Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 31, 2006, 131–142

THE EXISTENCE OF QUASIMEROMORPHIC MAPPINGS

Emil Saucan

Technion, Department of Mathematics and Department of Electrical Engineering
Haifa, Israel; semil@tx.technion.ac.il

Abstract. Let G be a Kleinian group G acting on $\mathbf{B}^n$, $n \geq 2$. We show that if the orders of the elliptic elements in G which have non-degenerate fixed set are bounded, then G carries non-constant G-automorphic quasimeromorphic mappings. This together with an earlier non-existence theorem by Srebro gives a complete characterization of Kleinian groups that admit non-constant quasimeromorphic automorphic mappings.

1. Introduction

Definition 1.1. Let $D \subseteq \mathbf{R}^n$ be a domain; $n \geq 2$ and let $f\colon D \to \mathbf{R}^n$ be a continuous mapping. f is called

(1) *quasiregular* if and only if (i) f belongs to $W^{1,n}_{\mathrm{loc}}(D)$ and (ii) there exists $K \geq 1$ such that:

$$|f'(x)|^n \leq K J_f(x) \quad \text{a.e.} \tag{1.1}$$

where $f'(x)$ denotes the formal derivative of f at x, $|f'(x)| = \sup_{|h|=1} |f'(x)h|$, and where $J_f(x) = \det f'(x)$.

(2) *quasiconformal* if and only if $f\colon D \to f(D)$ is a quasiregular homeomorphism.

(3) *quasimeromorphic* if and only if $f\colon D \to \widehat{\mathbf{R}^n}$, $\widehat{\mathbf{R}^n} = \mathbf{R}^n \bigcup \{\infty\}$ is quasiregular, where the condition of quasiregularity at $f^{-1}(\infty)$ can be checked by conjugation with auxiliary Möbius transformations.

The smallest number K that satisfies (1.1) is called the *outer dilatation* of f.

One can extend the definitions above to oriented, connected $\mathscr{C}^\infty$ Riemannian manifolds as follows:

Definition 1.2. Let M^n, N^n be oriented, connected $\mathscr{C}^\infty$ Riemannian n-manifolds, $n \geq 2$, and let $f\colon M^n \to N^n$ be a continuous function. f is called *locally quasiregular* if and only if for every $x \in M^n$, there exist coordinate charts (U_x, φ_x) and $(V_{f(x)}, \psi_{f(x)})$, such that $f(U_x) \subseteq V_{f(x)}$ and $g = \psi_{f(x)} \circ f \circ \varphi_x^{-1}$ is quasiregular.

If f is locally quasiregular, then $T_x f\colon T_x(M^n) \to T_{f(x)}N^n$ exist for a.e. $x \in M^n$.

2000 Mathematics Subject Classification: Primary 30C65, 57R05, 57M60.

This paper represents part of the authors Ph.D. Thesis written under the supervision of Professor Uri Srebro.

Definition 1.3. Let M^n, N^n be oriented, connected $\mathscr{C}^\infty$ Riemannian n-manifolds, $n \geq 2$, and let $f\colon M^n \to N^n$ be a continuous function. f is called *quasiregular* if and only if

(i) f is locally quasiregular

and

(ii) there exists K, $1 \leq K < \infty$, such that

$$|T_x f|^n \leq K J_f(x) \tag{1.2}$$

for a.e. $x \in M^n$.

Recall that a group G of homeomorphisms acts *properly discontinuously* on a locally compact topological space X if and only if the following conditions hold for any $g \in G_x$, $x \in X$: (a) the stabilizer $G_x = \{g \in G \mid g(x) = x\}$ of x is finite; and (b) there exists a neighbourhood V_x of x, such that (b$_1$) $g(V_x) \cap V_x = \emptyset$, for any $g \in G \setminus G_x$; and (b$_2$) $g(V_x) \cap V_x = V_x$.

Definition 1.4. A discontinuous group of orientation-preserving isometries of $\mathbf{B}^n$ is called a *Kleinian* group.

It is well known that a discontinuous group is discrete (see [Ms]).

Definition 1.5. Let $f\colon \mathbf{B}^n \to \widehat{\mathbf{R}^n}$, and let G be a Kleinian group acting upon $\mathbf{B}^n$. The function f is called G-automorphic if and only if:

$$f\big(g(x)\big) = f(x); \quad \text{for any } x \in \mathbf{B}^n \quad \text{and for all } g \in G; \tag{1.3}$$

Recall the definition of elliptic transformations:

Definition 1.6. A Möbius transformation $f\colon \mathbf{B}^n \to \mathbf{B}^n$, $f \neq \mathrm{Id}$ is called *elliptic* if and only if f has a fixed point in $\mathbf{B}^n$.

The existence of non-constant automorphic meromorphic functions in dimension $n = 2$ represents a classical result which follows from the existence of meromorphic functions on Riemann surfaces (see [Fo], [K]).

The question whether quasimeromorphic mappings (or qm-mappings, in short) exist in any dimension $n \geq 3$ was originally posed by Martio and Srebro in [MS1]; subsequently in [MS2] they proved the existence of the fore-mentioned mappings in the case of co-finite groups, i.e., groups such that $\mathrm{Vol}_{\mathrm{hyp}}(\mathbf{B}^n/G) < \infty$ (the important case of geometrically finite groups being thus included). Also, it was later proved by Tukia ([Tu]) that the existence of non-constant qm-mappings is assured in the case when G acts torsionless upon $\mathbf{B}^n$. Moreover, since for torsionless Kleinian groups G, $\mathbf{B}^n/G$ is an (analytic) manifold, the next natural question to ask is whether there exist non-constant qm-mappings $f\colon M^n \to \widehat{\mathbf{R}^n}$; where M^n is an orientable n-manifold. A partial affirmative answer to this question is due to Peltonen (see [Pe]); to be more precise she proved the existence of qm-mappings in the case when M^n is a complete, connected, orientable $\mathscr{C}^\infty$-Riemannian manifold.

Our main result is the following theorem:

Theorem 1.7. *Let G be a Kleinian group G acting on $\mathbf{B}^n$, $n \geq 2$. If the orders of the elliptic elements of G which have non-degenerate fixed set are bounded, then G admits non-constant G-automorphic quasimeromorphic mappings.*

In contrast with the above results it was proved by Srebro ([Sr]) that, if G is a Kleinian group acting on $\mathbf{B}^n$, $n \geq 3$, containing elliptic elements with non-degenerate fixed set, of arbitrarily large orders, then G does not admit non-constant G-automorphic qm-mappings; and showed that such groups exist in all dimensions $n \geq 3$.

This non existence result, together with Theorem 1.7 gives a complete characterization of those Kleinian groups which admit G-automorphic quasimeromorphic mappings. Namely:

Theorem 1.8. *Let G be a Kleinian group acting on $\mathbf{B}^n$. Then G admits non-constant automorphic qm-mappings if and only if*:

(1) $n = 2$;

or

(2) $n \geq 3$, *and the orders of the elliptic elements of G having non-degenerate fixed sets are uniformly bounded.*

Remark 1.9. Given any finitely generated Kleinian group acting on $\mathbf{B}^3$ the number of conjugacy classes of elliptic elements is finite (see [FM]). However, this is not true for Kleinian groups acting upon $\mathbf{B}^n$, $n \geq 4$; (for counterexamples, see [FM], [Po] and [H]).

Remark 1.10. Hamilton ([H, Theorem 4.1]) constructed examples of Kleinian groups G acting on $\mathbf{B}^4$ such that there exists an infinite sequence $\{f_n\}_{n\in\mathbf{N}} \subset G$ of elliptic transformations, with $\operatorname{ord}(f_n) \to \infty$ and such that the fixed set of each f_n is degenerate. (For the relevant definitions, see Section 2 below.) (Here $\operatorname{ord}(f_n)$ denotes the order of f_n.)

Note that by Remark 1.9 we have the following corollary:

Corollary 1.11. *Let G be a finitely generated Kleinian group acting upon $\mathbf{B}^3$. Then there exists a non constant G-automorphic qm-mapping $f\colon \mathbf{B}^3 \to \widehat{\mathbf{R}^3}$.*

The classical methods employed in proving the existence in the case $n = 2$ do not apply in higher dimensions—indeed, for $n \geq 4$, $\mathbf{B}^n/G$ is not even a manifold, but an orbifold. Therefore, different methods are needed. Following other researchers, we shall employ the classical "Alexander trick" (see [Al]).

A uniform bound for the dilatations can be attained (see [MS2], [Tu]) if the considered triangulation is *fat*, i.e. such that each of its individual simplices may be mapped onto a standard n-simplex, by a L-bilipschitz map, followed by a homothety, for a fixed L. (For a precise definition of fatness see Section 3 below.)

The idea of the proof of Theorem 1.7 is, in a nutshell, as follows: Based upon the geometry of the elliptic transformations construct a fat triangulation

$\mathscr{T}_1$ of N_e^*, where N_e^* is a certain closed neighbourhood of the singular set of $\mathbf{B}^n/G$. Since $M_p = \big(\mathbf{B}^n \setminus \mathrm{Fix}(G)\big)/G$, $\mathrm{Fix}(G) = \big\{x \in \mathbf{B}^n \mid$ there exists $g \in G \setminus \{\mathrm{Id}\},\ g(x) = x\big\}$ is an orientable analytic manifold, we can apply Peltonen's result to gain a triangulation $\mathscr{T}_2$ of M_p. Therefore, if the triangulations $\mathscr{T}_1$ and $\mathscr{T}_2$ are chosen properly, each of them will induce a triangulation of $N_e^* \setminus N_e^{*\prime}$, for a certain $N_e^{*\prime} \subsetneq N_e^*$ (see Section 2).

'Mash' $\mathscr{T}_1$ and $\mathscr{T}_2$ (in $N_e^* \setminus N_e^{*\prime}$) i.e. ensure that the given triangulations intersect into a new triangulation $\mathscr{T}_0$ (see [Mun, Theorem 10.4]). Modify $\mathscr{T}_0$ to receive a new fat triangulation $\mathscr{T}$ of $\mathbf{B}^n/G$.

In the presence of degenerate components $A_k = A(f_k)$ of the fixed set of G, where the transformations f_k may have arbitrarily large orders, a modification of this construction is needed; see Section 4.

Apply Alexander's trick to receive a quasimeromorphic mapping $f\colon \mathbf{B}^n/G \to \widehat{\mathbf{R}^n}$. The lift $\tilde{f}$ of f to $\mathbf{B}^n$ represents the required G-automorphic quasimeromorphic mapping.

In [S3] we showed how to build $\mathscr{T}_1$ using a generalization of a theorem of Munkres ([Mun, 10.6]) on extending the triangulation of the boundary of a manifold (with boundary) to the whole manifold. Munkres' technique also provided us with the basic method of mashing the triangulations $\mathscr{T}_1$ and $\mathscr{T}_2$. In this paper we present a more direct, geometric method of triangulating N_e^* and mashing the two triangulations. We already employed this simpler method in [S1], where we proved Theorem 1.7 in the case $n = 3$. The original technique used in [S1] for fattening the intersection of $\mathscr{T}_1$ and $\mathscr{T}_2$ is, however, restricted to dimension 3. Therefore here we make appeal to the method employed in [S3], which is essentially the one developed in [CMS].

This paper is organized as follows: in Section 2 we show how to triangulate the closed neighbourhood N_e^* of the singular set of $\mathbf{B}^n/G$. Section 3 is dedicated to the main task of mashing the triangulations and fattening the resulting common triangulation. In Section 4 we show how to apply the main result in the construction of a G-automorphic quasimeromorphic mapping from $\mathbf{B}^n$ to $\widehat{\mathbf{R}^n}$.

2. Geometric neigbourhoods

If G is a discrete Möbius group and if $f \in G$, $f \neq \mathrm{Id}$ is an elliptic transformation, then there exists $m \geq 2$ such that $f^m = \mathrm{Id}$. The smallest m satisfying this condition is called the *order* of f, and it is denoted by $\mathrm{ord}(f)$. In the 3-dimensional case the *fixed point set* of f, i.e. $\mathrm{Fix}(f) = \{x \in \mathbf{B}^3 \mid f(x) = x\}$, is a hyperbolic line and will be denoted by $A(f)$—the *axis of* f. In dimension $n \geq 4$ the fixed set (or *axis of* f) of an elliptic transformation is a k-dimensional hyperbolic plane, $0 \leq k \leq n-2$. An axis A is called *degenerate* if and only if $\dim A = 0$. In dimensions higher than $n = 3$, different elliptics may have fixed sets of different dimensions.

If G is a discrete group, G is countable and so is the set of elliptics and the set of connected components of $\mathrm{Fix}(G)$, which we denote by $\{f_i\}_{i\geq 1}$ and $\{C_j\}$, respectively.

Moreover, by the discreteness of G, the sets $\mathscr{A} = \{\mathscr{A}_{i\geq 0}\}$—and hence $\mathscr{S} = \{C_j\}$—have no accumulation points in $\mathbf{B}^n$.

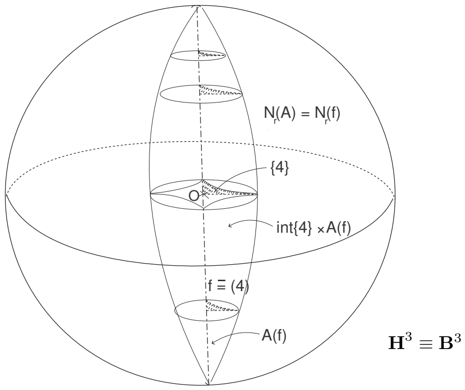

Figure 1. Geometric neighbourhood for $n = 3$ and $m = 4$. Here $\{4\}$ denotes the regular (hyperbolic) polygon with 4 sides.

Hence we can choose disjoint, G-invariant neighbourhoods N_j and N'_j of C_j, $N'_j \subsetneq N_j$. Indeed, first choose a neighbourhood N_1 of C_1, such that $\overline{N}_1 \cap \bigcup_{j\geq 2} C_j = \emptyset$; then recursively build a neighbourhood N_k of C_k, such that $N_k \subset \mathbf{B}^n \setminus (N_1 \cup \cdots \cup N_{k-1})$ and $\overline{N}_k \cap \bigcup_{j>k} C_j = \emptyset$, for all $k \geq 2$. Denote $N_e = \bigcup_{j\in\mathbf{N}} N_j$, $N'_e = \bigcup_{j\in\mathbf{N}} N'_j$. Define $N^*_e = (\overline{N}_e \cap \mathbf{B}^n)/G$, $N^{*\prime}_e = (\overline{N}'_e \cap \mathbf{B}^n)/G$.

To produce the desired closed neighbourhood N^*_e of the singular set of $\mathbf{B}^n/G$ and its triangulation $\mathscr{T}_1$, we first consider the case where $C_i = A(f)$, for some $f \in G$, and then construct a standard neighbourhood $N_f = N\bigl(A(f)\bigr)$ of the axis of each elliptic element of G such that $N_f \simeq A(f) \times I^{n-k}$, where $A(f) = \mathbf{S}^k$ and where I^{n-k} denotes the unit $(n-k)$-dimensional interval. The construction of N_f proceeds as follows: By [Cox, Theorem 11.23] the fundamental region for the local action of the stabilizer group of the axis of f, $G_f = G_{A(f)} = \{g \in G \mid g(x) = x\}$ is a simplex or a product a simplices. Let $\mathscr{S}_f$ be the fundamental region (see Figure 2).

Then we can define the generalized prism (or *simplotope*—see [Som, VII.25]) $\mathscr{S}_f^{\perp}$, defined by translating $\mathscr{S}_f$ in a direction perpendicular to $\mathscr{S}_f$, where the translation length is $\mathrm{dist}_{\mathrm{hyp}}\bigl(\mathscr{S}_f, A(f)\bigr)$. It naturally decomposes into simplices (see [Som, VII.25], [Mun, Lemma 9.4]). We have thus constructed an f-invariant triangulation of a prismatic neigbourhood N_f of $A(f)$. We can reduce the mesh of this triangulation as much as required, while controlling its fatness by dividing $\mathscr{S}_f$ into similar simplices and partitioning N_f into a finite number of radial strata of equal width ϱ. In the special case when the minimal distance between axes

$\delta = \min\{\mathrm{dist}_{\mathrm{hyp}}(A(f), A(g)) \mid g \text{ elliptic}, \ g \neq f\}$ is attained we can chose $\varrho = \delta/\kappa_0$, for some integer κ_0, and further partition it into 'slabs' of equal hight h. (In particular one can use this approach in the case when G acts on $\mathbf{B}^3$ and it contains no order two elliptics, since in this particular case, according to a result of Gehring and Martin [GM1], the minimum exists and is strictly positive.) Henceforth we shall call the neighbourhood thus produced, together with its fat triangulation, a *geometric neighbourhood*.

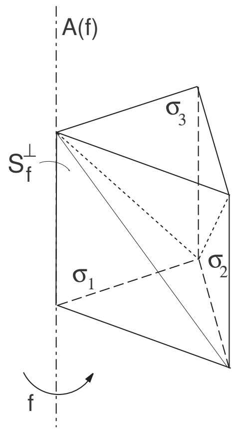

Figure 2. Canonical decomposition into simplices of $\mathscr{S}_f^{\perp}$, for $n = 3$.

Since the stabilizer $\mathrm{Stab}(A_{1,\dots,k})$ of the intersection of axes $A_{1,\dots,k} = A_{i_1} \cap \cdots \cap A_{i_k}$ is a finite subgroup of $O^+(n)$, and since in any dimension there exist only a finite number of such groups of orders $\leq M_0$, for any $M_0 \in \mathbf{N}$ (see [Cox, Chapter 11]), the angles between the axes of transformations of orders $\leq m_0$ admit a bound $\alpha = \alpha(m_0, n)$. Therefore, the intersection $N(A_{1,\dots,k}) = N_{f_1} \cap \cdots \cap N_{f_k}$ of the geometric neighbourhoods of several axes is also endowed with a natural fat triangulation, invariant under the group $G = \langle G_{f_1}, \dots, G_{f_k} \rangle$. (In the particular case $n = 3$ one can choose as a geometric neighbourhood of A a regular or a semi-regular polyhedron together with its interior (see Figure 3 below).

If $q \in \mathbf{B}^n, \dim q = 0$, is a degenerate element of the singular locus, we replace the tubular neighbourhood considered above by $\mathscr{P}_q \cup \mathrm{int}\mathscr{P}_q$, where $\mathscr{P}_q$ is a regular polytope invariant under the stabilizer G_q of q in G, together with its canonical simplicial subdivision (see [Cox, 7.6]). Indeed, every finite group generated by reflections is the symmetry group of a regular polyhedron $\mathscr{P}$ (see [Cox, p. 209]). Moreover, the rotation group of $\mathscr{P}$ has order $nl/2$, where l is the number of faces of $\mathscr{P}$ (see [Cox, pp. 227–231]).

Remark 2.1. As noted above, if G is a Kleinian group acting with torsion on $\mathbf{B}^n$, then $M_p = (\mathbf{B}^n \setminus \mathrm{Fix}(G))/G$ is a complete orientable manifold. Moreover, since the isotropy groups of any point in $Q_G = \mathbf{B}^n/G$ are subgroups of $O^+(n)$, it

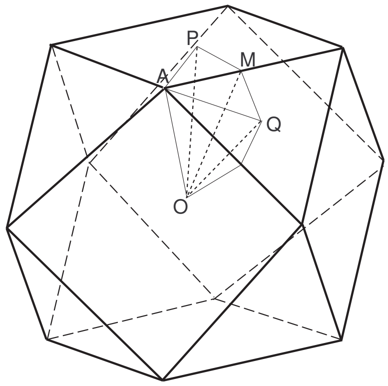

Figure 3. A Euclidean semi-regular polyhedron and two of its fundamental tetrahedra ($n = 3$).

follows that $\mathbf{B}^n/G$ is complete orientable orbifold (see [Dr, p. 46]). The singular locus $\Sigma_{Q_G} = \mathrm{Fix}(G)/G$ of Q_G contains all the non-manifold points of Q_G, yet the two sets are not equal. Indeed, in dimension $n = 2$ ($n = 3$) any orbifold (orientable orbifold) is homeomorphic to a manifold. The local structure of Σ_{Q_G} at a point $x_Q \in Q_G$ is determined by the stabilizer in G of its preimage in $\mathbf{B}^n$, i.e. by the finite subgroups of $O^+(n)$. (For instance, in dimension $n = 3$ only two infinite families and three more special cases of branching points (of $\mathrm{Fix}(G)$ and thus of Σ_{Q_G}) can occur—see [Th1, 5.6]). However, the global structure of Σ_{Q_G} can be very complicated (see [Th1, 5.6]).

3. Mashing and fattening triangulations

We present the main steps of the Munkres ([Mun, Chapter 10]) and Cheeger ([CMS, pp. 432–440]) techniques, and we indicate how to adapt them to our particular setting. First let us establish some definitions and notation:

Definition 3.1. Let M^n be a PL-manifold. Two triangulations $\mathcal{T}_1, \mathcal{T}_2$ of M^n intersect transversally if and only if for any $p \in M^n$, there exist neighbourhoods U_1, U_2, U_3 of p in $|\mathcal{T}_1|$, $|\mathcal{T}_2|$ and M^n, respectively, such that the triple (U_1, U_2, U_3) is PL-homeomorphic to a neighbourhood of 0 in $(\mathbf{R}^n \times 0, 0 \times \mathbf{R}^n, \mathbf{R}^n \times \mathbf{R}^n)$.

To ensure the fatness of the common triangulation we need to make appeal to a stronger notion of transversality, namely:

Definition 3.2. Let $\sigma_i \in K$, $\dim \sigma_i = k_i$, $i = 1, 2$; such that $\mathrm{diam}\, \sigma_1 \le \mathrm{diam}\, \sigma_2$. Denote by $[\sigma_i]$ the affine subspace of $\mathbf{R}^N$ generated by σ_i, and let $\langle \sigma_i \rangle$ denote the subspace parallel to $[\sigma_i]$, such that $0 \in \langle \sigma_i \rangle \subset \mathbf{R}^N$; $i = 1, 2$. We say that σ_1, σ_2 are δ*-transverse* if and only if

(i) $\dim([\sigma_1] \cap [\sigma_2]) = \max(0, k_1 + k_2 - n)$;

(ii) $0 < \delta < \measuredangle([\sigma_1], [\sigma_2])$, where $\measuredangle([\sigma_1], [\sigma_2]) = \measuredangle(\langle \sigma_1 \rangle, \langle \sigma_2 \rangle)$, and where $\measuredangle(\langle \sigma_1 \rangle, \langle \sigma_2 \rangle) = \min_{(e_1, e_2)} \arccos(e_1, e_2)$, $e_i \in (\langle \sigma_1 \rangle \cap \langle \sigma_1 \rangle)^{\perp} \cap \langle \sigma_i \rangle$, $\|e_i\| = 1$,

$i=1,2$; where (e_1,e_2) denotes the standard inner product on $\mathbf{R}^n$; and if $\sigma_3 \subsetneq \sigma_1$, $\sigma_4 \subsetneq \sigma_2$, such that $\dim\sigma_3+\dim\sigma_4 < n = \dim K$, then

(iii) $\mathrm{dist}(\sigma_3,\sigma_4) > \delta\cdot d_1$, where $d_1 = \mathrm{diam}\,\sigma_1$.

In this case we write: $\sigma_1 \pitchfork_\delta \sigma_2$.

Definition 3.3. Let $\tau \subset \mathbf{R}^n$; $0\le k\le n$ be a k-dimensional simplex. The *fatness* φ of τ is defined as being:

$$\varphi = \varphi(\tau) = \inf_{\substack{\sigma<\tau\\ \dim\sigma=l}} \frac{\mathrm{Vol}(\sigma)}{\mathrm{diam}^{\,l}\sigma}. \tag{3.1}$$

The infimum is taken over all the faces of τ, $\sigma<\tau$, and $\mathrm{Vol}_{\mathrm{eucl}}(\sigma)$ and $\mathrm{diam}\,\sigma$ stand for the Euclidean l-volume and the diameter of σ, respectively. (If $\dim\sigma=0$, then $\mathrm{Vol}_{\mathrm{eucl}}(\sigma)=1$, by convention.)

A simplex τ is *φ_0-fat*, for some $\varphi_0>0$, if $\varphi(\tau)\ge\varphi_0$. A triangulation (of a submanifold of $\mathbf{R}^n$) $\mathscr{T}=\{\sigma_i\}_{i\in\mathbf{I}}$ is *φ_0-fat* if all its simplices are φ_0-fat. A triangulation $\mathscr{T}=\{\sigma_i\}_{i\in\mathbf{I}}$ is *fat* if there exists $\varphi_0>0$ such that all its simplices are *φ_0-fat*.

Remark 3.4. There exists a constant $c(k)$ that depends solely upon the dimension k of τ such that

$$\frac{1}{c(k)}\cdot\varphi(\tau) \le \min_{\substack{\sigma<\tau\\ \dim\sigma=l}} \measuredangle(\tau,\sigma) \le c(k)\cdot\varphi(\tau), \tag{3.2}$$

and

$$\varphi(\tau) \le \frac{\mathrm{Vol}(\sigma)}{\mathrm{diam}^{\,l}\sigma} \le c(k)\cdot\varphi(\tau); \tag{3.3}$$

where $\measuredangle(\tau,\sigma)$ denotes the (*internal*) *dihedral angle* of $\sigma<\tau$. (For a formal definition, see [CMS, pp. 411–412], [Som].)

Remark 3.5. The definition above is the one introduced in [CMS]. For equivalent definitions of fatness, see [Ca1], [Ca2], [Mun], [Pe], [Tu].

The first step is that of mashing the triangulations $\mathscr{T}_1$, $\mathscr{T}_2$:

We approximate the triangulation $\mathscr{T}_2$ of M_p by a locally finite Euclidean triangulation, by means of the secant map (see [Mun, p. 90]). Also, the hyperbolic simplices of $\mathscr{T}_1$ can be approximated arbitrarily well by Euclidean simplices, by considering $\mathrm{diam}\,\sigma, \sigma\in\mathscr{T}_1$ small enough (see [Tu]). Therefore the mashing and fattening of triangulations reduces to that of Euclidean ones.

Next we ensure that the given triangulations intersect into a new triangulation $\mathscr{T}_0$. This is first done locally by modifying these local triangulations coordinate chart by chart, so they will be PL-compatible wherever they overlap. More precisely, we first apply infinitesimal moves of the vertices so that the two triangulations will intersect transversally. Next we perform suitable barycentric subdivisions of the closed, convex polyhedral cells $\bar{\gamma} = \bar{\sigma}_1 \cap \bar{\sigma}_2$, $\sigma_i \in \mathscr{T}_i$, $i = 1, 2$; in the following manner: suppose each cell $\beta \subset \partial\gamma$ already has a subdivision into simplices β_i, $i = 1, \dots, p$; choose an interior point $p_\gamma \in \operatorname{int}\gamma$, construct the joins $J(p_\gamma, \beta_i)$, $i = 1, \dots, p$; and consider all their simplices (see [Mun, 10.2–10.3]).

To extend the local triangulations to a global triangulation $\mathscr{T}_0$, we work in $\mathbf{R}^n$, by using the coordinate charts and maps. Here again we have to approximate the given triangulation by a PL-map, such that the given triangulation and the one we produce will be PL-compatible (see [Mun, Theorem 10.4]). The existence of the common triangulation $\mathscr{T}_0$ follows immediately (see [Mun, Theorem 10.5]).

We next present the main steps of the fattening process (for details see [CMS]):

One begins by triangulating and fattening the intersection of two individual simplices belonging to the two given triangulations, respectively. First one shows that if two individual simplices are fat and if they intersect δ-transversally, then one can choose the points p_γ such that the barycentric subdivision $\bar{\gamma}^*$ will be composed of fat simplices. (See [CMS, Lemma 7.1].)

Next one shows that given two fat Euclidean triangulations that intersect δ-transversally, it is possible to infinitesimally move any given point of one of the triangulations such that the resulting intersection will be δ^*-transversal, where δ^* depends only on δ, the common fatness of the given triangulations, and on the displacement length (see [CMS, Lemma 7.3]).

By repeatedly applying this results to the simplices of dimensions $0, \dots, n$, of the intersection of two fat triangulations, one can now prove the main fattening result:

Proposition 3.6 ([CMS, Lemma 6.3]). *Let $\mathscr{T}_1$, $\mathscr{T}_2$ be two fat triangulations of open sets $U_1, U_2 \subset \mathbf{R}^n$, $B_r(0) \subseteq U_1 \cap U_2$, having common fatness $\geq \varphi_0$ and such that $d_1 = \inf_{\sigma_1 \in \mathscr{T}_1} \operatorname{diam} \sigma_1 \leq d_2 = \inf_{\sigma_2 \in \mathscr{T}_2} \operatorname{diam} \sigma_2$. Then there exist φ_0^*-fat triangulations $\mathscr{T}_1'$, $\mathscr{T}_2'$, $\varphi_0^* = \varphi_0^*(\varphi_0)$, of open sets $V_1, V_2 \subseteq B_r(0)$, such that*

(1) $\mathscr{T}_i'|_{B_{r-8d_2}(0)} = \mathscr{T}_i|_{B_{r-8d_2}(0)}$, $i = 1, 2$;
(2) *$\mathscr{T}_1'$ and $\mathscr{T}_2'$ agree near their common boundary.*
Moreover:
(3) $\inf_{\sigma_1' \in \mathscr{T}_1'} \operatorname{diam} \sigma_1' \leq 3d_1/2$, $\inf_{\sigma_2' \in \mathscr{T}_2'} \operatorname{diam} \sigma_2' \leq d_2$.

We apply Proposition 3.6 above to our particular context in the following manner: Let $\mathscr{T}_1$, $\mathscr{T}_2$ be the triangulations of $N_e^* \setminus N_e^{*\prime}$ constructed above. To gain a globally fat triangulation from the mashing of $\mathscr{T}_1$ and $\mathscr{T}_2$, we start by partitioning $N_e^* \setminus N_e^{*\prime}$ into (almost) cubes Q. If the diameters of the sets Q are small enough we can apply Proposition 3.6, for Q instead $B_r(0)$. Extend $\mathscr{T}_0$ by $\mathscr{T}_2$ on the face included in ∂N_e and by $\mathscr{T}_1$ on the other faces, to receive the desired triangulation $\mathscr{T}$. (Further fattening of the triangulations induced on the lower dimensional faces may be necessary. However, by the locally finiteness of the triangulation, the number of steps required for fattening the lower dimensional intersections is finite and depends solely upon the dimension n.) This gives the required globally fat triangulation of $\mathbf{B}^n/G$.

4. The existence of quasimeromorphic mappings

We first prove the following lemma:

Lemma 4.1 ([MS1], [Pe]). *Let $M^n \subset \mathbf{R}^N$ be an orientable n-manifold, let $\mathscr{T}$ be a chessboard fat triangulation of M^n, let $\sigma \in \mathscr{T}$, $\sigma = (p_0, \dots, p_n)$ and let $\tau_0 = (p_{0,1}, \dots, p_{0,n})$ denote the equilateral n-simplex inscribed in the unit sphere $\mathbf{S}^{n-1}$. Then there exists a orientation-preserving homeomorphism $h = h_\sigma \colon |\sigma| \to \widehat{\mathbf{R}^n}$ such that*

(1) $h(|\sigma|) = |\tau_0|$, *if σ is positively oriented and $h(|\sigma|) = \widehat{\mathbf{R}^n} \setminus |\tau_0|$, otherwise.*
(2) $h(p_i) = p_{0,i}$, $i = 0, \dots, n$.
(3) $h|_{\partial|\sigma|}$ *is a PL-homeomorphism.*
(4) $h|_{\mathrm{int}|\sigma|}$ *is quasiconformal.*

Proof. If $\det(p_0, \dots, p_n) > 0$, then the PL-mapping h defined by condition (2) above also satisfies conditions (1), (3) and (4). If $det(p_0, \dots, p_n) < 0$, we define h as follows: $h = \varphi^{-1} \circ J \circ \varphi \circ h_0$, where φ is the *radial linear stretching* $\varphi \colon \tau_0 \to \mathbf{R}^n$, J denotes the reflection in the unit sphere $\mathbf{S}^{n-1}$ and $h_0 \colon |\sigma| \to |\tau_0|$ is the orientation-reversing PL-mapping defined by condition (2). Recall that φ is onto and bilipschitz (see [MS2]). Moreover, by a result of Gehring and Väisalä, φ is also quasiconformal (see [V]). We can extend φ to $\widehat{\mathbf{R}^n}$ by defining $\varphi(\infty) = \infty$. It follows that h indeed represents the required PL-homeomorphism. □

The existence theorem of quasimeromorphic mappings now follows immediately:

Proof of Theorem 1.7. Let $\mathscr{T}$ be the φ_0^*-fat chessboard triangulation of $\mathbf{B}^n/G$ constructed in Section 3 above. Let $f \colon \mathbf{B}^n/G \to \widehat{\mathbf{R}^n}$ be defined by: $f|_{|\sigma|} = h_\sigma$, where h is the homeomorphism constructed in the lemma above. Then f is a local homeomorphism on the $(n-1)$-skeleton of $\widetilde{\mathscr{T}}$ too, while its branching set B_f is the $(n-2)$-skeleton of $\widetilde{\mathscr{T}}$. By its construction f is quasiregular and its (outer) dilatation depends only φ_0^* and on the dimension n (see [Tu, Lemma E]). The lift $\tilde{f}$ of f to $\mathbf{B}^n$ represents the required G-automorphic quasimeromorphic mapping.

In the case of degenerate components A_k of the fixed set $\mathscr{S}$, the proof is essentially the same as in the classical case of Riemann surfaces (see, e.g. [Fo, pp. 233–238]). More precisely, we proceed as follows: We excise from $\mathbf{B}^n$ disjoint ball neighbourhoods B_k of A_k. Let $S_k = \partial B_k$. Then each of the quotients S_k/G admits a fat triangulation $\mathscr{T}_k$. The manifold $\big(\mathbf{B}^n \setminus \bigcup_{k \geq 1} B_k\big)/G$ admits a fat triangulation that extends the fat triangulation of S_k (see [CMS, p. 444] and [S3, Theorem 2.9]). We build the simplices P_k with vertex A_k/G and base

T_{kl}, where T_{kl} are simplices belonging to $\mathscr{T}_k$. Then each of the simplices P_k can be quasiconformally mapped onto a half-space, with bounded dilatation which depends only on n and not on the angles at the vertices A_i, even if the orders of the transformations f_k are not bounded from above (see [Car, Theorem 3.6.10 and Theorem 3.6.13]). □

Acknowledgment. The author would like to thank Professor Uri Srebro for posing this problem and for his enlightening remarks and suggestions. Many thanks are also due to Professor Pekka Tukia for his kind support and to Professor Bronislaw Wajnryb for his help and advice.

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Received 7 December 2004