Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 31, 2006, 213–238

ON HECKE L-FUNCTIONS ASSOCIATED WITH CUSP FORMS II: ON THE SIGN CHANGES OF $S_f(T)$

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In honour of Professor T. N. Shorey on his sixtieth birthday

Abstract. We study the number of sign changes of $S_f(t)$ (related to Hecke L-functions attached to holomorphic cusp forms of even positive integral weight with respect to the full modular group) over shorter intervals.

1. Introduction

Let

$$S(t) = \pi^{-1} \arg \zeta\big(\tfrac{1}{2} + it\big),$$

where the argument is obtained by continuous variation along the straight lines joining 2, $2 + it$ and $\frac{1}{2} + it$, starting with the value zero. When t is equal to the imaginary part of any zero of $\zeta(s)$, we put

$$S(t) = \lim_{\varepsilon \to 0} \tfrac{1}{2}\big\{S(t+\varepsilon) + S(t-\varepsilon)\big\}.$$

As for Atle Selberg's comment on a deep result of Littlewood on $S(t)$, A. Ghosh established that (see Theorem 1 of [5] and also the paper of Selberg [16]) $S(t)$ changes its sign at least

$$T(\log T) \exp\big(-A(\delta)(\log\log T)(\log\log\log T)^{-(1/2)+\delta}\big)$$

times in the interval $(T, 2T)$. Here δ is any arbitrarily small positive constant, and $A(\delta) > 0$ depending only on δ. In fact, he proved this result over shorter intervals.

Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be a holomorphic cusp form of even integral weight $k > 0$ with respect to the full modular group $\Gamma = SL(2, \mathbf{Z})$. We define the associated Hecke L-function

$$L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{1.1}$$

2000 Mathematics Subject Classification: Primary 11Mxx; Secondary 11M06, 11M41.

for $\operatorname{Re} s > (k+1)/2$. Throughout this paper, we assume that $f(z)$ is a Hecke eigenform with $a_1 = 1$. It is known (see [7]) that $L_f(s)$ admits analytic continuation to $\mathbf{C}$ as an entire function and it satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)L_f(s) = (-1)^{k/2}(2\pi)^{-(k-s)}\Gamma(k-s)L_f(k-s). \tag{1.2}$$

$L_f(s)$ has an Euler-product representation (for $\operatorname{Re} s > (k+1)/2$)

$$L_f(s) = \prod_p \bigl(1 - a_p p^{-s} + p^{k-1}p^{-2s}\bigr)^{-1}. \tag{1.3}$$

The non-trivial zeros of $L_f(s)$ lie within the critical strip $(k-1)/2 < \operatorname{Re} s < (k+1)/2$. These zeros are located symmetrically to the real axis and they are also symmetrical about the line $\operatorname{Re} s = k/2$. The Riemann hypothesis in this situation asserts that all the non-trivial zeros are on the critical line $\operatorname{Re} s = k/2$. From Deligne's proof of Ramanujan–Petersson's conjecture (see [1] and [2]), we have the bound for the coefficients

$$|a_n| \le d(n)n^{(k-1)/2}. \tag{1.4}$$

Several interesting deep results about the Hecke L-functions have been established lately. As a sample, a certain average growth of these L-functions in the weight aspect on the critical line has been investigated in the papers of Peter Sarnak (see [15]) and of Matti Jutila and Yoichi Motohashi (see [9]).

Let $N_f(T)$ denote the number of zeros $\beta + i\gamma$ of $L_f(s)$ for which $0 < \gamma < T$. If T is equal to the ordinate of any zero, then we define

$$N_f(T) := \lim_{\varepsilon\to 0} \tfrac{1}{2}\{N_f(T+\varepsilon) + N_f(T-\varepsilon)\}. \tag{1.5}$$

Now, one can show that (following Theorem 9.3 of [18])

$$N_f(T) = \frac{T}{\pi}\log\frac{T}{\pi} - \frac{T}{\pi} + 1 + S_f(T) + O\left(\frac{1}{T}\right), \tag{1.6}$$

where

$$S_f(t) = \frac{1}{\pi}\arg\ L_f\left(\frac{k}{2} + it\right). \tag{1.7}$$

The argument is obtained by a continuous variation along the straight lines joining the points $\frac{1}{2}k+1$, $\frac{1}{2}k+1+it$ and $\frac{1}{2}k+it$, starting with the value $\frac{1}{2}(k-1)$. Hence the variation of $S_f(t)$ is closely connected with the distribution of the imaginary parts of the zeros of $L_f(s)$.

We now define, for $\sigma \ge k/2$, $T \ge 1$ and $H \le T$,

$$N_f(\sigma, T, T+H) = \#\bigl\{\beta + i\gamma : L_f(\beta + i\gamma) = 0,\ \beta \ge \sigma,\ T \le \gamma \le T+H\bigr\}. \tag{1.8}$$

In [14], we proved the following two theorems:

Theorem A. *For $t \ge 2$, $2 \le x \le t^2$, we have*

$$\begin{aligned} S_f(t) = &-\frac{1}{\pi}\sum_{n<x^3}\frac{\Lambda_{x,f}(n)\sin(t\log n)}{n^{\sigma_{x,t}}\log n} + O\biggl((\sigma_{x,t}-k/2)\biggl|\sum_{n<x^3}\frac{\Lambda_{x,f}(n)}{n^{\sigma_{x,t}+it}}\biggr|\biggr) \\ &+ O\bigl((\sigma_{x,t}-k/2)\log t\bigr), \end{aligned}$$

where

$$\sigma_{x,t} = k/2 + 2\max(\beta - k/2, 2/\log x),$$

$\varrho = \beta + i\gamma$ running over those zeros for which

$$|t-\gamma| \le x^{3|\beta-k/2|}(\log x)^{-1},$$

and $\Lambda_{x,f}(n)$ is as in (2.6).

As corollaries we obtained (by choosing $x = \sqrt{\log t}$)

$$S_f(t) = O(\log t)$$

unconditionally, and assuming the Riemann hypothesis for $L_f(s)$, we got

$$S_f(t) = O\biggl(\frac{\log t}{\log\log t}\biggr).$$

Theorem A$'$. *Let B be any fixed small positive constant. Let*

$$B' = \frac{19}{20} + \frac{13.505}{5}B$$

and $B' < \alpha \le 1$. Then for $T^\alpha \le H \le T$, we have

$$N_f(\sigma, T, T+H) \ll H\biggl(\frac{H}{T^{B'}}\biggr)^{-(B/(1-B'))(\sigma-k/2)}\log T$$

uniformly for $k/2 \le \sigma \le (k+1)/2$.

As an application to the above Theorems A and A$'$, the object of this paper is now to prove

Main theorem. *Let B' be the constant as in Theorem A$'$. Let $B' < \alpha \le 1$. If $(T+1)^\alpha \le H \le T$ and $\delta > 0$ is an arbitrarily small real number, there is an $A = A(\alpha,\delta) > 0$ and a $T_0 = T_0(\alpha,\delta) > 0$ such that when $T > T_0$, $S_f(t)$ changes its sign at least*

$$H(\log T)\exp\bigl(-A(\log\log T)(\log\log\log T)^{-(1/2)+\delta}\bigr)$$

times in the interval $(T, T+H)$.

Remark 1. This main theorem is an analogous result of the theorem in the case of $S(t)$ related to the ordinary Riemann zeta-function, which was established by A. Ghosh (see Theorem 1 of [5]). In the case of $S(t)$, B' can be replaced by $\frac{1}{2}$ (or even by a better positive constant).

Remark 2. If we assume the Riemann hypothesis for $L_f(s)$, then the main theorem is true with $0 < \alpha \le 1$.

The proof requires asymptotic formulae for integrals of the type

$$\int_T^{T+H} |S_f(t)|^{2l}\, dt$$

and

$$\int_T^{T+H} |S_{1,f}(t+h) - S_{1,f}(t)|^{2l}\, dt,$$

where

$$S_{1,f}(t) := \int_0^t S_f(u)\, du$$

with the error terms uniform in integers $l \ge 1$ and $h > 0$ with a suitable value of h. It should be mentioned that the asymptotic formulae for higher moments of $S(t)$ over shorter intervals have been extensively studied earlier in [3], [4], [5] and [6].

In fact, first we establish the following theorems from which the main theorem follows. The constants B and B' occurring in the sequel are as in Theorem A$'$, which we do not mention hereafter.

Theorem 1. *Let $B' < \alpha \le 1$. If $T^\alpha \le H \le T$, then there is an absolute positive constant $A_1 = A_1(\alpha)$ such that for any integer l satisfying*

$$1 \le l \ll (\log\log T)^{1/3},$$

we have

$$\int_T^{T+H} |S_f(t)|^{2l}\, dt = \frac{(2l)!}{l!}\left(\frac{1}{2\pi}\right)^{2l} H(\log\log T)^l + O\big(A_1^l l^{l-(1/2)} H(\log\log T)^{l-(1/2)}\big),$$

where the implied constants depend at most on α.

Theorem 2. *Let $B' < \alpha \le 1$. If $(T+h)^\alpha \le H \le T$, then there is an absolute positive constant $A_2 = A_2(\alpha)$ such that for any integer l, with*

$$1 \le l \ll (\log\log T)^{1/3},$$

and any h *satisfying*

$$(\log T)^{1/2} < h^{-1} < \frac{1}{10l} \log T,$$

we have

$$\int_T^{T+H} |S_{1,f}(t+h) - S_{1,f}(t)|^{2l}\, dt = \frac{(2l)!}{l!}\left(\frac{h}{2\pi}\right)^{2l} H(\log h^{-1})^l + O\big(A_2^l l^{l-(1/2)} H h^{2l} (\log\log T)^{l-(1/2)}\big).$$

Remark 3. Theorems 1 and 2 are analogous results of Theorems 2 and 3 of [5]. However, here the range of l as well as the error terms have been improved. In fact, Theorems 2 and 3 of [5] hold with this range of l as well as with this error term, which can be easily noticed from our arguments.

As a consequence of Theorems 1 and 2, we obtain

Theorem 3. *Let* $B' < \alpha \le 1$. *If* $T^\alpha \le H \le T$, *then for any given* $\delta > 0$, *we have*

$$\int_T^{T+H} |S_f(t)|\, dt = \frac{2}{\sqrt{\pi}} \frac{H}{2\pi} (\log\log T)^{(1/2)} + O_\delta\Big(H\big((\log\log T)(\log\log\log T)^{-(1/2)+\delta}\big)^{(1/2)}\Big),$$

where the implied constants depend on α *and* δ.

Theorem 4. *Let* $B' < \alpha \le 1$. *If* $(T+h)^\alpha \le H \le T$, *then for any given* $\delta > 0$ *and any* h *satisfying*

$$(\log T)^{1/2} < h^{-1} < \varepsilon_1 \frac{\log T}{\log\log T},$$

for some suitable constant $\varepsilon_1 = \varepsilon_1(\alpha) > 0$, *we have*

$$\int_T^{T+H} |S_{1,f}(t+h) - S_{1,f}(t)|\, dt = \frac{2}{\sqrt{\pi}} \frac{Hh}{2\pi} (\log h^{-1})^{1/2} + O\Big(Hh\big((\log\log T)(\log\log\log T)^{-(1/2)+\delta}\big)^{1/2}\Big),$$

where the implied constants depend on α *and* δ.

Remark 4. We prove Theorems 1 and 2 in detail adapting the approach of [5] to our situation. However, we need an asymptotic estimate for the quantity $\sum_{p \le x} a_p^2 \log p / p^{k-1}$ which is proved in Section 4 using Shimura's split of the Rankin–Selberg L-function into the ordinary Riemann zeta-function and the symmetric square L-function associated to a Hecke eigenform f for the full modular group. Apart from this, Theorem A' plays a crucial role (on the whole) particularly in proving the main theorem over shorter intervals.

Acknowledgement. The author wishes to thank the anonymous referee for the careful reading of the manuscript and for valuable comments.

2. Notation and preliminaries

Throughout the paper, the implied constants A are effective absolute positive constants and they need not be the same at each occurrence. When k is even, it is known that a_n s are real. In fact, they are totally real algebraic numbers. Hence a_p is real from (1.1) and (1.3). By Deligne's estimate, we also have $|a_p| \le 2p^{(k-1)/2}$. We define a real number A'_p such that $a_p = 2A'_p p^{(k-1)/2}$, and hence, $|A'_p| \le 1$. Let α'_p and $\overline{\alpha'_p}$ be the roots of the equation $x^2 - 2A'_p x + 1 = 0$ and we note that $|\alpha'_p| = 1$. Therefore, from the Euler product of $L_f(s)$, we can write

$$L_f(s) = \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \overline{\alpha_p} p^{-s})^{-1} \tag{2.1}$$

with $|\alpha_p| = p^{(k-1)/2}$ and $a_p = \alpha_p + \overline{\alpha_p}$. Taking the logarithm and differentiating both sides of (2.1) with respect to s, we find that

$$-\frac{L'_f(s)}{L_f(s)} = \sum_{m\ge 1, p} (\alpha_p^m + \overline{\alpha_p}^m) p^{-ms} (\log p). \tag{2.2}$$

Now we define

$$\Lambda_f(n) = (\alpha_p^m + \overline{\alpha_p}^m)(\log p) \quad \text{if } n = p^m; \ 0 \text{ otherwise.} \tag{2.3}$$

Hence we obtain

$$-\frac{L'_f(s)}{L_f(s)} = \sum_{n=2}^{\infty} \Lambda_f(n) n^{-s} \quad (\text{in } \operatorname{Re} s > (k+1)/2). \tag{2.4}$$

Note that

$$\Lambda_f(n) \le 2(\log n) n^{(k-1)/2}. \tag{2.5}$$

For $x > 1$, we define

$$\Lambda_{x,f}(n) = \begin{cases} \Lambda_f(n), & \text{if } 1 \le n \le x, \\ \Lambda_f(n) \dfrac{\left\{ \left(\log\left(\dfrac{x^3}{n} \right) \right)^2 - 2\left(\log\left(\dfrac{x^2}{n} \right) \right)^2 \right\}}{2(\log x)^2}, & \text{if } x \le n \le x^2, \\ \Lambda_f(n) \dfrac{\left(\log\left(\dfrac{x^3}{n} \right) \right)^2}{2(\log x)^2} & \text{for } x^2 \le n \le x^3. \end{cases} \tag{2.6}$$

3. Some lemmas

Lemma 3.1. *Let τ be a real positive number and suppose that $\delta(n)$ are complex numbers satisfying*

$$|\delta(n)| \le C$$

for some fixed constant $C > 0$. Then, for any integer $l \ge 1$, we have

$$\begin{aligned} S_1 := \sum_{\substack{p_1,\cdots,p_l<y,\\ q_1,\cdots,q_l<y,\\ p_1\cdots p_l=q_1\cdots q_l}} \frac{\delta(p_1)\cdots\delta(p_l)\delta(q_1)\cdots\delta(q_l)}{(p_1\cdots p_l q_1\cdots q_l)^\tau} \\ = l!\Big(\sum_{p<y}\frac{\delta^2(p)}{p^{2\tau}}\Big)^l + O\Big(C^{2l} l!\Big(\sum_{p<y} p^{-2\tau}\Big)^{l-2}\Big(\sum_{p<y} p^{-4\tau}\Big)\Big). \end{aligned}$$

Proof. See, for example, Lemma 1 of [5]. □

For $x \ge 2$, $t > 0$, we define the number $\sigma_{x,t}$ by

$$\sigma_{x,t} = k/2 + 2\max\big(\beta - k/2, 2/\log x\big),$$

where $\varrho = \beta + i\gamma$ runs over all zeros of $L_f(s)$ for which

$$|t-\gamma| \le x^{3|\beta-k/2|}(\log x)^{-1}.$$

Lemma 3.2. *Suppose that $T^\alpha \le H \le T$, where $B' < \alpha \le 1$ and $x \ge 2$, $1 \le \xi \le x^{8l}$, $x^3\xi^2 \le (H/T^{B'})^{1/4}$. Then, for $0 \le \nu \le 8l$, we have*

$$\begin{aligned} I_1 := \int_T^{T+H} \Big(\sigma_{x,t} - \frac{k}{2}\Big)^\nu \xi^{\sigma_{x,t}-(k/2)}\, dt \ll A^l \frac{H}{(\log x)^\nu} \\ + A^l \Bigg(H\log T\Bigg((\nu)!\frac{\log T}{\log x}\Bigg(\frac{4}{\log\big(H/T^{B'}\big)}\Bigg)^{\nu+1} \\ + (\nu)!\frac{1}{\log x}\Bigg(\frac{4}{\log\big(H/T^{B'}\big)}\Bigg)^{\nu}\Bigg)\Bigg). \end{aligned}$$

Proof. The proof follows using Theorem A$'$ at the appropriate place of the proof of Lemma 12 of [16]. □

Lemma 3.3. *Let $H > 1$, $l \ge 1$ and $1 < y \le H^{1/l}$. Suppose that β_p are complex numbers satisfying*

$$|\beta_p| < B_1 \frac{\log p}{\log y} \quad \text{for } p < y. \tag{3.3.1}$$

Then, we have

$$\int_0^H \Big|\sum_{p<y} \beta_p p^{-(1/2)-it}\Big|^{2l} dt \ll (AB_1^2 l)^l H, \tag{3.3.2}$$

and if $|\beta_p| < B_1$, *then we have*

$$\int_0^H \Big|\sum_{p<y} \beta_p p^{-1-2it}\Big|^{2l} dt \ll (AB_1^2 l)^l H. \tag{3.3.3}$$

Proof. See, for example, Lemma 3 of [5]. □

Remark. It should be mentioned here that a general mean-value theorem for the Dirichlet polynomial with a better error term is also available, for which we refer to [10].

Lemma 3.4. *Let* $B' < \alpha \le 1$, $T^\alpha \le H \le T$ *and* $x = T^{(\alpha - B')/(60l)}$. *Then, for* $T \le t \le T + H$, *we have*

$$\begin{aligned} &S_f(t) + \frac{1}{\pi}\sum_{p<x^3} \frac{(\alpha_p + \overline{\alpha_p})\sin(t\log p)}{p^{k/2}} \\ &\quad = O\bigg(\Big|\sum_{p<x^3} \frac{\Lambda_f(p) - \Lambda_{x,f}(p)}{p^{k/2}\log p} p^{-it}\Big|\bigg) \\ &\qquad + O\bigg(\Big|\sum_{p<x^{3/2}} \frac{\Lambda_{x,f}(p^2)}{p^k \log p} p^{-2it}\Big|\bigg) + O\bigg(\Big(\sigma_{x,t} - \frac{k}{2}\Big)\log T\bigg) \\ &\qquad + O\bigg(\Big(\sigma_{x,t} - \frac{k}{2}\Big) x^{(\sigma_{x,t} - (k/2))} \int_{k/2}^\infty x^{(k/2)-\sigma} \Big|\sum_{p<x^3} \frac{\Lambda_{x,f}(p)\log(xp)}{p^{\sigma+it}}\Big| d\sigma\bigg). \end{aligned}$$

Proof. From Theorem A (stated in the introduction), we obtain

$$\begin{aligned} S_f(t) = &-\frac{1}{\pi}\sum_{p<x^3} \frac{\Lambda_{x,f}(p)\sin(t\log p)}{p^{\sigma_{x,t}}\log p} - \frac{1}{\pi}\sum_{p^2<x^3} \frac{\Lambda_{x,f}(p^2)\sin(t\log p^2)}{p^{2\sigma_{x,t}}(\log p^2)} \\ &+ O\bigg(\Big(\sigma_{x,t} - \frac{k}{2}\Big)\Big|\sum_{p<x^3} \frac{\Lambda_{x,f}(p)}{p^{\sigma_{x,t}+it}}\Big|\bigg) \\ &+ O\bigg(\Big(\sigma_{x,t} - \frac{k}{2}\Big)\Big|\sum_{p^2<x^3} \frac{\Lambda_{x,f}(p^2)}{p^{2\sigma_{x,t}+2it}}\Big|\bigg) \\ &+ O\bigg(\Big|\sum_{\substack{p^r<x^3 \\ r>2}} \frac{\Lambda_{x,f}(p^r)\sin(t\log p^r)}{p^{r\sigma_{x,t}}(\log p^r)}\Big|\bigg) \\ &+ O\bigg(\Big(\sigma_{x,t} - \frac{k}{2}\Big)\Big|\sum_{\substack{p^r<x^3 \\ r>2}} \frac{\Lambda_{x,f}(p^r)}{p^{r\sigma_{x,t}+rit}}\Big|\bigg) + O\bigg(\Big(\sigma_{x,t} - \frac{k}{2}\Big)\log t\bigg). \end{aligned} \tag{3.4.1}$$

Note that $\sigma_{x,t} \geq \frac{1}{2}k$ and

$$|\Lambda_{x,f}(n)| \leq |\Lambda_f(n)| \leq 2(\log n)n^{(k-1)/2}.$$

Now, it is easy to see that

$$\sum_{\substack{p^r<x^3\\ r>2}} \frac{\Lambda_{x,f}(p^r)\sin(t\log p^r)}{p^{r\sigma_{x,t}}(\log p^r)} = O(1) = O\biggl(\Bigl(\sigma_{x,t}-\frac{k}{2}\Bigr)\log T\biggr), \tag{3.4.2}$$

$$\begin{aligned} \Bigl(\sigma_{x,t}-\frac{k}{2}\Bigr)\biggl|\sum_{\substack{p^r<x^3\\ r>2}} \frac{\Lambda_{x,f}(p^r)}{p^{r\sigma_{x,t}+rit}}\biggr| &= O\biggl(\Bigl(\sigma_{x,t}-\frac{k}{2}\Bigr)\biggr) \\ &= O\biggl(\Bigl(\sigma_{x,t}-\frac{k}{2}\Bigr)\log T\biggr) \end{aligned} \tag{3.4.3}$$

and

$$\begin{aligned} \Bigl(\sigma_{x,t}-\frac{k}{2}\Bigr)\biggl|\sum_{p^2<x^3} \frac{\Lambda_{x,f}(p^2)}{p^{2\sigma_{x,t}+2it}}\biggr| &= O\biggl(\Bigl(\sigma_{x,t}-\frac{k}{2}\Bigr)\log x\biggr) \\ &= O\biggl(\Bigl(\sigma_{x,t}-\frac{k}{2}\Bigr)\log T\biggr). \end{aligned} \tag{3.4.4}$$

Now, we write the first four terms on the right-hand side of (3.4.1) in the following manner, namely,

$$\begin{aligned} S_f(t) &+ \frac{1}{\pi}\sum_{p<x^3} \frac{(\alpha_p+\overline{\alpha_p})\sin(t\log p)}{p^{k/2}} \\ &= O\biggl(\biggl|\sum_{p<x^3} \frac{\Lambda_f(p)-\Lambda_{x,f}(p)}{p^{k/2}\log p}p^{-it}\biggr|\biggr) \\ &\quad + O\biggl(\biggl|\sum_{p<x^3} \frac{\Lambda_{x,f}(p)}{p^{k/2}\log p}(1-p^{(k/2)-\sigma_{x,t}})p^{-it}\biggr|\biggr) \\ &\quad + O\biggl(\Bigl(\sigma_{x,t}-\frac{k}{2}\Bigr)\biggl|\sum_{p<x^3} \frac{\Lambda_{x,f}(p)}{p^{\sigma_{x,t}+it}}\biggr|\biggr) \\ &\quad + O\biggl(\biggl|\sum_{p<x^{3/2}} \frac{\Lambda_{x,f}(p^2)}{p^k\log p}p^{-2it}\biggr|\biggr) \\ &\quad + O\biggl(\biggl|\sum_{p<x^{3/2}} \frac{\Lambda_{x,f}(p^2)}{p^k\log p}(1-p^{k-2\sigma_{x,t}})p^{-2it}\biggr|\biggr) \\ &\quad + O\biggl(\Bigl(\sigma_{x,t}-\frac{k}{2}\Bigr)\log T\biggr). \end{aligned} \tag{3.4.5}$$

We note that

$$
\begin{aligned}
Q_1 &:= \Bigg| \sum_{p<x^{3/2}} \frac{\Lambda_{x,f}(p^2)}{p^k \log p}(1-p^{k-2\sigma_{x,t}})p^{-2it} \Bigg| \\
&< \sum_{p<x^{3/2}} \frac{2(\log p)p^{k-1}}{p^k \log p}(1-p^{k-2\sigma_{x,t}}) \\
&< \sum_{p<x^{3/2}} \frac{4\big(\sigma_{x,t}-\frac{1}{2}k\big)\log p}{p} = O\big(\big(\sigma_{x,t}-\tfrac{1}{2}k\big)\log T\big),
\end{aligned} \tag{3.4.6}
$$

since

$$
\sigma_{x,t} \geq \frac{k}{2} + \frac{4}{\log x}
$$

and $1-e^{-x} < x$. Further, we have

$$
\begin{aligned}
Q_2 &:= \Bigg| \sum_{p<x^3} \frac{\Lambda_{x,f}(p)}{p^{k/2}\log p}\big(1-p^{(k/2)-\sigma_{x,t}}\big)p^{-it} \Bigg| \\
&= \Bigg| \int_{k/2}^{\sigma_{x,t}} \sum_{p<x^3} \frac{\Lambda_{x,f}(p)}{p^{\sigma'+it}}\, d\sigma' \Bigg| \leq \int_{k/2}^{\sigma_{x,t}} \Bigg| \sum_{p<x^3} \frac{\Lambda_{x,f}(p)}{p^{\sigma'+it}} \Bigg| d\sigma'.
\end{aligned} \tag{3.4.7}
$$

If $\frac{1}{2}k \leq \sigma' \leq \sigma_{x,t}$, then

$$
\begin{aligned}
\Bigg| \sum_{p<x^3} \frac{\Lambda_{x,f}(p)}{p^{\sigma'+it}} \Bigg| &= \Bigg| x^{\sigma'-(k/2)} \int_{\sigma'}^{\infty} x^{(k/2)-\sigma} \sum_{p<x^3} \frac{\Lambda_{x,f}(p)(\log xp)}{p^{\sigma+it}}\, d\sigma \Bigg| \\
&\leq x^{\sigma_{x,t}-(k/2)} \int_{k/2}^{\infty} x^{(k/2)-\sigma} \Bigg| \sum_{p<x^3} \frac{\Lambda_{x,f}(p)(\log xp)}{p^{\sigma+it}} \Bigg| d\sigma,
\end{aligned} \tag{3.4.8}
$$

and therefore, from (3.4.7) and (3.4.8), we get

$$
Q_2 \leq \left(\sigma_{x,t}-\frac{k}{2}\right) x^{\sigma_{x,t}-(k/2)} \int_{k/2}^{\infty} x^{(k/2)-\sigma} \Bigg| \sum_{p<x^3} \frac{\Lambda_{x,f}(p)(\log xp)}{p^{\sigma+it}} \Bigg| d\sigma. \tag{3.4.9}
$$

Now, the lemma follows from (3.4.5), (3.4.6) and (3.4.9). □

Lemma 3.5. *Let $B' < \alpha \leq 1$ and suppose that $T^{\alpha} \leq H \leq T$. Put $x = T^{(\alpha-B')/(60l)}$. Then, for $l \ll \log T$, we have*

$$
\int_T^{T+H} \Bigg| S_f(t) + \frac{1}{\pi} \sum_{p<x^3} \frac{(\alpha_p + \overline{\alpha_p})\sin(t\log p)}{p^{k/2}} \Bigg|^{2l} dt \ll A^l l^{2l} H.
$$

Proof. Let

$$\sum_1(t) := \sum_{p<x^3} \frac{(\alpha_p + \overline{\alpha_p})\sin(t\log p)}{p^{k/2}}, \tag{3.5.1}$$

$$E_1(t) := \sum_{p<x^3} \frac{\Lambda_f(p) - \Lambda_{x,f}(p)}{p^{k/2}\log p} p^{-it}, \tag{3.5.2}$$

$$E_2(t) := \sum_{p<x^{3/2}} \frac{\Lambda_{x,f}(p^2)}{p^k \log p} p^{-2it}, \tag{3.5.3}$$

$$E_3(t) := \left(\sigma_{x,t} - \tfrac{1}{2}k\right)\log T, \tag{3.5.4}$$

and

$$E_4(t) := \left(\sigma_{x,t} - \tfrac{1}{2}k\right)x^{(\sigma_{x,t}-(k/2))} \int_{k/2}^{\infty} x^{(k/2)-\sigma} \left|\sum_{p<x^3} \frac{\Lambda_{x,f}(p)\log(xp)}{p^{\sigma+it}}\right| d\sigma. \tag{3.5.5}$$

Now, clearly from Lemma 3.4, we have

$$\left|S_f(t) + \pi^{-1}\sum_1(t)\right|^{2l} \ll A^l\left(|E_1(t)|^{2l} + |E_2(t)|^{2l} + |E_3(t)|^{2l} + |E_4(t)|^{2l}\right). \tag{3.5.6}$$

If we take

$$\beta_p = \frac{\Lambda_f(p) - \Lambda_{x,f}(p)}{p^{(k-1)/2}\log p},$$

then from the definition of $\Lambda_f(n)$ and $\Lambda_{x,f}(n)$, we easily find that

$$\beta_p = 0 \quad \text{for } 2 \le p \le x,$$
$$|\beta_p| \le 2\left(\frac{\log p}{\log x} - 1\right)^2 \le 2\frac{\log p}{\log x} \quad \text{for } x \le p \le x^2,$$

and

$$|\beta_p| \le 6\frac{\log p}{\log x} \quad \text{for } x^2 \le p \le x^3.$$

Therefore,

$$|\beta_p| \le B_1\frac{\log p}{\log x} \quad \text{for } p \le x^3$$

with some absolute positive constant B_1. Similarly, if we take

$$\beta'_p = \frac{\Lambda_{x,f}(p^2)}{p^{k-1}\log p},$$

then from the definition of $\Lambda_{x,f}(n)$, we find that

$$\Lambda_{x,f}(p^2) \leq 9p^{k-1}(\log p),$$

and so we get $|\beta'_p| < B_2$ with some absolute positive constant B_2. Therefore, from Lemma 3.3, ((3.3.2), (3.3.3), respectively), we obtain

$$\int_T^{T+H} |E_1(t)|^{2l}\, dt \ll (Al)^l H \tag{3.5.7}$$

and

$$\int_T^{T+H} |E_2(t)|^{2l}\, dt \ll (Al)^l H. \tag{3.5.8}$$

Note that we have fixed $x = T^{(\alpha-B')/(60l)}$. From Lemma 3.2, with $\xi = 1$ and $\nu = 2l$, we get

$$\int_T^{T+H} |E_3(t)|^{2l}\, dt \ll A^l\big(l(2l)! + l^{2l}\big)H \ll A^l l(2l)^{2l-1}H \ll A^l l^{2l} H, \tag{3.5.9}$$

since,

$$(2l)! \leq (2l)^{2l-1} \quad \text{for } l \geq 1, \tag{3.5.10A}$$

(3.5.10B)

$$S_2 := H\log T\left((\nu)!\frac{\log T}{\log x}\left(\frac{4}{\log\big(H/T^{B'}\big)}\right)^{\nu+1} + (\nu)!\frac{1}{\log x}\left(\frac{4}{\log\big(H/T^{B'}\big)}\right)^{\nu}\right)$$
$$\ll \nu!\frac{H}{(\log x)(\log T)^{\nu-1}}$$

and

$$\frac{H}{(\log x)^{\nu}} \ll A^l l^{2l} H. \tag{3.5.10C}$$

Now, we notice that

$$\int_T^{T+H} |E_4(t)|^{2l}\, dt \leq Q_3 Q_4 \tag{3.5.11}$$

where

$$Q_3 := \left(\int_T^{T+H} \big(\sigma_{x,t} - \tfrac{1}{2}k\big)^{4l} x^{4l(\sigma_{x,t}-(k/2))}\, dt\right)^{1/2}$$

and

$$Q_4 := \left(\int_T^{T+H} \left(\int_{k/2}^{\infty} x^{(k/2)-\sigma} \left| \sum_{p<x^3} \frac{\Lambda_{x,f}(p)(\log xp)}{p^{\sigma+it}} \right| d\sigma \right)^{4l} dt \right)^{1/2}.$$

From Lemma 3.2, (with $\xi = x^{4l}$, $\nu = 4l$), we obtain

$$Q_3 \ll \left(A^l (l^{4l} + l(4l)!) H (\log T)^{-4l} \right)^{1/2} \ll A^l l^{2l} H^{1/2} (\log T)^{-2l}. \tag{3.5.12}$$

By Hölder's inequality, we get

$$\begin{aligned} Q_4^2 &\le \int_T^{T+H} \left(\int_{k/2}^{\infty} x^{(k/2)-\sigma}\, d\sigma \right)^{4l-1} \\ &\quad \times \left(\int_{k/2}^{\infty} x^{(k/2)-\sigma} \left| \sum_{p<x^3} \frac{\Lambda_{x,f}(p)(\log xp)}{p^{\sigma+it}} \right|^{4l} d\sigma \right) dt \\ &\le (\log x)^{1-4l} \left(\int_{k/2}^{\infty} x^{(k/2)-\sigma} \right. \\ &\quad \left. \times \left(\int_T^{T+H} \left| \sum_{p<x^3} \frac{\Lambda_{x,f}(p)(\log xp)}{p^{\sigma+it}} \right|^{4l} dt \right) d\sigma \right). \end{aligned} \tag{3.5.13}$$

By taking

$$\beta_p = \frac{\Lambda_{x,f}(p)(\log xp)}{p^{(k-1)/2}(\log x)^2},$$

we observe that $|\beta_p| \le 10 \log p / \log x$. Now, by (3.3.2), we obtain

$$\int_T^{T+H} \left| \sum_{p<x^3} \frac{\Lambda_{x,f}(p)(\log xp)}{p^{\sigma+it}} \right|^{4l} dt \ll (AB_1^2 l)^{2l} H (\log x)^{8l}. \tag{3.5.14}$$

Therefore, we get from (3.5.13) and (3.5.14)

$$Q_4^2 \ll (AB_1^2 l)^{2l} H (\log x)^{4l}. \tag{3.5.15}$$

From (3.5.11), (3.5.12) and (3.5.15), with our choice of x, we get

$$\begin{aligned} \int_T^{T+H} |E_4(t)|^{2l}\, dt &\ll A^l l^{2l} H^{1/2} (\log T)^{-2l} (AB_1^2 l)^l H^{1/2} (\log x)^{2l} \\ &\ll A^l l^l H. \end{aligned} \tag{3.5.16}$$

This proves the lemma. □

Lemma 3.6. *Let $B' < \alpha \le 1$ and $T^\alpha \le H \le T$. Then, if $l \ge 1$ is an integer and*

$$x^3 = T^{(\alpha-B')/(20l)} \le z \le H^{1/l},$$

we have

$$Q_5 := \int_T^{T+H} \left| S_f(t) + \frac{1}{\pi} \sum_{p<z} \frac{(\alpha_p + \overline{\alpha_p}) \sin(t \log p)}{p^{k/2}} \right|^{2l} dt \ll A^l l^{2l} H. \tag{3.6.1}$$

Proof. We clearly have

$$\begin{aligned} Q_5 \ll 4^l \int_T^{T+H} \left| S_f(t) + \frac{1}{\pi} \sum_{p<x^3} \frac{(\alpha_p + \overline{\alpha_p}) \sin(t \log p)}{p^{k/2}} \right|^{2l} dt \\ + 4^l \int_T^{T+H} \left| \sum_{x^3 \le p < z} p^{-(1/2)-it} \right|^{2l} dt. \end{aligned} \tag{3.6.2}$$

From Lemma 3.5, we observe that

$$\int_T^{T+H} \left| S_f(t) + \frac{1}{\pi} \sum_{p<x^3} \frac{(\alpha_p + \overline{\alpha_p}) \sin(t \log p)}{p^{k/2}} \right|^{2l} dt \ll A^l l^{2l} H. \tag{3.6.3}$$

From (3.3.2) (with $B_1 = O(1)$), we have

$$\int_T^{T+H} \left| \sum_{x^3 \le p < z} p^{-(1/2)-it} \right|^{2l} dt \ll A^l l^l H. \tag{3.6.4}$$

In the notation of Lemma 3.3,

$$\beta_p = 1 = \frac{\log p}{\log z} \frac{\log z}{\log p} \ll \frac{\log p}{\log z}$$

so that (3.3.1) is satisfied with z in place of y. This proves the lemma. □

4. Prime number theorem related to the Dirichlet series $\sum_{n=1}^\infty a_n^2/n^s$

We know that

$$L_f(s) = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\overline{\alpha_p}}{p^s}\right)^{-1} = \sum_{n=1}^\infty \frac{a_n}{n^s} \tag{4.1}$$

is an entire function, $|\alpha_p| = p^{(k-1)/2}$, $\alpha_p\overline{\alpha_p} = p^{k-1}$ and $a_p = \alpha_p + \overline{\alpha_p}$. Now, let

$$L_{f^2}(s) := \sum_{n=1}^{\infty} \frac{a_n^2}{n^s} \tag{4.2}$$

and

$$L_{f\otimes f}(s) = \prod_p \Big(1 - \frac{\alpha_p^2}{p^s}\Big)^{-1} \Big(1 - \frac{\alpha_p\overline{\alpha_p}}{p^s}\Big)^{-1} \Big(1 - \frac{\overline{\alpha_p}^2}{p^s}\Big)^{-1}, \tag{4.3}$$

where the symbol $\otimes$ in (4.3) denotes the Rankin–Selberg convolution. The important relation between (4.2) and (4.3) is given by (see [12], [11], [17] and [13])

$$\zeta(s-k+1)L_{f\otimes f}(s) = \zeta(2s-2k+2)L_{f^2}(s), \tag{4.4}$$

where $\zeta(s)$ is the ordinary Riemann zeta-function. It has been proved by Rankin (see [12]) that $L_{f^2}(s)$ has a simple pole at $s = k$ with residue $k\alpha$ (α is a certain constant). Therefore, the series $-\big(L'_{f^2}(k-1+s)\big)/\big(L_{f^2}(k-1+s)\big)$ has a simple pole at $s = 1$ with residue 1.

We define

$$\Lambda^*(n) = \begin{cases} \dfrac{\big(\alpha_p^{2m} + \overline{\alpha_p}^{2m} + \big(\alpha_p\overline{\alpha_p}\big)^m + (-1)^{m+1}(\alpha_p\overline{\alpha_p})^m\big)\log p}{p^{m(k-1)}}, & \text{if } n = p^m, \\ 0, & \text{otherwise.} \end{cases} \tag{4.5}$$

We have the usual von Mangoldt's function, namely,

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m, \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

We also define $\Psi^*_{f^2}(x)$ and $\Psi_{f^2}(x)$ by

$$\Psi^*_{f^2}(x) = \sum_{n\le x} \Lambda^*(n)(x-n) \tag{4.7}$$

and

$$\Psi^*_{f^2}(x) = \int_0^x \Psi_{f^2}(u)\,du = \int_1^x \Psi_{f^2}(u)\,du. \tag{4.8}$$

It is obvious that

$$\Psi_{f^2}(x) = \sum_{n\le x} \Lambda^*(n). \tag{4.9}$$

The aim of this section is to prove:

Theorem 4.1. *For $x \geq x_0$, we have*

$$\Psi_{f^2}(x) = x + O\big(xe^{-C\sqrt{\log x}}\big).$$

To prove this theorem, we need the following lemmas.

Lemma 4.1. *There exists a positive constant C (>0) such that*

$$L_{f^2}(k-1+s) \neq 0 \text{ in } \sigma > 1 - \frac{C}{\log(|t|+2)}.$$

Proof. See, for example, [8]. □

Lemma 4.2. *Suppose that $L_{f^2}(s)$ has no zeros in the domain*

$$\sigma > 1 - \eta(|t|),$$

where $\eta(t)$, for $t \geq 0$, a decreasing function, has a continuous derivative $\eta'(t)$ and satisfies

$$\text{(i)} \quad 0 < \eta(t) < \tfrac{1}{2},$$
$$\text{(ii)} \quad \eta'(t) \to 0 \quad \text{as} \quad t \to \infty,$$
$$\text{(iii)} \quad \frac{1}{\eta(t)} = O(\log t) \quad \text{as} \quad t \to \infty.$$

Let α_1' be a fixed number satisfying $0 < \alpha_1' < 1$. Then,

$$-\frac{L'_{f^2}(s)}{L_{f^2}(s)} = O\big(\log^2(|t|)\big)$$

uniformly in the region $\sigma \geq 1 - \alpha_1'\eta(|t|)$ as $t \to \pm\infty$.

Proof. Since we have an Euler-product representation for $L_{f^2}(s)$ from (4.3) and (4.4), the proof of this lemma follows in a similar fashion to that of Theorem 20 of [8]. □

Lemma 4.3. *Under the conditions of Lemma 4.2, we have*

$$\Psi^*_{f^2}(x) = \tfrac{1}{2}x^2 + O\big(x^2 e^{-\alpha_1'\omega(x)}\big)$$

as $x \to \infty$, where $\omega(x)$ is the minimum of $\eta(t)\log x + \log t$ for $t \geq 1$.

Proof. First of all, we note that (for $C>1$)

(4.3.1)

$$\begin{aligned}\Psi^*_{f^2}(x) &= \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{x^{s+1}}{s(s+1)}\Big(-\frac{L'_{f^2}(k-1+s)}{L_{f^2}(k-1+s)}\Big)\,ds\\ &= \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{x^{s+1}}{s(s+1)}\Big(-\frac{L'_{f^2}(k-1+s)}{L_{f^2}(k-1+s)}-\frac{\zeta'(2s)}{\zeta(2s)}+\frac{\zeta'(2s)}{\zeta(2s)}\Big)\,ds\\ &= \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{x^{s+1}}{s(s+1)}\Big(-\frac{L'_{f\otimes f}(k-1+s)}{L_{f\otimes f}(k-1+s)}-\frac{\zeta'(s)}{\zeta(s)}+\frac{\zeta'(2s)}{\zeta(2s)}\Big)\,ds\\ &= \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{x^{s+1}}{s(s+1)}\Big(-\frac{L'_{f\otimes f}(k-1+s)}{L_{f\otimes f}(k-1+s)}-\frac{\zeta'(s)}{\zeta(s)}\Big)\,ds+O\big(x^{7/4}\big),\end{aligned}$$

since

$$\begin{aligned}\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{x^{s+1}}{s(s+1)}\Big(-\frac{\zeta'(2s)}{\zeta(2s)}\Big)\,ds &= \frac{1}{2\pi i}\int_{C-iT}^{C+iT}\frac{x^{s+1}}{s(s+1)}\Big(-\frac{\zeta'(2s)}{\zeta(2s)}\Big)\,ds\\ &\quad+O\Big(\frac{x^{C+1}}{T}\Big).\end{aligned}$$

Now, by moving the line of integration to $\sigma=\frac{3}{4}$, we see that the horizontal portions contribute an error which is in the absolute value at most $O(x^{C+1}/T)$, and the vertical portion contributes at most $O(x^{7/4})$. We can choose $C=1+\varepsilon$ (ε is a small positive constant) and $T=x^{1/2}$. From (4.3.1), we get

(4.3.2)

$$\frac{\Psi^*_{f^2}(x)}{x^2}=\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{x^{s-1}}{s(s+1)}\Big(-\frac{L'_{f\otimes f}(k-1+s)}{L_{f\otimes f}(k-1+s)}-\frac{\zeta'(s)}{\zeta(s)}\Big)\,ds+O(x^{-1/4}).$$

Now, we move the line of integration of the integral appearing on the right-hand side of (4.3.2) to $\sigma=1-\alpha'_1\eta(|t|)$. Therefore, this lemma follows when applying Lemmas 4.1 and 4.2.

Now, from Lemma 4.3, Theorem 4.1 follows by standard arguments (see, for example, [8]). □

5. Proof of Theorem 1

We fix $z=T^{\alpha/(5l)}$. Notice that $\alpha_p+\overline{\alpha_p}=a_p$. Let us write

(5.1) $$\Delta_z(t):=\Delta(t):=S_f(t)+\frac{1}{\pi}\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}.$$

Then, from the binomial theorem, we have

$$\begin{aligned}\big(S_f(t)\big)^{2l} &= \Big(\frac{1}{\pi}\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}\Big)^{2l}\\ &\qquad+\sum_{j=1}^{2l}\binom{2l}{j}\Delta^j(t)\Big(-\frac{1}{\pi}\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}\Big)^{2l-j}\\ &= Q_6+Q_7,\quad \text{say.}\end{aligned} \tag{5.2}$$

We observe that

$$Q_7 \ll 4^l\, l\,|\Delta(t)|\bigg(|\Delta(t)|^{2l-1}+\bigg|\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}\bigg|^{2l-1}\bigg).$$

Therefore, we obtain (using Hölder's inequality)

$$\begin{aligned}Q_8 &:= \int_T^{T+H}|S_f(t)|^{2l}\,dt-\frac{1}{\pi^{2l}}\int_T^{T+H}\bigg|\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}\bigg|^{2l}dt\\ &\ll A^l\int_T^{T+H}|\Delta(t)|^{2l}\,dt+A^l\int_T^{T+H}|\Delta(t)|\,\bigg|\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}\bigg|^{2l-1}dt\\ &\ll A^l\int_T^{T+H}|\Delta(t)|^{2l}\,dt\\ &\quad+A^l\Big(\int_T^{T+H}|\Delta(t)|^{2l}\,dt\Big)^{1/2l}\bigg(\int_T^{T+H}\bigg|\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}\bigg|^{2l}dt\bigg)^{1-(1/2l)}.\end{aligned} \tag{5.3}$$

Let $\eta_1 := \eta_1(t) := \sum_{p<z} a_p p^{-(k/2)-it}$, and hence,

$$\sum_{p<z} a_p p^{-k/2}\sin(t\log p) = \frac{i}{2}\big(\eta_1-\overline{\eta_1}\big). \tag{5.4}$$

Therefore, from the binomial expansion, we obtain

$$\begin{aligned}Q_9 &:= \int_T^{T+H}\bigg|\sum_{p<z}\frac{a_p\sin(t\log p)}{p^{k/2}}\bigg|^{2l}dt\\ &= \Big(\frac{1}{2}\Big)^{2l}\sum_{j=0}^{2l}(-1)^j\binom{2l}{j}\int_T^{T+H}\eta_1^j\,\overline{\eta_1}^{(2l-j)}\,dt\\ &= 2^{-2l}\frac{(2l)!}{(l!)^2}\int_T^{T+H}|\eta_1(t)|^{2l}\,dt\\ &\quad+O\bigg(4^{-l}\sum_{\substack{j=0,1,\dots,2l\\ j\neq l}}\binom{2l}{j}\bigg|\int_T^{T+H}\eta_1^j\,\overline{\eta_1}^{(2l-j)}\,dt\bigg|\bigg).\end{aligned} \tag{5.5}$$

We note that the integral in the error term of (5.5) is

$$\ll \sum_{\substack{p_1,\cdots,p_j<z\\ q_1,\ldots,q_{(2l-j)}<z}} \frac{a_{p_1}\cdots a_{p_j}a_{q_1}\cdots a_{q_{(2l-j)}}}{(p_1\cdots p_j q_1\cdots q_{(2l-j)})^{k/2}}\left|\log\left(\frac{p_1\cdots p_j}{q_1\cdots q_{(2l-j)}}\right)\right|^{-1}. \tag{5.6}$$

We note that $|a_p|\le 2p^{(k-1)/2}$ and $z=T^{\alpha/(5l)}$. Since

$$\min\left(\frac{1}{a},\frac{1}{b}\right)\le\left|\log\left(\frac{a}{b}\right)\right| \tag{5.7}$$

for any two distinct positive integers a and b, from (5.6) and (5.7) (for $j\neq l$), we get,

$$\int_T^{T+H}\eta_1^j\overline{\eta_1}^{(2l-j)}\,dt\ll z^{2l}\left(\sum_{p<z}|a_p|p^{-k/2}\right)^{2l}\ll A^l z^{3l}\ll A^l H. \tag{5.8}$$

Therefore, the error term in (5.5) is

$$\ll A^l H. \tag{5.9}$$

Now,

$$\begin{aligned} I_2 &:= \int_T^{T+H}|\eta_1(t)|^{2l}\,dt\\ &= H\sum_{\substack{p_1,\ldots,p_l<z\\ q_1,\ldots,q_l<z\\ p_1\cdots p_l=q_1\cdots q_l}}\frac{a_{p_1}\cdots a_{p_l}a_{q_1}\cdots a_{q_l}}{(p_1\cdots p_l q_1\cdots q_l)^{k/2}}\\ &\quad+O\left(\sum_{\substack{p_1,\ldots,p_l<z\\ q_1,\ldots,q_l<z\\ p_1\cdots p_l\neq q_1\cdots q_l}}\frac{a_{p_1}\cdots a_{p_l}a_{q_1}\cdots a_{q_l}}{(p_1\cdots p_l q_1\cdots q_l)^{k/2}}\left|\log\left(\frac{p_1\cdots p_l}{q_1\cdots q_l}\right)\right|^{-1}\right). \end{aligned} \tag{5.10}$$

Arguments similar to (5.6) yield the error term in (5.10) as

$$\ll A^l H. \tag{5.11}$$

Since $|a_p|\le 2p^{(k-1)/2}$, we have $|\delta(p)|:=|a_p/p^{(k-1)/2}|\le 2$. Therefore, choosing $C=2$ and $\tau=\frac{1}{2}$ in Lemma 3.1, we obtain the first term on the right-hand side of (5.10) as

$$= Hl!\left(\sum_{p<z}\frac{a_p^2}{p^k}\right)^l + O\left(H2^{2l}l!\left(\sum_{p<z}p^{-1}\right)^{l-2}\left(\sum_{p<z}p^{-2}\right)\right). \tag{5.12}$$

We note that (from Theorem 4.1),

$$\Psi_{f^2}(x) = \sum_{n\le x} \Lambda^*(n) = \sum_{p\le x} \frac{a_p^2 \log p}{p^{k-1}} + O\big(x^{1/2}\log x\big) = x + O\big(xe^{-C\sqrt{\log x}}\big), \tag{5.13}$$

and hence, using Abel's identity, we obtain

$$\sum_{p\le z} \frac{a_p^2}{p^k} = \log\log z + O(1) = \log\log T - \log(5l) + O(1). \tag{5.14}$$

Hence, from (5.10), (5.11), (5.12) and (5.14), we get

$$\int_T^{T+H} |\eta_1(t)|^{2l}\,dt = l!H(\log\log T)^l + O\big(A^l l!(\log l)H(\log\log T)^{l-1}\big). \tag{5.15}$$

Therefore, from (5.5), (5.9) and (5.15), we find that

$$\begin{aligned}\int_T^{T+H} \left|\sum_{p<z} \frac{a_p \sin(t\log p)}{p^{k/2}}\right|^{2l} dt &= \frac{(2l)!}{l!}4^{-l}H(\log\log T)^l \\ &\quad + O\big(A^l l!(\log l)H(\log\log T)^{l-1}\big) \\ &\ll A^l l! H(\log\log T)^l,\end{aligned} \tag{5.16}$$

since $1 \le l \ll (\log\log T)^{1/3}$. Note that we have used

$$\frac{(2l)!}{(l!)^2} = \binom{2l}{l} \le 2^{2l}.$$

From Lemma 3.6 and (5.16), we see that the right-hand side of (5.3) is

$$\ll (Al)^{2l}H + A^l l H^{1/2l}\big(A^l l^{l-1}H(\log\log T)^l\big)^{1-(1/2l)}, \tag{5.17}$$

since (for $l \ge 1$) we have

$$l! \le l^{l-1}. \tag{5.18}$$

Therefore, the right-hand side of (5.17) becomes the total error, which is

$$\ll (Al)^{2l}H + A^l l^{l-(1/2)}H(\log\log T)^{l-(1/2)}. \tag{5.19}$$

Note that

$$l^{2l} \ll l^{l-(1/2)}(\log\log T)^{l-(1/2)} \quad \text{provided } l \ll (\log\log T)^{(l-(1/2))/(l+(1/2))},$$

and

$$\min_{l\ge 1}\left(\frac{l-\frac12}{l+\frac12}\right) = \min_{l\ge 1}\left(1 - \frac{1}{l+\frac12}\right) = \frac13.$$

Hence, Theorem 1 holds with this error term

$$O\big(A^l l^{l-(1/2)}H(\log\log T)^{l-(1/2)}\big),$$

provided $1 \le l \ll (\log\log T)^{1/3}$. This proves Theorem 1.

6. Proof of Theorem 2

First, we write

$$\Delta_z(t) := \Delta(t) := S_f(t) + \pi^{-1} \sum_{p<z} \frac{a_p \sin(t \log p)}{p^{k/2}} := S_f(t) + \pi^{-1} \sum_2 (t).$$

Then,

$$S_{1,f}(t+h) - S_{1,f}(t) = \int_t^{t+h} S_f(u)\,du = -\pi^{-1} \int_t^{t+h} \sum_2 (u)\,du + \int_t^{t+h} \Delta(u)\,du.$$

Therefore,

$$\begin{aligned} \left|\int_t^{t+h} S_f(u)\,du\right|^{2l} = \frac{1}{\pi^{2l}} &\left|\int_t^{t+h} \sum_2 (u)\,du\right|^{2l} \\ &+ O\left(A^l \left|\int_t^{t+h} \Delta(u)\,du\right|^{2l}\right) \\ &+ O\left(A^l \left|\int_t^{t+h} \Delta(u)\,du\right| \left|\int_t^{t+h} \sum_2 (u)\,du\right|^{2l-1}\right) \end{aligned} \tag{6.1}$$

exactly as in (5.3). We notice that

$$\left|\int_t^{t+h} \Delta(u)\,du\right|^{2l} \le h^{2l-1} \int_t^{t+h} |\Delta(u)|^{2l}\,du,$$

and hence, by Hölder's inequality, we get

$$\begin{aligned} Q_{10} &:= \int_T^{T+H} \left|\int_t^{t+h} S_f(u)\,du\right|^{2l} dt \\ &= \frac{1}{\pi^{2l}} \int_T^{T+H} \left|\int_t^{t+h} \sum_2 (u)\,du\right|^{2l} dt \\ &\quad + O\left(A^l h^{2l-1} \int_T^{T+H} \int_t^{t+h} |\Delta(u)|^{2l}\,du\right) \\ &\quad + O\left(A^l \left(h^{2l-1} \int_T^{T+H} \int_t^{t+h} |\Delta(u)|^{2l}\,du\,dt\right)^{1/2l} \right. \\ &\qquad \left. \times \left(\int_T^{T+H} \left|\int_t^{t+h} \sum_2 (u)\,du\right|^{2l} dt\right)^{1-(1/2l)}\right). \end{aligned} \tag{6.2}$$

We notice that

$$\int_T^{T+H} \int_t^{t+h} |\Delta(u)|^{2l}\, du\, dt = \int_0^h du \int_{T+u}^{T+u+H} |\Delta(t)|^{2l}\, dt, \tag{6.3}$$

and hence, by Lemma 3.6, with $(T+h)^\alpha \le H \le T$, $B' < \alpha \le 1$ and

$$(T+h)^{(\alpha-B')/(20l)} \le z \le H^{1/l},$$

we have

$$\int_T^{T+H} |\Delta(t)|^{2l}\, dt \ll (Al)^{2l} H. \tag{6.4}$$

With these restrictions, we have

$$\begin{aligned} Q_{10} &:= \int_T^{T+H} \left| \int_t^{t+h} S_f(u)\, du \right|^{2l} dt \\ &= \frac{1}{\pi^{2l}} \int_T^{T+H} \left| \int_t^{t+h} \sum_2 (u)\, du \right|^{2l} dt \\ &\quad + O\left((Al)^{2l} h^{2l} H + A^l l H^{1/2l} h \left(\int_T^{T+H} \left| \int_t^{t+h} \sum_2 (u)\, du \right|^{2l} dt \right)^{1-(1/2l)} \right). \end{aligned} \tag{6.5}$$

Now, the main term on the right-hand side of (6.5) (apart from the constant π^{-2l}) is

$$\int_T^{T+H} \left| \sum_{p<z} \frac{a_p\big(\cos\big((t+h)\log p\big) - \cos(t\log p)\big)}{p^{k/2}\log p} \right|^{2l} dt. \tag{6.6}$$

We put

$$\eta_2 = \eta_2(t) = \sum_{p<z} a_p p^{-(k/2)-it} (\log p)^{-1} (p^{-ih} - 1), \tag{6.7}$$

so that

$$\sum_{p<z} \frac{a_p\big(\cos\big((t+h)\log p\big) - \cos(t\log p)\big)}{p^{k/2}\log p} = \frac{\eta_2 + \overline{\eta_2}}{2}. \tag{6.8}$$

The integral in (6.6) becomes equal to

$$2^{-2l} \frac{(2l)!}{(l!)^2} \int_T^{T+H} |\eta_2(t)|^{2l}\, dt + O\left(4^{-l} \sum_{\substack{j=0,1,\ldots,2l \\ j\ne l}} \binom{2l}{j} \left| \int_T^{T+H} \eta_2^j \overline{\eta_2}^{(2l-j)}\, dt \right| \right). \tag{6.9}$$

Now, (for $j \neq l$)

$$
\begin{aligned}
Q_{11} &:= \int_T^{T+H} \eta_2^j \overline{\eta_2}^{(2l-j)}\, dt \\
&\ll \sum_{\substack{p_1,\dots,p_j<z \\ q_1,\dots,q_{(2l-j)}<z}} \frac{a_{p_1}\cdots a_{p_j} a_{q_1}\cdots a_{q_{(2l-j)}}}{(p_1\cdots p_j q_1\cdots q_{(2l-j)})^{k/2}} \\
&\quad \times \prod_{m=1}^{j} \frac{|p_m^{ih}-1|}{(\log p_m)} \times \prod_{n=1}^{2l-j} \frac{|q_n^{ih}-1|}{(\log q_n)} \times \left|\log\left(\frac{p_1\cdots p_j}{q_1\cdots q_{(2l-j)}}\right)\right|^{-1} \\
&\ll A^{2l} z^{2l} h^{2l} \left(\sum_{p<z} p^{-1/2}\right)^{2l},
\end{aligned}
\tag{6.10}
$$

since $|a_p| \leq 2p^{(k-1)/2}$ and

$$
|p^{ih}-1| = 2\left|\sin\left(\frac{h\log p}{2}\right)\right| \leq h\log p.
$$

Hence, the error term in (6.9) is

$$
\ll A^l h^{2l} H, \tag{6.11}
$$

by taking $z = T^{\alpha/(5l)}$.

Now, we have

$$
\begin{aligned}
Q_{12} &:= \int_T^{T+H} |\eta_2(t)|^{2l}\, dt \\
&= H \sum_{\substack{p_1,\dots,p_l<z \\ q_1,\dots,q_l<z \\ p_1\cdots p_l = q_1\cdots q_l}} \frac{a_{p_1}\cdots a_{p_l} a_{q_1}\cdots a_{q_l}}{(p_1\cdots p_l q_1\cdots q_l)^{k/2}} \\
&\quad \times \prod_{j=1}^{l} \frac{(p_j^{ih}-1)(q_j^{-ih}-1)}{(\log p_j)(\log q_j)} + O(A^l h^{2l} H),
\end{aligned}
\tag{6.12}
$$

in the exact way as we obtained (5.10) and (5.11). Now, by Lemma 3.1, with $\tau = \frac{1}{2}$,

$$
\delta(p_j) = \begin{cases} \dfrac{a_{p_j}(p_j^{ih}-1)}{p_j^{(k-1)/2}(\log p_j)} & \text{for } 1 \leq j \leq l, \\[2ex] \dfrac{a_{p_j}(p_j^{-ih}-1)}{p_j^{(k-1)/2}(\log p_j)} & \text{for } l+1 \leq j \leq 2l, \end{cases}
$$

and $C=2h$, the main term in (6.12) becomes equal to

$$\begin{aligned} Q_{13} &:= l!\Big(\sum_{p<z}\frac{a_p^2|p^{ih}-1|^2}{p^k(\log p)^2}\Big)^l H + O\Big(2^{2l}h^{2l}l!\Big(\sum_{p<z}p^{-1}\Big)^{l-2}H\Big) \\ &= l!H\bigg(4\sum_{p<z}\frac{a_p^2}{p^k}\bigg(\frac{\sin\big(\frac{1}{2}h\log p\big)}{\log p}\bigg)^2\bigg)^l \\ &\quad + O\big(A^l l^l h^{2l} H(\log\log T)^{l-2}\big). \end{aligned} \tag{6.13}$$

Now, assuming that $1<h^{-1}<\log T/(10l)<\log z$, we write the sum on the right-hand side of (6.13) as

$$\Big(\sum_{p<e^{1/h}}+\sum_{e^{1/h}\le p<z}\Big)\frac{a_p^2}{p^k}\bigg(\frac{\sin\big(\frac{1}{2}h\log p\big)}{\log p}\bigg)^2. \tag{6.14}$$

The first sum in (6.14) is

$$\frac{h^2}{4}\sum_{p<e^{1/h}}\frac{a_p^2}{p^k}+O\Big(h^2\sum_{p<e^{1/h}}\frac{a_p^2}{p^k}h^2(\log p)^2\Big)=\frac{h^2}{4}\log h^{-1}+O(h^2), \tag{6.15}$$

since

$$\sum_{p\le x}\frac{a_p^2}{p^k}=\log\log x+O(1) \tag{6.16}$$

as in (5.14), and in the error term, we have used $a_p^2\le 4p^{k-1}$ and

$$\sum_{p\le x}\frac{(\log p)^2}{p}=O\big((\log x)^2\big). \tag{6.17}$$

The second sum in (6.14) is

$$\ll\sum_{e^{1/h}\le p<z}\frac{\log p}{p(\log p)^3}\ll\int_{1/h}^{\infty}v^{-3}\,dv\ll h^2. \tag{6.18}$$

Hence, we obtain from (6.14), (6.15) and (6.18)

$$\sum_{p<z}\frac{a_p^2}{p^k}\bigg(\frac{\sin\big(\frac{1}{2}h\log p\big)}{\log p}\bigg)^2=\frac{h^2}{4}\log h^{-1}+O(h^2). \tag{6.19}$$

Therefore, from (6.12), (6.13) and (6.19), we get

$$(6.20)\quad \int_T^{T+H} |\eta_2(t)|^{2l}\,dt = l!Hh^{2l}\big(\log h^{-1} + O(1)\big)^l + O\big(A^l l^l h^{2l} H(\log\log T)^{l-2}\big).$$

Substituting (6.11) and (6.20) in (6.5) and using the inequality $l! \le l^{l-1}$ for $l \ge 1$, we arrive at

$$(6.21)\quad \begin{aligned} Q_{10} &:= \int_T^{T+H} \left| \int_t^{t+h} S_f(u)\,du \right|^{2l} dt \\ &= \frac{(2l)!}{l!}\left(\frac{h}{2\pi}\right)^{2l} H(\log h^{-1})^l + O(A^l l^{2l} h^{2l} H) \\ &\quad + O\big(A^l H h^{2l} l^{l-(1/2)}(\log\log T)^{l-(1/2)}\big) \end{aligned}$$

from which Theorem 2 follows.

7. Completion of the proof of the main theorem

The proofs of Theorems 3 and 4 are verbatim the same as in [5] (see Section 5 and 6 of [5]). Hence, the proof of the main theorem follows from the arguments similar to those in Section 7 of [5].

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Received 3 January 2005