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SIMPLY CONNECTED QUASIREGULARLY ELLIPTIC 4-MANIFOLDS

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Abstract. We construct a quasiregular map of R^4 onto the connected sum of $S^2 \times S^2$ with itself. The proof is based on a symmetric splitting of T^4 to two pieces together with some technique from Piergallini's article [9].

1. Introduction

An oriented and connected Riemannian n-manifold N is (K)-*quasiregularly elliptic* if there exists a nonconstant (K)-quasiregular map of R^n into N. This terminology was introduced by Bonk and Heinonen in [1] and was suggested from the discussion by Gromov in [4, pp. 63–67]. In this paper we prove the following result.

Theorem 1. *The connected sum $S^2 \times S^2 \# S^2 \times S^2$ is quasiregularly elliptic.*

Theorem 1 gives the first example of a nontrivial simply connected closed quasiregularly elliptic 4-manifold and it solves one question posed by Gromov [3, p. 200], [4, 2.41] and the author [12, p. 183].

In the other direction, a break-through was recently obtained by Bonk and Heinonen [1], namely, if N is a connected, closed, and K-quasiregularly elliptic n-manifold, then the dimension of the de Rham cohomology ring $H^*(N)$ of N has an upper bound depending only on n and K. Earlier results on the closed case were essentially based on the behavior of the first homotopy group and are contained in [5], [8], and [18]. For example, from [18, pp. 146–147] it follows, as observed in [1, Corollary 1.6] that $\dim H^1(N) \le n$. This gives for $n = 2$ and $n = 3$ the bound 2^n for $\dim H^*(N)$. This is optimal as is seen from $\dim H^*(T^n) = 2^n$ for the n-torus T^n, which equality is true for every $n \ge 2$. It is an interesting question whether one has a bound for $\dim H^*(N)$ independent of K also for $n \ge 4$. Furthermore, if this is the case, is the bound again 2^n? We refer to [1] for a more detailed discussion on the question of quasiregular ellipticity. For the theory of quasiregular mappings, see [13].

The proof of Theorem 1 is reduced to showing the existence of a branched covering $T^4 \to S^2 \times S^2 \# S^2 \times S^2$ and composing it with the projection $R^4 \to T^4$.

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An outline of the various steps is as follows. We use a 2-fold branched covering $T^2 \to S^2$ to perform the product map $F\colon T^2 \times T^2 \to S^2 \times S^2$. With the help of a handle decomposition of T^4 we split T^4 in a symmetric way into two pieces V and W such that F takes V onto $S^2 \times S^2$ with a 4-ball deleted and such that boundary goes to boundary. By a preliminary shifting of coordinates in T^4 we get similarly a map G which takes W to another copy of $S^2 \times S^2$ with a 4-ball deleted. On the common boundary M of V and W we then get two 4-fold branched coverings onto 3-spheres. The task is reduced to the construction of a branched covering $M \times [0,1] \to S^3 \times [0,1]$ which coincides with given maps on $M \times 0$ and $M \times 1$ respectively. Here we make use of the symmetry property of F and G and apply Section 2 from Piergallini's paper [9]. Piergallini later used [9] in [10] to show that each closed oriented PL 4-manifold is a 4-fold covering of S^4.

Acknowledgements. I thank Juha Heinonen for inspiring discussions connected to the subject and to the proof of Theorem 1.

2. Maps F and G

We let T^n be $[-1,1]^n$ with opposite sides identified. We define a handle decomposition of T^4 consisting of one 0-handle H^0, four 1-handles $H^1_i, i = 1,\dots,4$, six 2-handles H^2_{ij}, $1 \le i < j \le 4$, four 3-handles $H^3_i, 1,\dots,4$, and one 4-handle H^4. These are fixed as follows:

$$\begin{aligned}
H^0 &= \bigl\{x \in T^4 : |x_k| \le 1/3,\ k = 1,\dots,4\bigr\},\\
H^1_i &= \bigl\{x \in T^4 : 1/3 \le |x_i| \text{ and } |x_k| \le 1/3 \text{ for } k \ne i\bigr\},\\
H^2_{ij} &= \bigl\{x \in T^4 : 1/3 \le |x_i|, |x_j| \text{ and } |x_k| \le 1/3 \text{ for } k \ne i,j\bigr\},\\
H^3_i &= \bigl\{x \in T^4 : |x_i| \le 1/3 \text{ and } 1/3 \le |x_k| \text{ for } k \ne i\bigr\},\\
H^4 &= \bigl\{x \in T^4 : 1/3 \le |x_k|,\ k = 1,\dots,4\bigr\}.
\end{aligned}$$

We work throughout with PL maps. Therefore our constructed map in the end will be quasiregular. First we define a 2-to-1 branched cover $f\colon T^2 \to S^2$ such that the square $[0,1]^2$ is mapped onto the upper half sphere S^2_+ of S^2 and such that the extension to the rest of T^2 satisfies $f(x_1,x_2) = f(-x_1,-x_2) = r \circ f(-x_1,x_2)$, where r is the reflection in the equator of S^2. The map f is shown in Figure 1, where points a_i and $b_i = f(a_i)$, $i = 1,\dots,4$, are indicated. Then set

$$F = f \times f\colon T^2 \times T^2 \to S^2 \times S^2. \tag{2.1}$$

Let $\lambda\colon T^4 \to T^4$ be the map $(x_1,x_2,x_3,x_4) \mapsto (x_1,x_3,x_4,x_2)$ and define

$$G\colon T^4 \to S^2 \times S^2 \tag{2.2}$$

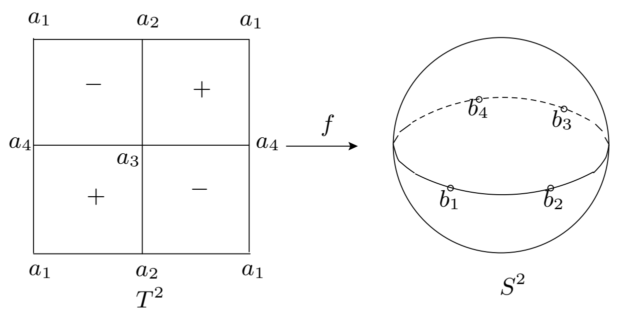

Figure 1.

to be $F \circ \lambda$ followed by the identity to another copy of $S^2 \times S^2$. The degree of F and G is clearly four.

Next let

$$\begin{aligned} X &= H^0 \cup H_1^1 \cup \cdots \cup H_4^1 \cup H_{12}^2 \cup H_{34}^2, \\ Y &= H^4 \cup H_1^3 \cup \cdots \cup H_4^3 \cup H_{13}^2 \cup H_{24}^2, \\ V &= X \cup H_{23}^2, \\ W &= Y \cup H_{14}^2. \end{aligned}$$

We first study $F \,|\, X$ and $F \,|\, V$. For this we write

$$E = \{x \in T^2 : |x_1|, |x_2| \le 1/3\}$$

and observe that fE is a disk with $f\partial E = \partial f E$. We have

$$\begin{aligned} X &= (T^2 \times E) \cup (E \times T^2), \\ T^4 \setminus X &= (T^2 \setminus E) \times (T^2 \setminus E), \\ FX &= S^2 \times S^2 \setminus U, \end{aligned}$$

where U is the open 4-ball $(S^2 \setminus fE) \times (S^2 \setminus fE) = F(T^4 \setminus X)$. The boundary of X is the 3-manifold

$$\partial X = \big((T^2 \setminus \operatorname{int} E) \times \partial E) \cup (\partial E \times (T^2 \setminus \operatorname{int} E)\big)$$

and

$$F\partial X = \big((S^2 \setminus \operatorname{int} fE) \times \partial fE) \cup (\partial fE \times (S^2 \setminus \operatorname{int} fE)\big) = \partial U. \tag{2.3}$$

Hence $F\partial X = \partial FX$, which means that $F\,|\,X$ is a closed map. Moreover, $F\,|\,\partial X\colon \partial X \to \partial U$ is a branched cover.

In (2.3) we see the 3-sphere ∂U splitted into two solid tori. The image of the branch set of $F\,|\,\partial X$ (notice the difference of terminology in literature) is the union of the following six circles:

$$s_i = b_i \times \partial fE, \quad S_i = \partial fE \times b_i, \quad i = 1,2,4.$$

We write $H^2_{23} = A_2 \times A_1$, where

$$\begin{aligned} A_1 &= \{x \in T^2 : |x_2| \le 1/3 \le |x_1|\}, \\ A_2 &= \{x \in T^2 : |x_1| \le 1/3 \le |x_2|\}. \end{aligned}$$

Clearly $\partial FH^2_{23} = F\partial H^2_{23}$, so $F\,|\,H^2_{23}$ is closed. Moreover, FH^2_{23} is a 4-ball contained in $\overline{U}$. The common boundary of X and H^2_{23} is

$$X \cap H^2_{23} = \bigl(A_2 \times (E \cap A_1)\bigr) \cup \bigl((E \cap A_2) \times A_1\bigr), \tag{2.4}$$

which is a solid torus. The map F takes this onto the 3-ball $V_1 \cup V_2 \subset \partial U$ where V_1 and V_2 are the following closed 3-balls:

$$\begin{aligned} V_1 &= fA_2 \times f(E \cap A_1), \\ V_2 &= f(E \cap A_2) \times fA_1. \end{aligned}$$

These have disjoint interior and

$$V_1 \cap V_2 = f(E \cap A_2) \times f(E \cap A_1)$$

is a disk. The circle s_2 has the arc $l^1_2 = b_2 \times f(E \cap A_1)$ in V_1 and the circle S_4 the arc $L^2_4 = f(E \cap A_2) \times b_4$ in V_2.

We have

$$Y = (C_2 \times C_2) \cup (C_1 \times C_1),$$

where

$$C_i = \{x \in T^2 : 1/3 \le |x_i|\}, \quad i = 1,2.$$

The common boundary of Y and H^2_{23} is the solid torus

$$Y \cap H^2_{23} = \bigl(A_2 \times (C_2 \cap A_1)\bigr) \cup \bigl((C_1 \cap A_2) \times A_1\bigr). \tag{2.5}$$

The map F takes this onto the 3-ball $V_3 \cup V_4 \subset \overline{U}$, where

$$\begin{aligned} V_3 &= fA_2 \times f(C_2 \cap A_1), \\ V_4 &= f(C_1 \cap A_2) \times fA_1. \end{aligned}$$

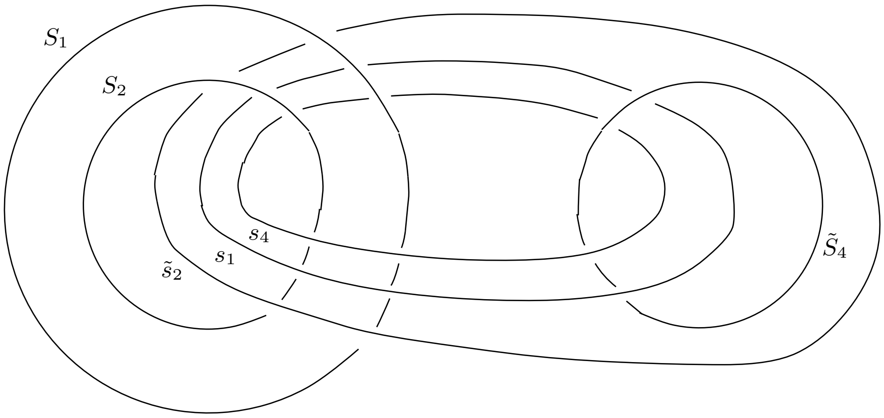

Figure 2. Link L.

The intersection is now the disk

$$V_3 \cap V_4 = f(C_1 \cap A_2) \times f(C_2 \cap A_1).$$

The boundary $\partial H^2_{23} \simeq S^3$ is the union of the solid tori given in (2.4) and (2.5) with a common 2-torus. The map F takes this 2-torus onto the common 2-sphere of $V_1 \cup V_2$ and $V_3 \cup V_4$.

We get $F\,|\,\partial V$ by replacing in $F\,|\,\partial X$ the part $F\,|\,X \cap H^2_{23}$ by $F\,|\,Y \cap H^2_{23}$. In particular, $F(X \cap H^2_{23}) = V_1 \cup V_2$ will be replaced by $F(Y \cap H^2_{23}) = V_3 \cup V_4$. We observe that $F\,|\,V$ is a closed map and that $F\,|\,\partial V$ is a branched cover onto a 3-sphere.

To get a picture of the link diagram of the image of the branch set of $F\,|\,\partial V$ we observe the following. The image $F(H^2_{23})$ gives a cobordism between $(V_1 \cup V_2, l^1_2 \cup L^2_4)$ and $(V_3 \cup V_4, l^3_2 \cup L^4_4)$. This affects the link diagram for $F\,|\,\partial X$ by changing the crossing between s_2 and S_4 so that the resulting link L for $F\,|\,\partial V$ has the form given in Figure 2. There we have denoted the modified s_2 and S_4 by $\tilde{s}_2$ and $\widetilde{S}_4$ respectively.

To study $G\,|\,Y$ and its extension $G\,|\,W$ we write

$$E' = \{x \in T^2 : |x_1|, |x_2| \geq 1/3\}.$$

We have

$$GY = S^2 \times S^2 \setminus U',$$

where U' is the open 4-ball $(S^2 \setminus fE') \times (S^2 \setminus fE') = G(T^4 \setminus Y)$. Instead of (2.3) we have in an analogous way

$$G\partial Y = \big((S^2 \setminus \operatorname{int} fE') \times \partial fE'\big) \cup \big(\partial fE' \times (S^2 \setminus \operatorname{int} fE')\big) = \partial U'.$$

The image of the branch set of $G\,|\,\partial Y$ now consists of the circles

$$s'_i = b_i \times \partial fE', \quad S'_i = \partial fE' \times b_i, \quad i = 2, 3, 4.$$

A treatment similar to the one for F shows that the image of the branch set of $G\,|\,\partial W$ is the link L' given in Figure 3. There $\tilde{s}'_4$ and $\widetilde{S}\,'_4$ are obtained from s'_4 and S'_4 when $G(Y\cap H^2_{14})$ is replaced by $G(X\cap H^2_{14})$. We denote the set $\partial V=\partial W$ by M. The sets V and W induce opposite orientations on M, and $F\,|\,V$ and $G\,|\,W$ induce orientations on FM and GM.

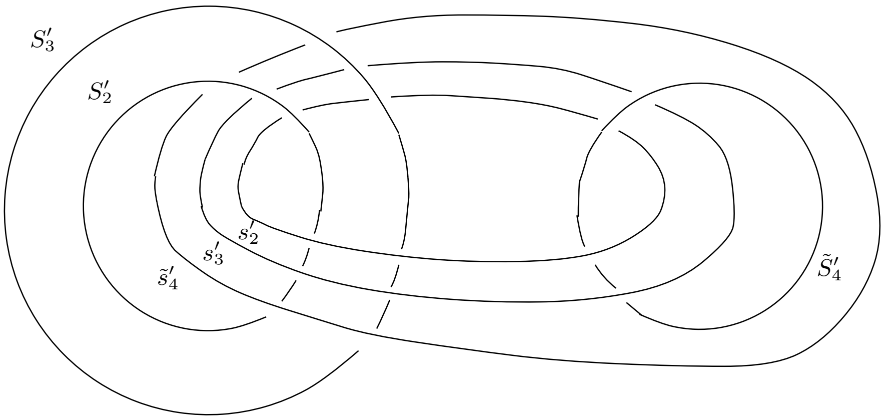

Figure 3. Link L'.

3. Monodromies and change to simple maps

In order to relate $F\,|\,M$ and $G\,|\,M$ as explained in the introduction we will in this section present preparatory material in order to apply Piergallini's paper [9].

To present the monodromy of $F\,|\,M$ we choose $q\in FM\simeq S^3$ for which $F^{-1}(q)=\{p_1,p_2,p_3,p_4\}$, where

$$\begin{aligned}p_1&=(-1/3,0,0,-1/3),\\ p_2&=(1/3,0,0,-1/3),\\ p_3&=(-1/3,0,0,1/3),\\ p_4&=(1/3,0,0,1/3).\end{aligned}$$

Let α_i, β_i, $i=1,2,4$, be paths in FM with base point q shown in Figure 4. We observe that these paths stay in $F(M\cap\partial X)$ so that we see the various lifts by looking at $F\,|\,\partial X$. When we identify p_j with j in the presentation of elements of the symmetric group of degree four, the monodromy map takes these paths as follows:

$$\alpha_i\mapsto(12)(34),\quad \beta_i\mapsto(13)(24),\quad i=1,2,4.$$

Paths α_i, β_i represent a part of the Wirtinger generators of $\pi_1(FM\setminus L)$ (see [14, p. 56]). By studying the behavior at the bridges of L we find that the other Wirtinger generators give permutations so that permutations stay constant

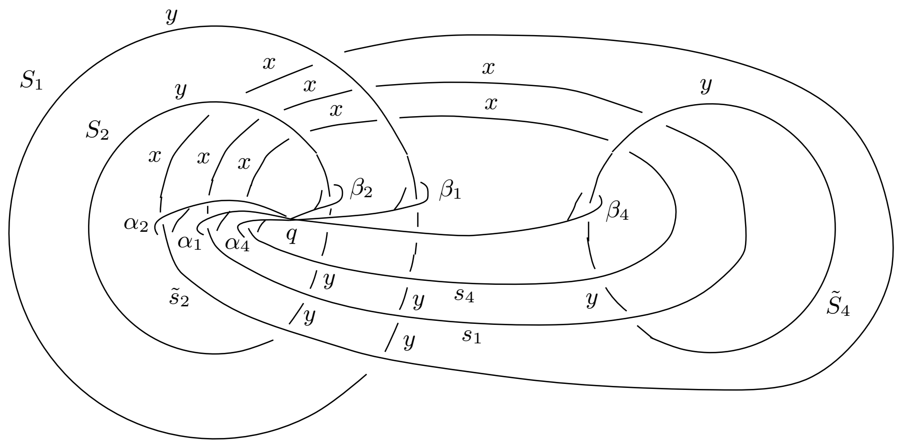

Figure 4. Link L.

for each circle of L. In Figure 4 we have indicated this by placing the corresponding permutations at each subarc and by using abbreviations $x = (12)(34)$, $y = (13)(24)$.

The monodromy of $G\,|\,M$ is similar. This time we take $q' \in GM \simeq S^3$ for which $G^{-1}(q') = \{p'_1, p'_2, p'_3, p'_4\}$, where

$$\begin{aligned} p'_1 &= (-1/3, 1, 1, -1/3), \\ p'_2 &= (1/3, 1, 1, -1/3), \\ p'_3 &= (-1/3, 1, 1, 1/3), \\ p'_4 &= (1/3, 1, 1, 1/3), \end{aligned}$$

and paths α'_i, β'_i, $i = 2, 3, 4$, in GM with base point q' shown in Figure 5. Then we can repeat word by word for the link L' what was said for the link L above. Without loss of generality we can therefore assume that $F\,|\,V$ and $G\,|\,W$ are maps onto two copies of $S^2 \times S^2 \setminus B^4$ with a common boundary S^3 where L and L' are identified together with their monodromies. Because of this latter property the boundary maps $F\,|\,M$ and $G\,|\,M$ are conjugated by a homeomorphism of M. We cannot use directly this fact because such a homeomorphism is not necessarily isotopic to identity. The solution to this problem is given in the next section by the technique presented in [9, Section 2]. The rest of this section is devoted to perform moves for the maps that result into so called normalized forms. Note that the induced orientations on S^3 are opposite, hence $F\,|\,M\colon M \to S^3$ and $G\,|\,M\colon M \to S^3$ are orientation preserving because M is equipped with opposite orientations for $F\,|\,M$ and $G\,|\,M$.

By isotopy on S^3 we then move L together with monodromy labelling to the link presented in Figure 6. There we have decomposed S^3 into two 3-balls B_1

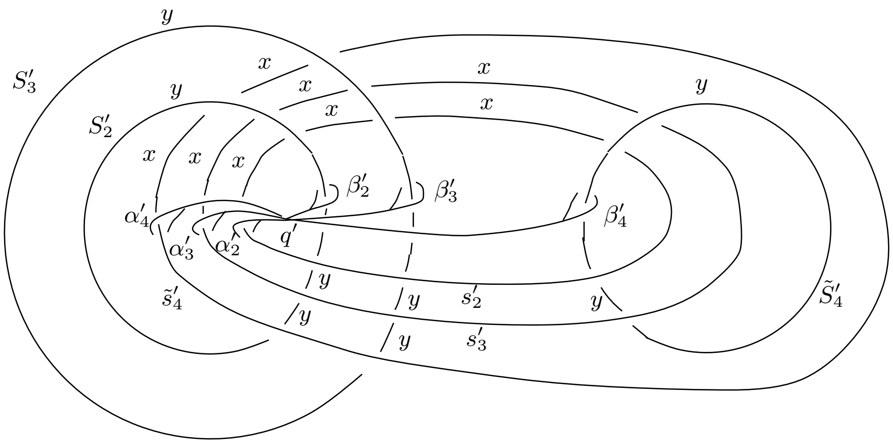

Figure 5. Link L'.

and B_2 and a ring $R = S^2 \times [0,1]$. The monodromies stay in our case fixed on circles in the link during the isotopy. The part in R of the link is called a *braid.* We denote $F\,|\,M$ and $G\,|\,M$ followed by the isotopies described above by φ and ψ respectively.

By slightly perturbing maps φ and ψ in small tubes arround the preimages of the circles of the link in Figure 6 we change φ and ψ to simple maps, i.e., the inverse of a point consists of at least three points. The monodromy is then represented by transpositions. Each arc of the link in B_1 and B_2 is then replaced by two arcs. In Figure 7 we see the part in B_1 of the new link together with the corresponding transpositions. There are simple rules for the behavior of the transpositions at each bridge of the braid. To see this we look at a bridge of the braid with transpositions u, v, w as in Figure 8. Then u and v determine w by the following rules. If u and v have no common index or if $u = v$, then $w = v$. If $u = (ij)$ and $v = (kj)$ with $k \neq i$, then $w = (ik)$.

Next we perform isotopies on S^3 in order to rearrange the braid into a so called normalized form. First we move each pair of arcs in B_1 and B_2 coming from a single circle in Figure 6 as shown in Figure 9 for the first pair from left in B_1. Then, by successive use of the isotopies presented in [9, Figure 8, p. 910] in a way that only three indices are taken at a time, we obtain a new braid diagram where the parts B_1 and B_2 are shown in Figure 10. This we call a *normalized form.* We call the new φ and ψ by ξ and η respectively.

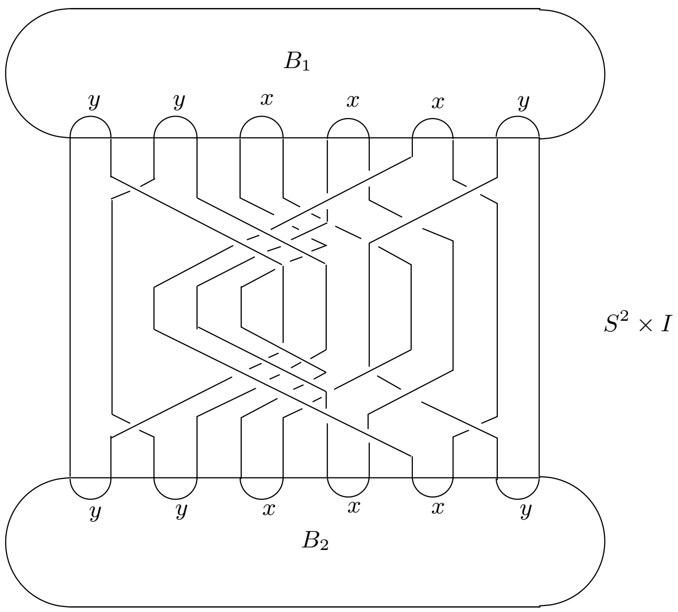

Figure 6.

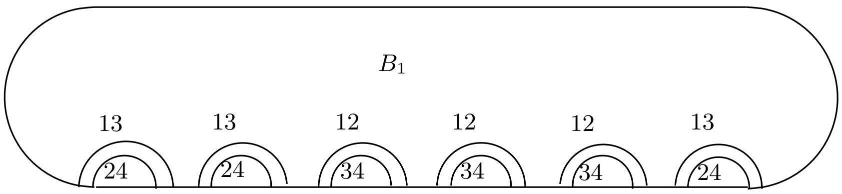

Figure 7.

4. Stable equivalence and its realization

An easy argument shows that the preimages

$$H_1 = \xi^{-1}B_1, \quad H_2 = \xi^{-1}B_2$$
$$H_1' = \eta^{-1}B_1, \quad H_2' = \eta^{-1}B_2$$

are handlebodies. Furthermore, $P = \xi^{-1}R$ and $P' = \eta^{-1}R$ have product presentations

$$P = \partial H_1 \times [0,1],$$
$$P' = \partial H_1' \times [0,1],$$

such that $\xi^{-1}(S^2 \times t)$ and $\eta^{-1}(S^2 \times t)$ are identified with $\partial H_1 \times t$ and $\partial H_1' \times t$ respectively. Maps ξ and η define Heegard splittings $(H_1,\ P \cup H_2)$ and $(H_1',\ P' \cup H_2')$ of M with same genus g.

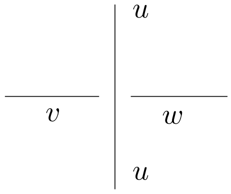

Figure 8.

Next we fix a handlebody $T_g \subset R^3$ of genus g with boundary surface F_g and a standard map s with degree four of T_g onto B_1 (with B_1 deformed slightly) as shown in Figure 11. Without loss of generality we may assume that there are homeomorphisms $\theta_i\colon H_i \to T_g$ and $\theta_i'\colon H_i' \to T_g$, $i = 1, 2$, such that $\xi \,|\, H_1 = s \circ \theta_1$, $\eta \,|\, H_1' = s \circ \theta_1'$, $\xi \,|\, H_2 = \varrho \circ s \circ \theta_2$, $\eta \,|\, H_2' = \varrho \circ s \circ \theta_2'$, where $\varrho\colon B_1 \to B_2$ is the obvious homeomorphism. In [9, Figure 4, p. 906] a map similar to s is shown for degree three.

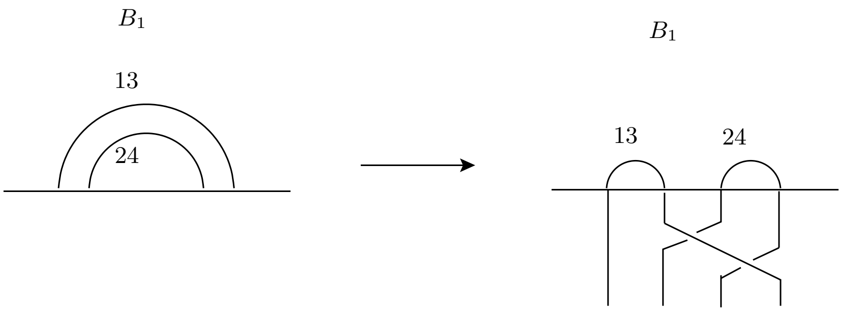

Figure 9.

The braid in Figure 10 can be realized by an isotopy on ∂B_1. Such an isotopy gives a homeomorphism $h\colon F_g \to F_g$ through lifting by s.

Let W be a closed oriented 3-manifold and (W_1, W_2) a Heegard splitting of W. Let $U \subset W_2$ be a disk such that ∂U consists of two arcs $\Gamma_1 \subset \partial W_1$ and $\Gamma_2 \subset W_2$. We form a new Heegard splitting of W by adding to W_1 a closed tubular neighborhood V of Γ_2 so that $(W_1 \cup V, \overline{W_2 \setminus V})$ is a Heegard splitting of one genus higher. A Heegard splitting (Z_1, Z_2) is obtained from (W_1, W_2) by *stabilization* if it is obtained by a finite number of steps as above.

Given two Heegard splittings (W_1, W_2) and (W_1', W_2') of W, then they are *stably equivalent*, that is, there exist stabilizations of these to splittings (Z_1, Z_2) and (Z_1', Z_2') of same genus and a homeomorphism $\zeta\colon W \to W$ isotopic to identity such that $\zeta Z_i = Z_i'$, $i = 1, 2$. This result is known as the stabilization theorem by Reidemeister [11] and Singer [16]. See the paper [19] by Waldhausen. Later proofs are given by Craggs [2] and by Lei [6].

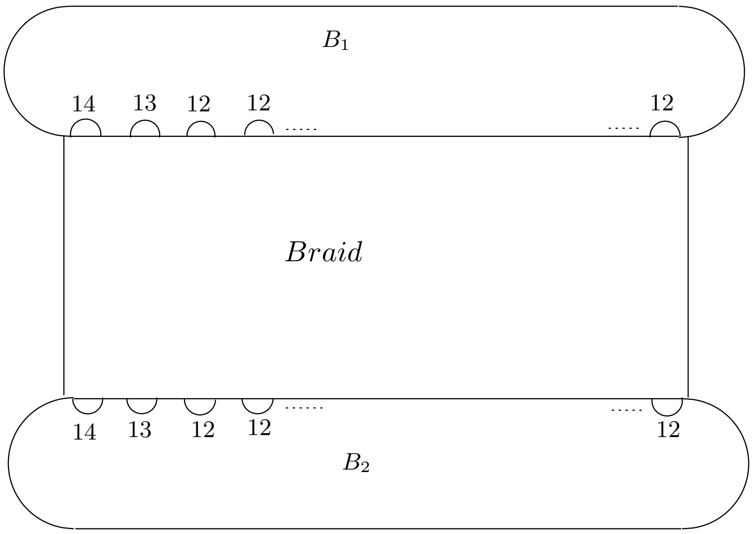

Figure 10. Normalized form.

We apply the stabilization theorem to Heegard splittings $(H_1,\, P \cup H_2)$ and $(H_1',\, P' \cup H_2')$. We may assume that in the first step of the stabilization process the new map corresponding to s takes the form obtained by adding one (12)-part to the braid in Figure 10 as shown in Figure 12. For this, see [9, Figure 9] and [15, 3.1]. We continue similarly. We retain the original notation for the stabilized case, like H_i, H_i', g, s etc. The map $h\colon F_g \to F_g$ is now a lift with respect to $s\,|\,F_g\colon F_g \to \partial B_1$ of some isotopy realizing the braid diagram D for the stabilized case corresponding to the braid diagram in Figure 10. So we have an isotopy $\zeta^t\colon M \to M$, $t \in [0,1]$, with $\zeta^0 = \mathrm{id}$, $\zeta^1 H_1 = H_1'$, $\zeta^1(P \cup H_2) = P' \cup H_2'$. We may also assume $\zeta^1 H_2 = H_2'$ and $\zeta^1 P = P'$.

Next we transfer ζ^1 to T_g via our maps $\theta_i\colon H_i \to T_g$, $\theta_i'\colon H_i' \to T_g$, and set $\zeta_i = \theta_i' \circ \zeta^1 \circ \theta_i^{-1}$, $i = 1, 2$. Then we have the following diagram:

$$\begin{tikzcd}
T_g \arrow[d, "\zeta_1"'] & \supset & F_g \arrow[r, "h"] \arrow[d, "\zeta_1"'] & F_g \arrow[d, "\zeta_2"] & \subset & T_g \arrow[d, "\zeta_2"] \\
T_g & \supset & F_g \arrow[r, "h"'] & F_g & \subset & T_g
\end{tikzcd}$$

This corresponds to the diagram in [9, p. 910] where instead of one map h one has two different maps. We emphasize here that our map h in both places in the

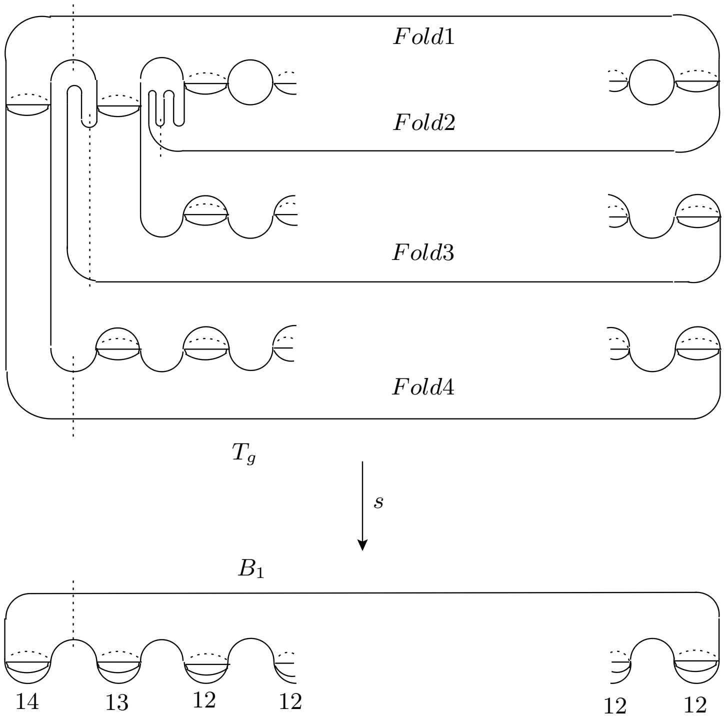

Figure 11.

above diagram is the result of lifting the *same* braid diagram D by s.

Our final task is to find moves that correct the difference presented by homeomorphisms ζ_1 and ζ_2. The idea is to replace η by another map $\tilde{\eta}$ through adding braids D_1 and D_2 to top and bottom of D that lift by s and $\varrho \circ s$ to homeomorphisms ζ_1 and ζ_2. Suppose we have done this. Let λ_1^t and λ_2^t, $t \in [0,1]$, be isotopies on B_1 and B_2 realizing the braids D_1 and D_2 on ∂B_1 and ∂B_2. We use the isotopies ζ^t, λ_1^t, and λ_2^t to define a level preserving branched covering $\sigma\colon M \times [0,1] \to S^3 \times [0,1]$ of degree four such that $\sigma^0 = \xi$, $\sigma^1 = \tilde{\eta}$ with the notation $\sigma^t(x) = \mathrm{pr}_1\big(\sigma(x,t)\big)$. In particular,

$$\sigma^t \,|\, \zeta^t H_1 = \lambda_1^t \circ s \circ \theta_1 \circ (\zeta^t)^{-1} \,|\, \zeta^t H_1,$$
$$\sigma^t \,|\, \zeta^t H_2 = \lambda_2^t \circ \varrho \circ s \circ \theta_2 \circ (\zeta^t)^{-1} \,|\, \zeta^t H_2,$$

$t \in [0,1]$. To obtain the braids D_1 and D_2 we apply [9, Section 2] in a slightly modified form.

Following the notation in [9] let $M^*(g)$ be the subgroup of the mapping class group $M(g)$ of F_g whose elements extend to T_g. The isotopy classes of ζ_1 and ζ_2 belong to $M^*(g)$. The task is to find moves that result in adding braids that generate, through lifting by s, homeomorphisms of F_g which give generators for $M^*(g)$. Generators for $M^*(g)$ are given in [17] and listed in [9, p. 912]. The

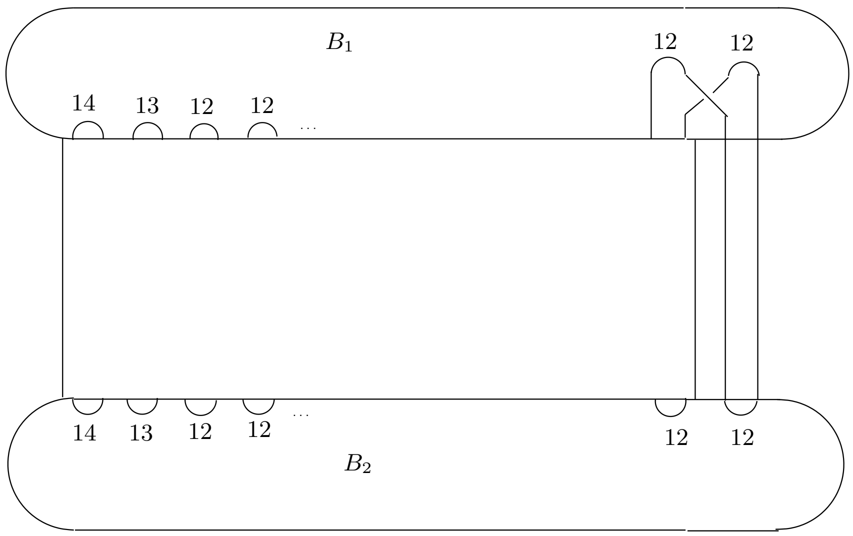

Figure 12.

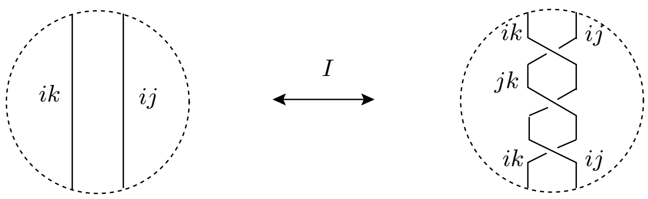

Figure 13. Move I.

solution for the case of degree three is given in [9, pp. 912–916]. By looking at the steps given there, we see that we can in our case be limited to the restriction of s to the right of the dashed lines in Figure 11. The reason is that the braid generator involving the transposition (14) is not needed and that the parts to the left of the dashed lines remain unchanged under braids that we need. We also observe that the action on the 4th sheet is trivial for our needed braids: They lead to compositions of disk twists in that sheet. According to [9] our conclusion therefore is that we need to use only isotopy on S^3 with transposition labelling (usually called colored isotopy in the case of degree three) and move I (Figure 13) according to the terminology in [9], see third paragraph on p. 912 in [9]. Details how move I is sitting in our map w is presented in [7]. Note that move I is called move $C^{\pm}$ in [7] and [10]. Observe that move I is local and is restricted to a part where the degree is three. This finishes the proof of Theorem 1.

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