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THE HYPERBOLIC GEOMETRY OF CONTINUED FRACTIONS $\mathbf{K}(1 \mid b_n)$

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Abstract. The Stern–Stolz theorem states that if the infinite series $\sum |b_n|$ converges, then the continued fraction $\mathbf{K}(1 \mid b_n)$ diverges. On p. 33 of [7], H. S. Wall asks whether just convergence, rather than absolute convergence of $\sum b_n$ is sufficient for the divergence of $\mathbf{K}(1 \mid b_n)$. We investigate the relationship between $\sum |b_n|$ and $\mathbf{K}(1 \mid b_n)$ with hyperbolic geometry and use this geometry to construct a sequence b_n of real numbers for which both $\sum b_n$ and $\mathbf{K}(1 \mid b_n)$ converge, thereby answering Wall's question.

1. Introduction

An *infinite complex continued fraction* is a formal expression

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}},$$

where the a_i and b_j are complex numbers and no a_i is equal to 0. This continued fraction will be denoted by $\mathbf{K}(a_n \mid b_n)$. We define Möbius transformations $t_n(z) = a_n/(b_n + z)$, for $n = 1, 2, \ldots$, and let $T_n = t_1 \circ \cdots \circ t_n$. The continued fraction is said to *converge classically* if the sequence $T_1(0), T_2(0), \ldots$ converges, else it is said to *diverge classically*. Each of the Möbius transformations T_n acts on the extended complex plane $\mathbf{C}_\infty$ and, through identifying $\mathbf{C}_\infty$ with the boundary $\{(x, y, t) : t = 0\} \cup \{\infty\}$ of three-dimensional hyperbolic space $\mathbf{H}^3 = \{(x, y, t) : t > 0\}$, the T_n may also be considered to act on $\mathbf{H}^3$ (see Section 2 for details). In this paper we restrict to continued fractions $\mathbf{K}(1 \mid b_n)$ for which every $a_n = 1$, and examine the relationship between divergence of the series $\sum |b_n|$, classical convergence of T_n and convergence of T_n within $\mathbf{H}^3$. Throughout the rest of this exposition, $t_n(z) = 1/(b_n + z)$ and $T_n = t_1 \circ \cdots \circ t_n$, although the $\circ$ symbol will in future be omitted from all functional compositions.

The chordal metric σ of $\mathbf{C}_\infty$ is defined for distinct points z and w in $\mathbf{C}$ by

$$\sigma(z, w) = \frac{2|z - w|}{\sqrt{1 + |z|^2}\,\sqrt{1 + |w|^2}}\,, \quad \sigma(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}\,.$$

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This definition is also valid for points z and w in $\mathbf{H^3}$. Let

$$\sigma_0(f,g) = \sup_{z\in\mathbf{C}_\infty} \sigma\big(f(z), g(z)\big),$$

for Möbius transformations f and g. This is the metric of uniform convergence on $\mathbf{C}_\infty$, with respect to σ.

In Section 3, we completely classify the behaviour of T_n when $\sum |b_n|$ converges. This section consists of joint work with A. F. Beardon. The following general theorem is the main result of Section 3.

Theorem 1.1. *Suppose that the series $\sum_{n=1}^\infty \sigma_0(s_n, s)$ converges, where $s, s_1, s_2, \dots$ are Möbius transformations such that s^q is the identity map for some $q\in\mathbf{N}$. Then there is another Möbius map f such that $s_1 s_2 \cdots s_{nq+r} \to f s^r$ as $n\to\infty$, where $0 \leqslant r < q$ and the convergence is uniform with respect to the chordal metric.*

We can apply Theorem 1.1 to gain information about continued fractions $\mathbf{K}(1 \mid b_n)$. Let $s_n(z) = t_n(z) = 1/(b_n+z)$ and $s(z) = \iota(z) = 1/z$. The map ι is an isometry in the chordal metric. Therefore

$$\sigma\big(s_n(z), s(z)\big) = \sigma(b_n + z, z) \le 2|b_n|.$$

Thus if the series $\sum |b_n|$ converges then Theorem 1.1 may be applied with $q = 2$ to show that $T_{2n} \to f$ for some Möbius transformation f, and $T_{2n+1} \to f\iota$. Thus T_n converges only at the fixed points of ι. The behaviour of T_n is thereby fully understood in the instance when $\sum |b_n| < \infty$.

One important aspect of Theorem 1.1 is that it contains a proof of the following well-known classical theorem.

The Stern–Stolz Theorem. *Let $b_1, b_2, \dots$ be complex numbers. If the infinite series $\sum_{n=1}^\infty |b_n|$ converges then the continued fraction $\mathbf{K}(1 \mid b_n)$ diverges classically.*

Using the notation of Theorem 1.1, $T_{2n}(0) \to f(0)$ and $T_{2n+1}(0) \to f(\infty) \neq f(0)$, therefore $T_n(0)$ does not converge, so that the Stern–Stolz theorem is proven. H. S. Wall asks on p. 33 of [7], 'whether or not the simple convergence of the series $\sum b_p$ is sufficient for the divergence of the continued fraction'. In Section 4, we answer this question by providing an example, motivated by hyperbolic geometry, of a sequence of *real* numbers $b_1, b_2, \dots$ for which both $\sum b_n$ and $\mathbf{K}(1 \mid b_n)$ converge. The reason Wall's simple question has gone so long unanswered is not because the algebra necessary to produce an example is difficult, rather it is because without the guidance provided by hyperbolic geometry, it is not clear how one should proceed in constructing such an example.

Whilst the example of Section 4 demonstrates that the converse to the Stern–Stolz theorem does not hold even for real numbers b_n, *the Seidel–Stern theorem* says that the converse does hold when the b_n are all positive numbers. In Section 5 we investigate the Seidel–Stern theorem in terms of hyperbolic geometry. The geometry uncovers more than the classical algebraic analyses and we prove the next more general theorem (in which $j = (0,0,1)$, $H = \{(x,0,t) : t > 0\}$ and the usual notation $t_n(x) = 1/(b_n + x)$ and $T_n = t_1 \cdots t_n$ are assumed).

Theorem 1.2. *Let $b_1, b_2, \ldots$ be non-negative real numbers. The sequence $T_1(j), T_2(j), \ldots$ converges in the chordal metric to a value ζ in $\overline{H}$ (closure taken in $\mathbf{R}^3_\infty$). Moreover, the following are equivalent,*

(i) $\sum_{n=1}^{\infty} b_n$ *converges;*
(ii) $\zeta \in H$;
(iii) $\mathbf{K}(1 \mid b_n)$ *diverges.*

If $\zeta \in \partial H$ and $T_n(j) \to \zeta$ as $n \to \infty$, the Möbius sequence T_n is said to *converge generally* to ζ (see [2] for a discussion of general convergence in a geometric context). Preservation of hyperbolic distance ensures that the choice of the point $j \in H$ in the definition of general convergence to ζ is not important.

Our proof of Theorem 1.2 is not necessarily shorter or simpler than existing proofs of the Seidel–Stern theorem, but the hyperbolic geometry provides us with a deeper understanding of the sequence T_n and its orbits, whereas the algebra conceals the general picture. Theorem 1.2 is an improvement on the Seidel–Stern theorem in that we also derive conclusions about the orbit of j under T_n. Moreover, the geometric techniques we utilise are not dependent on remaining in two dimensions; thus our result can easily be generalised to many dimensions. Note also that we only require the b_n not to be negative; they need not necessarily be positive.

We conclude this introduction by briefly discussing the role of the hyperbolic plane H (a more detailed treatment is supplied in Section 2). Real Möbius maps $f(x) = (ax+b)/(cx+d)$ fix the upper half-plane $\mathbf{H} = \{z \in \mathbf{C} : \mathrm{Im}[z] > 0\}$ if and only if $ad - bc > 0$ (if f does not fix $\mathbf{H}$, it interchanges $\mathbf{H}$ with $\{z : \mathrm{Im}[z] < 0\}$). We are working with Möbius maps of the form $t(x) = 1/(x+b)$, which do not satisfy this criterion, hence we cannot exploit the two-dimensional hyperbolic geometry of $\mathbf{H}$. To circumvent this problem, we extend the action of the real Möbius map t, not to $\mathbf{C}_\infty$, but to a half-plane $H = \{(x,0,t) : t > 0\}$ in $\mathbf{R}^3_\infty$ that is perpendicular to $\mathbf{C}$. With this alternative two-dimensional action, the half-plane H *is* fixed, because the transformation $\iota(x) = 1/x$ fixes j, and it also fixes H. Since t is a Möbius map, the action on H is isometric with respect to the hyperbolic metric on H. In fact, the action of t can be extended to the entire plane $\Pi \cup \{\infty\}$ that contains H, where $\Pi = \{(x,0,t) : x,\ t \in \mathbf{R}\}$. We write points $(x,0,t)$ of $\mathbf{R}$ in the form $x + tj$ to correspond with the usual real and imaginary part notation for elements of $\mathbf{C}$. This plane Π may be viewed as another model of the complex plane, and we manipulate it accordingly.

2. Möbius transformations and hyperbolic geometry

We quickly describe the geometric machinery and notation necessary to implement our methods. Rigorous details can be found in either [1] or [6].

The complex plane may be embedded as the $t=0$ plane of $\mathbf{R}^3$. Through this embedding, known as the Poincaré extension, the Möbius transformation group $\mathscr{M}$ acts on both the extended complex plane $\mathbf{C}_\infty$ and three-dimensional upper half-space, $\mathbf{H}^3 = \{(x,y,t) : t > 0\}$. The Möbius action on the latter space is isometric with respect to the hyperbolic metric $\varrho_{\mathbf{H}^3}$ of $\mathbf{H}^3$. We denote the hyperbolic metric of any suitable space D by ϱ_D.

Let $\phi\colon \mathbf{R}^3 \to \mathbf{R}^3$ be inversion in the sphere centred on $j=(0,0,1)$ of radius $\sqrt{2}$. This inversion maps $\mathbf{C}_\infty$ to the unit sphere $\mathbf{S}^2$ and maps $\mathbf{S}^2$ to $\mathbf{C}_\infty$. Furthermore, ϕ is its own inverse and satisfies $\phi(0)=-j$ and $\phi(\infty)=j$. If f is a Möbius map fixing $\mathbf{C}_\infty$, then $f^*=\phi f\phi$ is a Möbius map fixing $\mathbf{S}^2$. Likewise we use the notation $0^*=\phi(0)$, and more generally $z^*=\phi(z)$, to denote images of points under ϕ (this notation is only used at the end of Section 4). The Euclidean metric restricted to $\mathbf{S}^2$ can be transferred via ϕ to a metric σ on $\mathbf{C}_\infty$ defined by $\sigma(z,w):=|\phi(z)-\phi(w)|$. This is the chordal metric introduced in Section 1.

The *supremum metric* σ_0 was also defined in the introduction. The space $(\mathscr{M},\sigma)$ is complete. The metric σ_0 is right-invariant and satisfies

$$\sigma_0(hf,hg) \le L(h)\sigma_0(f,g),$$

for a certain positive constant $L(h)=\exp\varrho_{\mathbf{H}^3}\big(j,h(j)\big)$, although we do not make use of the exact value of $L(h)$. If $f_n\to f$ uniformly then,

$$\sigma_0(f_n^{-1},f^{-1}) = \sigma_0(I,f^{-1}f_n) \le L(f^{-1})\sigma_0(f,f_n),$$

so that $f_n^{-1}\to f^{-1}$ uniformly. If also $g_n\to g$ uniformly then

$$\sigma_0(f_ng_n,fg) \le \sigma_0(f_ng_n,fg_n)+\sigma_0(fg_n,fg) \le \sigma_0(f_n,f)+L(f)\sigma_0(g_n,g),$$

so that $f_ng_n\to fg$ uniformly.

Let f be a real Möbius transformation (a Möbius transformation with real coefficients) acting on $\mathbf{C}_\infty$. This map f fixes the extended real axis $\mathbf{R}_\infty$. Now consider f to act on $\mathbf{H}^3$. It fixes the half-plane $H=\{(x,0,t):t>0\}$ and this action is isometric with respect to the two-dimensional hyperbolic metric on H. Let $D=\{x+tj : x^2+t^2<1\}$ so that ϕ maps H to D. In fact, ϕ is an isometry from (H,ϱ_H) to (D,ϱ_D), so that $f^*=\phi f\phi$ is an isometry of (D,ϱ_D). We now have three distinct actions of f on $\mathbf{R}_\infty$, H and D, and we make use of all three of them. The hyperbolic metric on H satisfies the formula,

$$\cosh\varrho_H(z,w) = 1+2\sinh^2\left(\frac{1}{2}\varrho_H(z,w)\right) = 1+\frac{|z-w|^2}{2(z\cdot j)(w\cdot j)}. \tag{2.1}$$

This formula can be found in [1], as can the following lemma in hyperbolic geometry.

Lemma 2.1. *Choose two points a and b on the boundary of the unit disc D, separated by an angle $\theta \leqslant \pi$. Denote the hyperbolic line joining a and b by δ. Then* $\cosh \varrho_D(0,\delta)\sin(\theta/2) = 1$.

Notice that $|a-b| = 2\sin(\theta/2)$.

3. Continued fractions $\mathbf{K}(1 \mid b_n)$ for which $\sum_{n=1}^{\infty} |b_n| < \infty$

The crux of Theorem 1.1 can be found in [2, Theorem 3.7] in which A. F. Beardon credits an earlier source [5]. The identity map is denoted by I.

Theorem 3.7 from [2]. *Let u_n be a sequence of Möbius transformations for which $\sum \sigma_0(u_n, I)$ converges. Then $u_1 \cdots u_n$ converges uniformly on $\mathbf{C}_\infty$ to a Möbius transformation.*

The proof of this theorem from [2] is sufficiently short that we reproduce it here.

Proof of Theorem 3.7 from [2]. Right-invariance of σ_0 ensures that for $m < n$,

$$\sigma_0(u_n^{-1}\cdots u_1^{-1}, u_m^{-1}\cdots u_1^{-1}) = \sigma_0(u_n^{-1}\cdots u_{m+1}^{-1}, I).$$

This latter distance is equal to or less than

$$\begin{aligned}\sigma_0(&u_n^{-1}\cdots u_{m+1}^{-1}, u_{n-1}^{-1}\cdots u_{m+1}^{-1})\\ &+ \sigma_0(u_{n-1}^{-1}\cdots u_{m+1}^{-1}, u_{n-2}^{-1}\cdots u_{m+1}^{-1})\\ &+\cdots\\ &+\sigma_0(u_{m+1}^{-1}, I)\end{aligned}$$

which is equal to

$$\sigma_0(u_n^{-1}, I) + \sigma_0(u_{n-1}^{-1}, I) + \cdots + \sigma_0(u_{m+1}^{-1}, I).$$

Since $\sigma_0(u_k^{-1}, I) = \sigma_0(I, u_k)$, this sum can be forced to be arbitrarily small, provided m and n are restricted to being sufficiently large. Hence the sequence $(u_1\cdots u_n)^{-1} = u_n^{-1}\cdots u_1^{-1}$ is Cauchy with respect to the σ_0 metric, therefore it converges and hence so does $u_1\cdots u_n$. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Define $u_n = s_{(n-1)q+1}s_{(n-1)q+2}\cdots s_{nq}$ then, again using right-invariance of σ_0,

$$\begin{aligned}\sigma_0(u_n, I) &= \sigma_0(u_n, s^q)\\ &\le \sigma_0(u_n, ss_{(n-1)q+2}\cdots s_{nq}) + \sigma_0(ss_{(n-1)q+2}\cdots s_{nq}, s^q)\\ &\le \sigma_0(s_{(n-1)q+1}, s) + L(s)\sigma_0(s_{(n-1)q+2}\cdots s_{nq}, s^{q-1}).\end{aligned}$$

This argument may be repeated $q-2$ times with $s_i\cdots s_{nq}$ replacing $s_{(n-1)q+1}\cdots s_{nq}$ for $(n-1)q+1<i<nq$, to obtain

$$\sigma_0(u_n,I)\le \max\{1,L(s)^{q-1}\}\big(\sigma_0(s_{(n-1)q+1},s)+\cdots+\sigma_0(s_{nq},s)\big).$$

Therefore $\sum\sigma_0(u_n,I)$ converges so that [2, Theorem 3.7] applies to show that $s_1\cdots s_{nq}=u_1\cdots u_n\to f$ for some Möbius map f. Since $s_n\to s$ uniformly we also see that $s_1\cdots s_{nq+r}\to fs^r$. □

It was demonstrated in the introduction that when $s_n(z)=t_n(z)=1/(b_n+z)$ and $\sum|b_n|<\infty$, Theorem 1.1 shows that

$$T_{2n}\to f,\quad T_{2n+1}\to f\iota,$$

where $\iota(z)=1/z$. Therefore

$$T_{2n}^{-1}\to f^{-1},\quad T_{2n+1}^{-1}\to \iota f^{-1}.$$

In particular, this accounts for the well-known result that

$$T_{n-1}^{-1}(\infty)T_n^{-1}(\infty)\to 1$$

as $n\to\infty$.

4. Examples to answer Wall's question

For each $n=1,2,\ldots$, let

$$\hat b_{2n-1}=(-1)^n\big(\sqrt{n}-\sqrt{n-1}\big),\quad \hat b_{2n}=\frac{-2(-1)^n\sqrt{n}}{n+1}. \tag{4.1}$$

This sequence shows that *absolute* convergence of the series $\sum b_n$ is strictly necessary in the Stern–Stolz theorem, as for this particular sequence, $\sum\hat b_n$ converges and so does $\mathbf{K}(1\mid\hat b_n)$. That $\sum\hat b_n$ converges is clear. The remainder of this section consists of a proof that $\mathbf{K}(1\mid b_n)$ converges. Our proof uses the hyperbolic geometry of $H=\{x+tj: t>0\}$ as the real Möbius transformations t_n and T_n are isometries of this plane.

Let $\gamma=\{tj:t>0\}$ denote the hyperbolic line in H that joins 0 and ∞. It will be seen that $\mathbf{K}(1\mid b_n)$ converges for any real sequence $b_1,b_2,\ldots$ that satisfies the two conditions,

(i) $b_1<0,b_2>0,b_3>0,b_4<0,b_5<0,b_6>0,b_7>0,\ldots$;
(ii) $\varrho_H\big(T_n^{-1}(j),\gamma\big)$ is a positive unbounded increasing sequence.

Our particular sequence $\hat b_n$ was chosen to satisfy conditions (i) and (ii). That it satisfies (i) is evident. Before proving rigorously that $\hat b_n$ gives rise to a sequence

T_n that satisfies condition (ii), we supply a brief geometric explanation of how the $\hat{b}_n$ were chosen so that $\varrho_H\big(T_n^{-1}(j),\gamma\big) \to \infty$.

A *horocircle* $\mathscr{C}$ in H is a line or circle in $\overline{H}$ that is tangent to the boundary ∂H of H at one point ζ. As a point z in H approaches ζ along $\mathscr{C}$, the hyperbolic distance $\varrho_H(z,\delta) \to \infty$, where δ is any hyperbolic line with one end point at ζ. We define two particular horocircles in H. The first is a Euclidean line $l = \{x+j : x \in \mathbf{R}\}$ (tangent to ∂H at ∞) and the second is a Euclidean circle $C = \big\{x+tj : \big|x+\big(t-\frac{1}{2}\big)j\big| = \frac{1}{2}\big\}$ (tangent to ∂H at 0). The inversion $\iota(x) = 1/x$ maps l to C and C to l. We choose $\hat{b}_n$ so that $T_n^{-1}(j)$ remains on $l \cup C$. Notice that $t_n^{-1}(z) = -\hat{b}_n + 1/z$ and $T_n^{-1}(j) = t_n^{-1}T_{n-1}^{-1}(j)$. Thus if $T_{n-1}^{-1}(j) \in l$ then $\iota\big(T_{n-1}^{-1}(j)\big) \in C$, and $\hat{b}_n$ is chosen to be the unique non-zero real number such that $T_n^{-1}(j) = -\hat{b}_n + \iota\big(T_{n-1}^{-1}(j)\big)$ also lies on C. Similarly, if $T_{n-1}^{-1}(j) \in C$ then $T_n^{-1}(j) \in l$. The $\hat{b}_n$ are suitably fashioned so that the backwards orbits $T_n^{-1}(j)$ converge to ∞ on l and converge to 0 on C. Since l and C are horocircles and γ is a hyperbolic line connecting 0 and ∞, we see that $\varrho_H\big(T_n^{-1}(j),\gamma\big) \to \infty$ as $n \to \infty$.

Condition (ii) will now be verified rigorously for $b_n = \hat{b}_n$. It is easily proven by induction that for this choice of b_n,

$$T_{2n-1}^{-1}(j) = -(-1)^n\sqrt{n} + j, \quad T_{2n}^{-1}(j) = \frac{(-1)^n\sqrt{n}}{n+1} + \frac{j}{n+1}$$

for every n. The hyperbolic line (Euclidean half-circle) through a point $z \in H$ that is symmetric about the j axis and hence orthogonal to γ, intersects γ at $|z|j$. This is the point in γ of least hyperbolic distance from z, thus $\varrho_H(z,\gamma) = \varrho_H(z,|z|j)$. Using equation (2.1), it is a simple matter of algebra to prove that

$$\cosh \varrho_H\big(T_n^{-1}(j),\gamma\big) = \cosh \varrho_H\big(T_n^{-1}(j), |T_n^{-1}(j)|j\big) = \sqrt{n+1},$$

for every n. Condition (ii) has thereby been established for $\hat{b}_1, \hat{b}_2, \dots$.

It remains to demonstrate that $\mathbf{K}(1 \mid b_n)$ converges for any sequence $b_1, b_2, \dots$ satisfying conditions (i) and (ii). A preliminary lemma is required.

Lemma 4.1. *Let $T_1, T_2, \dots$ be a sequence of Möbius transformations arising from a continued fraction $\mathbf{K}(1 \mid b_n)$ that satisfies the above condition* (i). *The set $\mathbf{R}_\infty \backslash \{T_{n-1}(0), T_n(0)\}$ consists of two open components. The points $T_{n-2}(0)$ and $T_{n+1}(0)$ each lie in one of these open components, and they lie in the same component if and only if b_n and b_{n+1} differ in sign.*

Proof. Observe that $t_n(\infty) = 0$, $t_n^{-1}(\infty) = -b_n$ and $t_{n+1}(0) = 1/b_{n+1}$ from which it follows that

(a) $T_n(\infty) = T_{n-1}(0)$;

(b) $T_n(-b_n) = T_{n-2}(0)$ and $T_n(1/b_{n+1}) = T_{n+1}(0)$.

Equation (a) shows that the open components separated by $T_{n-1}(0)$ and $T_n(0)$ are $T_n(\mathbf{R}^+)$ and $T_n(\mathbf{R}^-)$. Equations (b) show that $T_{n-2}(0)$ and $T_{n+1}(0)$ both lie in a component (since the b_i are non-zero) and that they lie in the same component if and only if b_n and b_{n+1} differ in sign. □

For the remainder of this section, we make frequent use of the unit disc model of hyperbolic space described at the end of Section 2. Let $\theta_n \leqslant \pi$ be the acute angle between $T_{n-1}^*(0^*)$ and $T_n^*(0^*)$ (where $\theta_1 < \pi$ is the angle between $0^* = e^{-\pi j/2}$ and $T_1^*(0^*) = (1/b_1)^*$). Using Lemma 2.1 and preservation of hyperbolic distance under T_n, we see that,

$$\sin(\theta_n/2) = 1/\cosh \varrho_H\big(T_n^{-1}(j), \gamma\big) = 1/\cosh \varrho_H\big(j, T_n(\gamma)\big),$$

thus $\theta_1, \theta_2, \dots$ is a sequence in $(0, \pi)$ that decreases towards 0.

We encapsulate the convergence of $T_n(0)$ in a theorem and even obtain an explicit series formula for $T_n(0)$.

Theorem 4.2. *Let $T_1, T_2, \dots$ be a sequence of Möbius maps arising from a continued fraction $\mathbf{K}(1 \mid b_n)$ satisfying the above conditions* (i) *and* (ii). *Then*

$$\phi\big(T_n(0)\big) = T_n^*(0^*) = \exp\left[\left(-\frac{\pi}{2} + \sum_{k=1}^{n} \varepsilon_k \theta_k\right) j\right],$$

where $\theta_n < \pi$ is the acute angle between $T_{n-1}^*(0^*) \in \partial D$ and $T_n^*(0^*) \in \partial D$, and

$$\varepsilon_k = \begin{cases} -1 & \text{if } k = 1, 2 \pmod 4, \\ 1 & \text{if } k = 3, 4 \pmod 4). \end{cases}$$

Hence $T_1(0), T_2(0), \dots$ converges.

Proof. Let $z_0 = -j$ (where $-j = e^{-j\pi/2} = 0^*$) and $z_n = {T_n}^*(0^*)$, for $n \geqslant 1$. Since all z_n lie on ∂D and the angle between z_{n-1} and z_n is $\theta_n < \pi$, there is a unique sequence $\eta_1, \eta_2, \eta_3, \dots$, where $\eta_k \in \{-1, 1\}$, such that for every n,

$$z_n = \exp\left[\left(-\frac{\pi}{2} + \sum_{k=1}^{n} \eta_k \theta_k\right) j\right].$$

Recall that θ_n decreases to 0.

We denote the open segment of ∂D of angle $\theta_n < \pi$ between z_{n-1} and z_n by I_n and the other open component of $\partial D \backslash \{z_{n-1}, z_n\}$ by J_n. Notice that were z_{n-2} to lie in I_n, then $I_{n-1} \subsetneq I_n$, which is impossible as $\theta_{n-1} \geqslant \theta_n$. Therefore $z_{n-2} \in J_n$. Lemma 4.1 shows that if n is odd then $z_{n+1} \in J_n$ (since b_n and b_{n+1}

differ in sign) and if n is even then $z_{n+1} \in I_n$ (since b_n and b_{n+1} share the same sign).

We have that

$$I_n = \{z_n e^{-j\eta_n t} : 0 < t < \theta_n\},$$

$z_{n+1} = z_n e^{j\eta_{n+1}\theta_{n+1}}$ and $\theta_{n+1} \leqslant \theta_n < \pi$. Therefore the point z_{n+1} lies in I_n if and only if $\eta_{n+1} = -\eta_n$. Thus

$$\eta_{n+1} = \begin{cases} \eta_n & \text{if } n \text{ is odd,} \\ -\eta_n & \text{if } n \text{ is even.} \end{cases}$$

It is easy to check that $\eta_1 = \eta_2 = -1$, therefore by induction $\eta_n = \varepsilon_n$ for every n.

Finally, the series $\sum_{k=1}^{\infty} \varepsilon_k \theta_k$ must converge, hence $T_1(0), T_2(0), \ldots$ converges also. □

5. The proof of Theorem 1.2

The following observation of A. F. Beardon was used in [2] to prove the Stern–Stolz theorem. The numbers b_n may be complex, although we shortly lose this generality after Corollary 5.2.

Lemma 5.1. *Suppose that $b_n \to 0$ as $n \to \infty$. Given $\varepsilon > 0$ there is an integer N such that if $n \geqslant N$ then*

$$(1-\varepsilon)|b_n| \leqslant \varrho_{\mathbf{H}^3}\big(j, t_n(j)\big) \leqslant (1+\varepsilon)|b_n|.$$

Proof. Since $\iota(z) = 1/z$ is a hyperbolic isometry of $\mathbf{H}^3$ that fixes j,

$$\sinh^2\left(\frac{1}{2}\varrho_{\mathbf{H}^3}\big(j, t_n(j)\big)\right) = \sinh^2\left(\frac{1}{2}\varrho_{\mathbf{H}^3}(j, b_n + j)\right) = \frac{|b_n|^2}{4},$$

using equation (2.1). The result follows as $\sinh x$ and x are asymptotic as $x \to 0$. □

Lemma 5.1 has the following corollary, from which the Stern–Stolz theorem is easily derived (see [2] for details).

Corollary 5.2. *The two series $\sum |b_n|$ and $\sum \varrho_{\mathbf{H}^3}\big(j, t_n(j)\big)$ either both converge or both diverge.*

Observe that $\varrho_{\mathbf{H}^3}\big(T_{n-1}(j), T_n(j)\big) = \varrho_{\mathbf{H}^3}\big(j, t_n(j)\big)$, therefore if $b_n \to 0$, we can use Lemma 5.1 to estimate the hyperbolic distance between the terms $T_{n-1}(j)$ and $T_n(j)$. In this section we focus on the significance in terms of hyperbolic geometry of the restraint that each b_n is not negative. We shall see that this condition ensures that $\varrho_H\big(j, T_n(j)\big)$ is an increasing sequence (Theorem 5.4). We then show that the limit of this sequence is finite if and only if $\sum \varrho_H\big(j, t_n(j)\big)$ converges (Theorem 5.5). Only after these theorems do we focus on classical convergence and prove Theorem 5.7, which says that for continued fractions $\mathbf{K}(1 \mid b_n)$ with $b_n \geqslant 0$, classical and general convergence are equivalent. Finally, Theorem 1.2 is proven. All results other than those conclusions present in the Seidel–Stern theorem, and all proofs are due to the author (see [4] for an account of the Seidel–Stern theorem).

For the rest of this section we retain the notation $t_n(z) = 1/(b_n + z)$, but now all the b_n are required to be non-negative real numbers. None of the theorems in this section are true when the b_n are allowed to be negative. Let $L = \{x + tj \in H : x \leq 0\}$ and let $T_n^{-1}(j) = x_n + \tau_n j$. We also write ϱ_n for $\varrho_H\big(j, T_n(j)\big)$. The following lemma contains the essential hyperbolic geometry behind Theorem 5.4.

Lemma 5.3. *Let $b \geqslant 0$. If $z = u + tj \in L$ then $z - b \in L$ and*

$$\cosh \varrho(z - b, j) - \cosh \varrho(z, j) \geqslant \frac{-u}{t} b.$$

Proof. That $z - b \in L$ is clear. Using (2.1),

$$\begin{aligned}\cosh \varrho(z - b, j) - \cosh \varrho(z, j) &= \frac{|z - b - j|^2}{2t} - \frac{|z - j|^2}{2t} \\ &= \frac{b^2 - 2bu}{2t} \\ &\geqslant \frac{-u}{t} b. \ \square\end{aligned}$$

Theorem 5.4. *Each point $T_n^{-1}(j)$ is contained within L and the sequence $\varrho_n = \varrho_H\big(j, T_n(j)\big)$ is increasing.*

Proof. The first claim is true by induction as certainly $j \in L$, and if $x_n + \tau_n j = T_n^{-1}(j) \in L$ then $T_{n+1}^{-1}(j) = -b_{n+1} + 1/T_n^{-1}(j) \in L$. To prove the second claim, observe that

$$\varrho_n = \varrho_H\big(T_n^{-1}(j), j\big) = \varrho_H\big(\iota\big(T_n^{-1}(j)\big), j\big)$$

and

$$\varrho_{n+1} = \varrho_H\big(T_{n+1}^{-1}(j), j\big) = \varrho_H\big(-b_{n+1} + \iota\big(T_n^{-1}(j)\big), j\big),$$

then apply Lemma 5.3 to see that

$$\cosh \varrho_{n+1} - \cosh \varrho_n \geqslant \frac{-x_n}{\tau_n} b_{n+1} \geqslant 0, \tag{5.1}$$

from which the claim follows. $\square$

Either $\varrho_n \to \infty$, in which case $T_n(j)$ accumulates only on $\mathbf{R}_\infty$, or $\varrho_n \to k$ where $k < \infty$, in which case $T_n(j)$ remains within a compact subset of H. The next theorem relates the convergence of $T_n(j)$ to convergence of the series $\sum \varrho_H\big(j, t_n(j)\big)$.

Theorem 5.5. *If* $b_n \geqslant 0$, *then from Theorem* 5.4, $\varrho_H\big(j, T_n(j)\big) \nearrow k$, *where* $k \in (0, \infty]$. *The sum* $\sum \varrho_H\big(j, t_n(j)\big)$ *converges if and only if* $k < \infty$.

Proof. Suppose that $K = \sum \varrho_H\big(j, t_n(j)\big)$ is finite. Since T_{n-1} is a ϱ_H isometry, $\varrho_H\big(j, t_n(j)\big) = \varrho_H\big(T_{n-1}(j), T_n(j)\big)$. Therefore

$$\varrho_H\big(j, T_n(j)\big) \leq \varrho_H\big(j, T_1(j)\big) + \varrho_H\big(T_1(j), T_2(j)\big) + \cdots + \varrho_H\big(T_{n-1}(j), T_n(j)\big),$$

and this latter sum is equal to or less than K. Now suppose that $k < \infty$. The points $x_n + \tau_n j = T_n^{-1}(j)$ all lie within a compact subset of H so there is a positive constant M such that $|x_n/\tau_n| > M$ for every n. If $\varrho_n = \varrho_{n-1}$ then we see from equation (5.1) that $b_n = 0$. If $\varrho_n \neq \varrho_{n-1}$, apply the mean value theorem with the function cosh and points ϱ_{n-1} and ϱ_n to see that there is a value q_n between ϱ_{n-1} and $\varrho_n \leq k$ such that

$$\varrho_n - \varrho_{n-1} = \frac{\cosh \varrho_n - \cosh \varrho_{n-1}}{\sinh q_n}.$$

Therefore using equation (5.1), whether or whether not ϱ_n is equal to ϱ_{n-1},

$$\varrho_n - \varrho_{n-1} \geqslant \frac{-x_{n-1}}{\tau_{n-1} \sinh q_n} b_n \geqslant \frac{M}{\sinh k} b_n.$$

Summing these equations for values of n from 1 to m we obtain,

$$\varrho_m \geqslant \frac{M}{\sinh k} \sum_{n=1}^{m} b_n,$$

therefore $\sum b_n$ converges. Corollary 5.2 shows that $\sum \varrho_H\big(j, t_n(j)\big)$ converges. ▫

We must now relate convergence of $T_n(j)$ to convergence of $T_n(0)$. The following theorem is well known, although the proof is our own.

Theorem 5.6. *The sequence* $T_1(0), T_3(0), \ldots$ *decreases to* $\beta \geqslant 0$. *The sequence* $T_2(0), T_4(0), \ldots$ *increases to* $\alpha \geqslant 0$, *where* $\alpha \leqslant \beta$.

Proof. Let $l = [0, \infty]$ denote the closure of the positive real axis within $\mathbf{R}_\infty$. Notice that $t_n(l) \subseteq l$ for every n. Hence

$$l \supseteq T_1(l) \supseteq T_2(l) \supseteq \cdots.$$

Write $T_n(l) = [\alpha_n, \beta_n]$, then α_n increases to a limit α and β_n decreases to a limit β, where $\alpha \leqslant \beta$. Since α_n and β_n are the end-points of $T_n(l)$, either $\alpha_n = T_n(0)$ and $\beta_n = T_n(\infty)$, or $\beta_n = T_n(0)$ and $\alpha_n = T_n(\infty)$. Suppose the former situation occurs. Then

$$T_{n+1}(\infty) = T_n(0) = \alpha_n \leqslant \alpha_{n+1} < \beta_{n+1},$$

therefore $\beta_{n+1} = T_{n+1}(0)$ and $\alpha_{n+1} = T_{n+1}(\infty)$. Similarly, if $\beta_n = T_n(0)$ and $\alpha_n = T_n(\infty)$ then $\alpha_{n+1} = T_{n+1}(0)$ and $\beta_{n+1} = T_{n+1}(\infty)$. Since $\beta_1 = T_1(0)$ and $\alpha_1 = T_1(\infty)$, the result follows by induction. ▫

If $\alpha = \beta$ then the continued fraction $\mathbf{K}(1 \mid b_n)$ converges classically, otherwise it diverges classically.

Let γ denote the hyperbolic line in H from 0 to ∞. Then $T_n(\gamma)$ is the hyperbolic line joining $T_n(0)$ and $T_n(\infty)$. Lemma 2.1 shows that

$$\sigma\big(T_n(0), T_n(\infty)\big) = 2/\cosh \varrho_H\big(j, T_n(\gamma)\big).$$

Either $\alpha = \beta$, in which case T_n converges classically to α, or $\alpha \neq \beta$. In the former case, $T_n(0) \to \alpha$ and $T_n(\infty) = T_{n-1}(0) \to \alpha$ therefore also $T_n(j) \to \alpha$, as $T_n(j)$ lies on the hyperbolic line $T_n(\gamma)$ joining $T_n(0)$ and $T_n(\infty)$, which is a semi-circle of diminishing radius. In the latter case, $\sigma\big(T_n(0), T_n(\infty)\big)$ decreases towards $\sigma(\alpha, \beta) > 0$. We subsequently show that in this case $\varrho_H\big(j, T_n(j)\big)$ is bounded so that T_n does not converge generally. The next theorem encapsulates certain aspects of this information.

Theorem 5.7. *The sequence T_n converges generally if and only if it converges classically.*

Proof. We have seen that classical convergence entails general convergence (this is true of any Möbius sequence associated with a continued fraction). Suppose that T_n diverges classically. Theorem 5.6 ensures that the sequence with nth term $\sigma\big(T_n(0), T_n(\infty)\big)$ decreases to a positive constant. Hence $\cosh \varrho_H\big(T_n^{-1}(j), \gamma\big) = 2/\sigma\big(T_n(0), T_n(\infty)\big)$ increases to a positive constant $k > 1$. Let $z_n = x_n + \tau_n j = T_n^{-1}(j)$. The closest point on γ to z_n is $|z_n| j$, therefore

$$\cosh \varrho_H(z_n, \gamma) = 1 + \frac{|z_n - |z_n| j|^2}{2\tau_n |z_n|} = \frac{|z_n|}{\tau_n},$$

by equation (2.1). Therefore

$$\frac{x_n^2}{\tau_n^2} = \frac{x_n^2 + \tau_n^2}{\tau_n^2} - 1 \leqslant k - 1.$$

This shows that $|x_n/\tau_n| \leqslant K$ for every n, where $K = \sqrt{k-1}$. Recall from Theorem 5.4 that all $z_n \in L$, therefore we have shown that all z_n lie in the sector $S = \{x + tj \in H : x \leqslant 0,\ |x| \leqslant Kt\}$. We now show that no τ_n has value greater than 1. This is certainly true of τ_1 and τ_2. Suppose that τ_n is the smallest counterexample to this posit, for some $n > 2$. A short computation shows that

$$z_{n-2} = \frac{\kappa_n + \tau_n j}{\kappa_n^2 + \tau_n^2} = \frac{\kappa_n + \tau_n j}{1 + 2b_n\kappa_n - b_n^2\big((x_n + b_n)^2 + \tau_n^2\big)},$$

where $\kappa_n = (x_n + b_n) + \big((x_n + b_n)^2 + \tau_n^2\big)b_n$. Since $x_{n-2} \leqslant 0$, also $\kappa_n \leqslant 0$, hence

$$\tau_{n-2} = \frac{\tau_n}{1 + 2b_n\kappa_n - b_n^2\big((x_n + b_n)^2 + \tau_n^2\big)} \geqslant \tau_n > 1,$$

which is a contradiction. Hence all $\tau_n \le 1$. On the other hand, if $\tau_n < 1/(1+K^2)$ then

$$\tau_{n+1} = \frac{\tau_n}{x_n^2+\tau_n^2} \geqslant \frac{\tau_n}{K^2\tau_n^2+\tau_n^2} = \frac{1}{(K^2+1)\tau_n} > 1,$$

which is again a contradiction. Hence z_n is restricted to that portion of S with j component between $1/(K^2+1)$ and 1 (a compact set). Thus $\varrho_H\bigl(j, T_n^{-1}(j)\bigr)$ and hence $\varrho_H\bigl(T_n(j), j\bigr)$ are bounded sequences, therefore $T_n(j)$ does not converge generally. □

We remark that the above proof shows that when T_n diverges classically, the sequence $\varrho_H\bigl(T_n^{-1}(j), j\bigr)$ is bounded. Theorem 5.5 and Corollary 5.2 then demonstrate that $\sum |b_n|$ converges, hence the work of Section 3 applies to show that $T_{2n}^{-1} \to g$ and $T_{2n+1}^{-1} \to \iota g$, for some Möbius map g. Thus, not only is $T_n^{-1}(j)$ restricted to a compact subset of H, in fact $T_{2n}^{-1}(j) \to \xi$ and $T_{2n+1}^{-1}(j) \to \iota(\xi)$ for some $\xi \in H$.

It remains to supply a proof of Theorem 1.2.

Proof of Theorem 1.2. The comments preceding this proof along with the work of Section 3 show that when T_n does not converge generally, we have $T_{2n} \to f$ and $T_{2n+1} \to f\iota$, for some Möbius map f. Since j is a fixed point of ι, the sequence $T_n(j)$ converges. Hence we have proved the statement that $T_n(j) \to \zeta$, for some $\zeta \in \overline{H}$. Theorem 5.5 demonstrates the equivalence of (i) and (ii) and Theorem 5.7 demonstrates the equivalence of (ii) and (iii). □

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