{[0,1]})^{**}(t)\phi_X(t) \leq \|P_{1-2/n}\chi_{[0,1]}\|_X < \infty, \tag{3}$$

and easy computations show that (3) implies that

$$1 \le \sup_{t\ge 1} \frac{\phi_X(t)}{t^{1-2/n}} = c < \infty, \tag{4}$$

then, by (4)

$$\begin{aligned} M_X(a) &= \max\left(\sup_{t\ge 1/a} \frac{\phi_X(ta)}{\phi_X(t)}, \sup_{t<1/a} \frac{\phi_X(ta)}{\phi_X(t)} \right) \\ &= \max\left(\sup_{t\ge 1/a} \frac{\phi_X(ta)}{(at)^{1-2/n}} \frac{(at)^{1-2/n}}{\phi_X(t)}, \sup_{t<1/a} \frac{\phi_X(ta)}{\phi_X(t)} \right) \\ &\simeq \max\left(a^{1-2/n} \sup_{t\ge 1/a} \frac{t^{1-2/n}}{\phi_X(t)}, \sup_{t<1/a} \frac{\phi_X(ta)}{\phi_X(t)} \right). \end{aligned}$$

Thus, if $a<1$, using again (4) we get

$$M_X(a) \ge a^{1-2/n} \sup_{t\ge 1/a} \frac{t^{1-2/n}}{\phi_X(t)} \ge a^{1-2/n}$$

which implies that

$$\underline{\beta}_X \le 1 - 2/n. \ \square$$

Let us see that the converse in the previous theorem is not true.

Proposition 3.11. *There are rearrangement invariant spaces X such that*

(1) $Y \neq \{0\}$ *and* $\bar{\beta}_X \ge 1 - 2/n$.
(2) $Y = \{0\}$ *and* $\underline{\beta}_X < 1 - 2/n$.

Proof. Let $\varphi(t) = t^a \chi_{[0,1]}(t) + t^b \chi_{[1,\infty)}(t)$, with $0 \le a, b \le 1$. Let

$$X = \Big\{ f \in L^0\big([0,\infty)\big) : \sup_{t>0} f^{**}(t)\varphi(t) < \infty \Big\}.$$

Since φ is a quasi-concave function, we have that

$$\varphi(t) = \phi_X(t)$$

and

$$\underline{\beta}_X = \min(a,b), \quad \bar{\beta}_X = \max(a,b).$$

On the other hand, the space Y defined by (2) is not trivial if and only if

$$b \le 1 - 2/n.$$

Now, to prove (1) take $b \le 1 - 2/n$ and $a \ge 1 - 2/n$. And to see (2) take $b > 1 - 2/n$ and $a \le 1 - 2/n$. $\square$

Remark 3.12. If we consider

$$X_0 = \Big\{ f \in L^0\big([0,\infty)\big) : \sup_{t>0} f^{**}(t) t^{1-2/n}(1+\log^+ t) < \infty \Big\}$$

and

$$X_1 = \Big\{ f \in L^0\big([0,\infty)\big) : \sup_{t>0} f^{**}(t) \frac{t^{1-2/n}}{(1+\log^+ t)} < \infty \Big\}$$

then $\underline{\beta}_{X_i} = \bar{\beta}_{X_i} = 1-2/n$, $Y_0 = \{0\}$ and $Y_1 \neq \{0\}$.

Remark 3.13. It was proved in [7] that if we consider the strong maximal function (i.e., the maximal operator associated to centered intervals in $\mathbf{R}^n$), then there were no fixed points in any $L^p(\mathbf{R}^n)$ space, regardless of the dimension. The same argument works to show that $L^p(\mathbf{R}^n)$ cannot be replaced by any different r.i. space. Also, if we study this question for other kind of sets, like, e.g., Buseman–Feller differentiation bases (see [6]), then the only possible fixed points are the constant functions. This observation applies to any non-centered maximal operator (with respect to balls, cubes, etc.).

References

[1] Bennett, C., and K. Rudnick: On Lorentz–Zygmund Spaces. - Dissertationes Math. (Rozprawy Mat.) 175, 1980.

[2] Bennett, C., and R. Sharpley: Interpolation of Operators. - Academic Press, Boston, 1988.

[3] Carro, M., L. Pick, J. Soria, and V. Stepanov: On embeddings between classical Lorentz spaces. - Math. Inequal. Appl. 4, 2001, 397–428.

[4] Edmunds, D. E., R. Kerman, and L. Pick: Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms. - J. Funct. Anal. 170, 2000, 307–355.

[5] García-Cuerva, J., and J. L. Rubio de Francia: Weighted Norm Inequalities and Related Topics. - North-Holland Math. Stud. 116, North-Holland, Amsterdam, 1985.

[6] de Guzmán, M.: Differentiation of Integrals in $\mathbf{R}^n$. - Lecture Notes in Math. 481, Springer-Verlag, Berlin–New York, 1975.

[7] Korry, S.: Fixed points of the Hardy–Littlewood maximal operator. - Collect. Math. 52, 2001, 289–294.

[8] Lieb, E. H., and M. Loss: Analysis. - American Mathematical Society, Providence, 2001.

[9] Sawyer, E.: Boundedness of classical operators on classical Lorentz spaces. - Studia Math. 96, 1990, 145–158.

[10] Stein, E.: Singular Integrals and Differentiability Properties of Functions. - Princeton University Press, New Jersey, 1970.

Received 9 September 2004