Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 31, 2006, 31–38

SIMPLE GEODESICS ON SURFACES OF GENUS 2

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Abstract. We derive various identities satisfied by the spectrum of lengths of simple geodesics of a surface of genus 2.

1. Introduction

In previous articles [McS3], [McS2] we demonstrated the existence of several curious identities satisfied by the lengths of simple geodesics on a punctured torus T equipped with a hyperbolic structure, i.e., a complete metric of curvature -1. These identities took the form of infinite series over the collection of all simple geodesics, namely (*cusp identity*):

$$\sum_{\gamma} \frac{1}{1+e^{|\gamma|}} = \frac{1}{2},$$

where the sum extends over all closed simple geodesics and (*Weierstrass identities*):

$$\sum_{\gamma} \sin^{-1}\left(\operatorname{sech}\left(\frac{|\gamma|}{2}\right)\right) = \frac{\pi}{2},$$

where the sum runs over all closed simple geodesics in the same class $\mathbf{Z}/2\mathbf{Z}$-homology class. Recall that $H_1(T,\mathbf{Z}/2\mathbf{Z}) \simeq \mathbf{Z}/2\mathbf{Z}+\mathbf{Z}/2\mathbf{Z}$; each closed simple geodesic γ on M belongs to one of the three non-trivial $\mathbf{Z}/2\mathbf{Z}$-homology classes and each non-trivial homology class contains infinitely many closed simple geodesic representatives.

These identities are proved using a description of *Birman series set* at a Weierstrass point ω. Let us recall the necessary definitions from [McS2]. The Birman series set is the point set of all complete simple geodesics on the surface. Every punctured torus admits an order 2 isometry called the *elliptic involution*, denoted J, with the property that if a is an element of $H_1(T,\mathbf{Z})$, and $J_\star\colon H_1(T,\mathbf{Z}) \to H_1(T,\mathbf{Z})$ the induced homomorphism, then $J_\star(a) = -a$. The elliptic involution J has exactly three fixed points; we call these the *Weierstrass*

2000 Mathematics Subject Classification: Primary 57M50.

The author was supported by an EC grant no. ERBCHGCT930251 while preparing this work.

points. Each closed simple geodesic passes through exactly two of the three Weierstrass points and which pair it passes through depends only on its $\mathbf{Z}/2\mathbf{Z}$ homology class.

A natural question is: are there identities for length functions for surfaces of higher genus surfaces equipped with hyperbolic structures. Here we answer this question for the surface of genus 2 using a (well-known) trick.

Every surface of genus 2 equipped with a hyperbolic structure does have an isometry of order 2 called the *hyper-elliptic* involution; note that the automorphism group of the generic surface of genus $g \geq 3$ is trivial. We examine the relationship between the fixed points of this involution, which we shall still call Weierstrass points, and the closed simple geodesics of the surface of genus 2. We note that Haas and Susskind [SH] have already investigated certain aspects of the hyper-elliptic involution in genus 2 with respect to simple geodesics.

We develop a picture of the Birmann series set in a neighborhood of a Weierstrass point analogous to that in [McS1] (actually we will not work with the Birmann series set itself but with the set of vectors in the tangent space at ω—we think of this as being the "infinitesimal BS set"). As in [McS1] we will see that each *gap* (see below the definition) corresponds to a certain geometric object—*a dual class*—and that the size of the gap is a function of the lengths of a certain pair of closed geodesics. Let ω be a Weierstrass point of M; note that there are exactly 6 Weierstrass points. Define the *Weierstrass class* of ω to be the set of all closed simple geodesics which pass through ω. Define the *dual class of* ω to be the collection of pairs (γ, δ) of disjoint simple closed geodesics in $M - \omega$ such that γ is non-separating in M, δ separating and furthermore ω and γ are in the same component of $M - \delta$.

We shall show how to construct gaps in the infinitesimal Birman series set at the point ω and how to calculate the size of these gaps in terms of the lengths of simple geodesics. From this we shall deduce:

Theorem 1.1. *Let M be a surface of genus 2 and fix a dual class A then we have the following identity*:

$$\sum_{(\gamma,\delta)\in A} \tan^{-1}\left(\frac{2\cosh(|\gamma|/2 - |\delta|/4)}{\sinh(|\gamma|) + \sinh(|\delta|/2)}\right) = \frac{\pi}{2}.$$

Remarks. Identities of this type have recently been proved by a different method by Ser Tan Peow [ZTW]. Recently Mirzikhani [Mi] has used identities to compute Weil–Petersson volumes of moduli space. Sakuma et al. [AMS], [S] following Bowditch [Bo] have used a version of these identities for complex lengths to study functions on the quasi-Fuchsian deformation space of surface groups.

We remark that there is a simple generalisation of the Weierstrass identities (which we do not prove here but include for comparison) to a torus with a hole (that is a (hyperbolic) torus minus an open disc which has a convex core bounded by a closed geodesic, δ):

$$\sum_{\gamma} \tan^{-1}\left(\frac{\cosh(|\delta|/4)}{\sinh(|\gamma|/2)}\right) = \frac{\pi}{2},$$

where, once again, the sum is over all closed simple geodesics in the same $\mathbf{Z}/2\mathbf{Z}$ homology class.

Note that this identity can be used to "recover" the identities for the punctured torus and the 4-punctured sphere in the following way. Fix a Weierstrass point on M a surface of genus 2. To obtain the identity for the punctured torus choose a separating simple geodesic δ and consider a sequence of metrics on M such that the length of δ tends to 0 (one says that δ *gets pinched off*.) The pointed metric space (M, ω) converges (after passing to a subsequence if necessary) to a once punctured torus and, moreover, any hyperbolic punctured torus may be obtained from M by choosing the sequence of metrics suitably. Every separating simple geodesic $\delta' \neq \delta$ meets δ transversely and so by the collar lemma the length of δ' tends to infinity as δ gets pinched off. One sees (formally) that all terms in the above series vanish except for those involving our fixed δ; since the length of δ tends to 0 the remaining terms are of the form

$$\tan^{-1}\left(\frac{1}{\sinh(\gamma/2)}\right),$$

thus the series for M "converges" to the series for the punctured torus. To obtain the series for a 4-punctured sphere one must choose a pair of disjoint non-separating geodesics which do not pass through the Weierstrass point and then pinch these off, the limit space in this case is a 4-punctured sphere. We emphasise that this process of deduction is *formal* as we checked convergence of the series term by term though this process can be made rigorous using estimates on the size of the Margulis tube round a short geodesic [McS4].

Acknowledgements. I thank Troels Jorgensen, Igor Rivin and Ser Tan Peow for useful comments and the necessary encouragement to finally publish this result.

2. Preliminaries

We collect some definitions and elementary results from the theory of surfaces which serve as a basis for the sequel; for details see [CEG]. Throughout M denotes some surface of genus 2 together with a complete hyperbolic structure which one can construct as follows. The fundamental group of M has the standard presentation $a, b, c, d : aba^{-1}b^{-1}cdc^{-1}c^{-1}$. By the uniformisation theorem the universal cover of M is the upper half plane $\mathbf{H}^2$. The half plane admits a hyperbolic metric $|dz|/\operatorname{Im} z$; the covering transformations for M are isometries for this metric. This gives a representation $\varrho\colon \pi_1(M) \to \operatorname{Isom}(\mathbf{H}^2)$ such that the image is a discrete subgroup and one identifies M with the quotient $\mathbf{H}^2/\varrho$ the latter being a surface with a hyperbolic metric inherited from $\mathbf{H}^2$. The *Teichmueller space* of a surface is the space of all marked hyperbolic structures [A]; for M this means that not only are we given the representation $\varrho\colon \pi_1(M) \to \mathrm{Isom}(\mathbf{H}^2)$ (up to conjugation) but also the images $\varrho(a), \varrho(b), \varrho(c), \varrho(d)$; $a, b, c, d \in \pi_1$ the standard generators.

There are, up to homeomorphism, two classes of closed simple curve on M. If the complement of the curve is connected, and in this case it is invariably a torus minus two disjoint discs, then the geodesic is said to be *non-separating.* On the other hand, if the complement of the curve is disconnected, i.e. it consists of a pair of tori each minus a disc, then the geodesic is said to be *separating.* Evidently, by the classification of surfaces, the two classes are inequivalent under homeomorphism of M.

Construction of the hyperelliptic involution. We now show how to construct the hyperelliptic involution, J. Choose a separating geodesic δ and cut along it. Each of the two tori, T_1, T_2, in the complement admits an (elliptic) involution J_1, J_2 respectively. The restriction of J to T_1 (T_2, respectively) is J_1 (J_2, respectively). One checks that on reglueing this defines a map on the whole of $M = T_1 \cup T_2 \cup \delta$. It is easy to see from this construction, that the induced map on the homology J_* is $[v] \mapsto -[v]$ whatever the choice of δ and so, by Serre's lemma, the resulting map J is independent of the choice of δ.

We list the principal properties of J (see [SH]) which concern us here;

- J has six fixed points—the *Weierstrass points* of the surface of genus 2. The quotient surface M/J is conformally equivalent to a sphere with six cone points of angle π.
- No separating simple geodesic meets a Weierstrass point, whereas every non-separating simple geodesic passes through exactly two Weierstrass points.
- Every unoriented closed simple geodesic is mapped to itself by J. More precisely, each oriented separating geodesic is mapped to itself, whilst each oriented non-separating geodesic, γ, is mapped to $-\gamma$.

Finally, let γ be a complete simple geodesic, which passes through some Weierstrass point, ω, but which is not closed; since ω is a Weierstrass point the hyperelliptic involution leaves γ invariant as a pointset but swaps its ends.

3. Closest approach to the boundary

Consider a torus with a hole T. One knows from elementary hyperbolic geometry that the convex core of such a surface is bounded by a single simple closed geodesic, the *boundary geodesic*, denoted δ. Evidently no closed geodesic may cross δ though there are closed geodesics which approach arbitrarily close. This second observation is no longer true if we restrict to simple geodesics. That is, as was observed by Haas [H], there is a collar (i.e. a regular neighborhood) around δ which meets no other closed simple geodesic; we call the complement of this collar the *Haas core* of the torus.

To see that such a collar exists take the convex core, choose a simple geodesic γ and cut along it—the resulting surface is a pair of pants with two boundary components of equal length ($= |\gamma|$) and another of length $|\delta|$. One can cut this pants into four congruent right angled pentagons (see Figure 1) (see Buser [Bu]). The minimal distance between γ and δ is just the length of the common perpendicular, which is the length of the side marked CP lying between the side of length $\delta/4$ and $\gamma/2$. The trigonometry of the right angled pentagon yields that:

$$\cosh CP = \coth(\delta/4)\coth(\gamma/2).$$

So the minimum distance between γ and δ is bounded below by $\cosh^{-1}\coth(\delta/4)$.

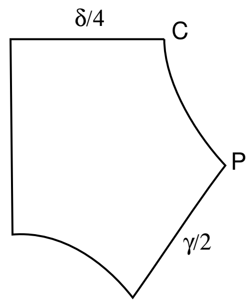

Figure 1.

4. Finding gaps

Finding gaps in the infinitesimal BS-set on a surface of genus 2 is a little more problematic than in the cases previously treated (compare [McS3], [McS2]). We give a construction as follows (we prove below that this is a genuine gap): choose a Weierstrass point ω and a separating geodesic δ. The separating geodesic does not go through ω and when we cut along it we find that the component containing ω is a torus with a hole. Choose any closed simple geodesic γ in this component which does not pass through our Weierstrass point. There are exactly two complete geodesics, γ^+, γ^-, which pass through ω and which spiral to γ. We note that, as in [McS2], the geodesics γ^+, γ^- are each tangent to the boundary of the Haas core. There are also exactly two complete geodesics δ^-, δ^+ through ω which spiral to δ and are disjoint from γ. Our gap is bounded by γ^+ and δ^+. Formally:

Theorem 4.1. *Let ω be a Weierstrass point on a surface of genus 2 and v a unit tangent vector at ω. If the geodesic tangent to v is not simple then there exists a pair of simple geodesics γ^+, δ^+ such that for every simple geodesic μ which passes through ω the unit tangent vector of μ at ω is separated from v by the tangent vectors to γ^+, δ^+. Furthermore*

(1) *γ^+ spirals to a closed simple geodesic non-separating geodesic γ.*
(2) *δ^+ spirals to a closed simple separating geodesic δ.*
(3) *γ, δ are disjoint.*

We present below a short argument which shows that this is indeed a gap; that these are the only gaps can be seen by modifying the methods described in [McS3] as follows. The key step is to show that a version of Theorem 13 is true. Let γ be a complete simple geodesic passing through ω and suppose that the ω-limit set of γ is an irrational lamination λ (that is λ is not a closed simple geodesic). To show that the gaps described above are the only gaps one must show that for each such lamination λ the corresponding geodesic γ can be *approximated from both sides* by closed simple geodesics passing through ω. The version of Theorem 13 that we require can be stated formally:

Theorem 13$'$. *If v is tangent to γ at ω then it is not isolated nor a boundary point in the infinitesimal BS-set.*

As before the difficulty is in showing the v is not a boundary point: this can be proved by modifying the proof of Theorem 13 [McS3] appropriately. We list the key points:

– Replace β' by a small geodesic segment leaving ω and perpendicular to γ.
– For each intersection point $y \in \gamma \cap \beta'$ define the quantity $\psi(y)$ to be the distance from ω to y along β'.
– Define *closest approach, joins* and *doublejoins* using ψ. One must modify the "tortuous path" used in Theorem 13 by replacing the horocyclic segment $(\beta', \gamma'$ in Figure 13) by a suitable short arc.

Once this is done the argument used in Theorem 13 proceeds as before.

Now we explain why the construction presented earlier yields gaps. We define a *wedge* to be a positive cone in the tangent space at some Weierstrass point ω. We say a wedge lies between two geodesics which meet at ω if and only if the tangent vectors to these geodesics are extremal vectors in the wedge. Note that we only have to verify that the wedge between γ^+ and δ^+ contains no vectors tangent to closed simple geodesics as the closed simple geodesics are dense in the BS-set.

Consider a closed simple geodesic, μ, passing through ω; there are just two ways in which this geodesic might behave. Firstly, if μ does not meet δ then it stays inside the torus with a hole bounded by δ and containing ω. Arguments similar to those in [McS2] apply and we see that μ never meets some wedge, denoted A, between γ^+, γ^-. Secondly, if μ is a simple closed geodesic passing through ω and with a tangent vector inside this wedge then it must meet δ (since γ^+, γ^- are tangent to the boundary of the Haas core and any closed geodesic entering the Haas collar either crosses δ or has a self intersection). Since γ is simple it must initially lie in a subwedge, B, contained in A and bounded by δ^+, δ^-. In summary: if a geodesic is closed simple ($\neq \gamma$) and furthermore does not meet the chosen separating geodesic δ then it must meet γ and so is disjoint from A; if a geodesic is closed simple and does not meet γ then it meets δ and furthermore initially it lies in A, therefore in the subwedge B.

The complement of this B in A consists of the union of two of the gaps described in the construction above. □

It remains to calculate the size of the two gaps in $B - A$ in terms of the lengths of δ and γ:

Lemma 4.2 (size of a gap). *The angle of each of the gaps appearing in the statement of Theorem* 4.1 *is*

$$\tan^{-1}\left(\frac{2\cosh(\gamma/2-\delta/4)}{\sinh(\gamma)+\sinh(\delta/2)}\right),$$

where γ, δ *are as in the statement of Theorem* 4.1.

Proof. We will first calculate the size of the wedge A. One cuts along δ and decomposes the resulting torus into four congruent right angled pentagons as before; we embed the pentagon in $\mathbf{H}^2$. The sides are labelled clockwise by their lengths $\delta/4$, x, $\gamma/2$, a, a'; the Weierstrass point ω lies between a and a'. One can represent γ^+ (respectively δ^+) on this diagram by a semi-infinite line leaving ω crossing the pentagon and then approaching the extension of the side labelled $\gamma/2$ (respectively $\delta/4$) asymptotically.

The extension of the side $\gamma/2$, the geodesic γ^+ and the side a form a right angled triangle so one has:

$$\tan\theta = 1/\sinh a,$$

where θ denotes the angle between γ^+ and a. Similarly, the extension of the side $\delta/4$, the geodesic δ^+ and the side a' form a right-angled triangle for which:

$$\tan\theta' = 1/\sinh a',$$

where θ' denotes the angle between δ^+ and a'. It is worth noting at this point that the size of the wedge B (see above) is $\pi - 2\theta$ and that the size of A is $2\theta'$. Since $B-A$ in fact consists of two gaps we need to calculate half the size of $B-A$ that is $\pi/2-\theta-\theta'$.

One has

$$\sinh(\gamma/2)\sinh(a) = \cosh(\delta/4)$$

and

$$\sinh(\delta/4)\sinh(a') = \cosh(\gamma/2);$$

thus

$$\tan(\pi/2-\theta) = \frac{\cosh(\delta/4)}{\sinh(\gamma/2)} \qquad \tan(\pi/2-\theta') = \frac{\cosh(\gamma/2)}{\sinh(\delta/4)}.$$

Recall that

$$\tan\big((\pi/2-\theta)+(\pi/2-\theta')\big) = \tan(\pi-\theta-\theta') = \cot(\pi/2-\theta-\theta')$$

so finally one has:

$$\tan(\pi/2-\theta-\theta') = \frac{2\cosh(\gamma/2-\delta/4)}{\sinh(\gamma)+\sinh(\delta/2)}. \ \square$$

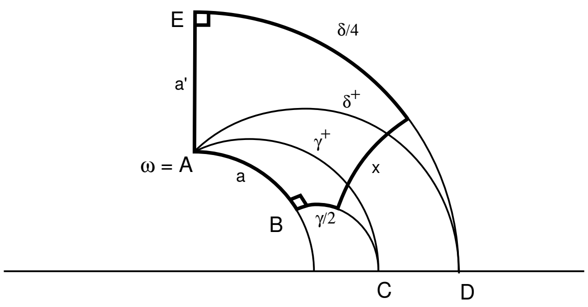

Figure 2.

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Received 31 May 2004