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COMPOSITION OPERATORS IN HYPERBOLIC Q-CLASSES

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Abstract. Function theoretic characterizations are given of when a composition operator mapping from a weighted Dirichlet space $\mathscr{D}_q$ into a holomorphic Q_s-space is bounded or compact. If X^* stands for the hyperbolic class corresponding to the space X, it is shown that a composition operator mapping from $\mathscr{D}_q$ into Q_s is bounded if and only if it is bounded from $\mathscr{D}_q^*$ into Q_s^*, provided $q \le 0$ and $s \le 1$.

1. Introduction and statements of results

Let $H(\mathbf{D})$ denote the space of all analytic functions in the open unit disc $\mathbf{D}$ of the complex plane, and let $B(\mathbf{D})$ be the subset of $H(\mathbf{D})$ consisting of those $h \in H(\mathbf{D})$ for which $|h(z)| < 1$ for all $z \in \mathbf{D}$. Every $\varphi \in B(\mathbf{D})$ induces a linear composition operator $C_\varphi(f) = f \circ \varphi$ from $H(\mathbf{D})$ or $B(\mathbf{D})$ into itself. For the general theory of composition operators in analytic function spaces, see [4] and [13].

A function $f \in H(\mathbf{D})$ belongs to the *α-Bloch space* $\mathscr{B}_\alpha$, $0 < \alpha < \infty$, if

$$\|f\|_{\mathscr{B}_\alpha} = \sup_{z \in \mathbf{D}} |f'(z)|(1 - |z|^2)^\alpha < \infty.$$

The *little α-Bloch space* $\mathscr{B}_{\alpha,0}$ consists of those $f \in H(\mathbf{D})$ for which $|f'(z)|(1 - |z|^2)^\alpha \to 0$ as $|z| \to 1$. Denoting $h^*(z) = |h'(z)|/\bigl(1 - |h(z)|^2\bigr)$, the *hyperbolic derivative* of $h \in B(\mathbf{D})$, the *hyperbolic α-Bloch classes* $\mathscr{B}_\alpha^*$ and $\mathscr{B}_{\alpha,0}^*$ are defined as the sets of those $h \in B(\mathbf{D})$ for which

$$\|h\|_{\mathscr{B}_\alpha^*} = \sup_{z \in \mathbf{D}} h^*(z)(1 - |z|^2)^\alpha < \infty$$

and $\lim_{|z|\to 1} h^*(z)(1-|z|^2)^\alpha = 0$, respectively. If $\alpha = 1$, it is simply denoted $\mathscr{B}^* = \mathscr{B}_1^*$ and $\mathscr{B}_0^* = \mathscr{B}_{1,0}^*$. Clearly $\mathscr{B}_\alpha^*$ and $\mathscr{B}_{\alpha,0}^*$ are not linear spaces. Moreover, the

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Schwarz–Pick lemma implies $\mathscr{B}_\alpha^* = B(\mathbf{D})$ if $\alpha \ge 1$, and therefore the hyperbolic α-Bloch classes are only considered when $0 < \alpha \le 1$.

For $s > -1$, the *weighted Dirichlet space* $\mathscr{D}_s$ (respectively *weighted hyperbolic Dirichlet class* $\mathscr{D}_s^*$) consists of those $f \in H(\mathbf{D})$ (respectively $h \in B(\mathbf{D})$) for which

$$\|f\|_{\mathscr{D}_s} = \left(\int_{\mathbf{D}} |f'(z)|^2 \left(\log\frac{1}{|z|}\right)^s dA(z)\right)^{1/2} < \infty$$

(respectively

$$\|h\|_{\mathscr{D}_s^*} = \left(\int_{\mathbf{D}} \big(h^*(z)\big)^2 \left(\log\frac{1}{|z|}\right)^s dA(z)\right)^{1/2} < \infty),$$

where $dA(z)$ denotes the element of the Lebesgue area measure on $\mathbf{D}$. The Schwarz–Pick lemma implies $\mathscr{D}_s^* = B(\mathbf{D})$ for $s > 1$, and therefore the class $\mathscr{D}_s^*$ is considered only when $-1 < s \le 1$. In this range the class $\mathscr{D}_s^*$ contains no inner functions by [14, Theorem 1.1].

Let the Green's function of $\mathbf{D}$ be defined as $g(z,a) = -\log|\varphi_a(z)|$, where $\varphi_a(z) = (a-z)/(1-\bar{a}z)$ is the automorphism of $\mathbf{D}$ which interchanges the points zero and $a \in \mathbf{D}$. For $0 \le s < \infty$, the Möbius invariant subspace (respectively subclass) Q_s (respectively Q_s^*) of $\mathscr{D}_s$ (respectively $\mathscr{D}_s^*$) consists of those $f \in H(\mathbf{D})$ (respectively $h \in B(\mathbf{D})$) for which

$$\|f\|_{Q_s} = \left(\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f'(z)|^2 g^s(z,a)\, dA(z)\right)^{1/2} < \infty \tag{1.1}$$

(respectively

$$\|h\|_{Q_s^*} = \left(\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \big(h^*(z)\big)^2 g^s(z,a)\, dA(z)\right)^{1/2} < \infty). \tag{1.2}$$

The space $Q_{s,0}$ (respectively class $Q_{s,0}^*$) consists of those $f \in H(\mathbf{D})$ (respectively $h \in B(\mathbf{D})$) for which the integral expression in (1.1) (respectively (1.2)) tends to zero as $|a| \to 1$. If $s = 0$, then Q_0 is the classical Dirichlet space $\mathscr{D} = \mathscr{D}_0$. If $s > 1$, then, by [2, Theorem 1], the spaces Q_s and $Q_{s,0}$ coincide with the Bloch space $\mathscr{B}$ and the little Bloch space $\mathscr{B}_0$, respectively, and the class Q_s^* reduces to $B(\mathbf{D})$ by the Schwarz–Pick lemma.

The following characterization of bounded composition operators mapping from $\mathscr{B}_\alpha$ into Q_s can be found in [17, Theorem 2.2.1(i)].

Theorem A. *Let $0 < \alpha < \infty$, $0 \le s < \infty$ and $\varphi \in B(\mathbf{D})$. Then the following statements are equivalent*:

(1) $C_\varphi\colon \mathscr{B}_\alpha \to Q_s$ *is bounded*;
(2) $C_\varphi\colon \mathscr{B}_{\alpha,0} \to Q_s$ *is bounded*;
(3) $\displaystyle \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \frac{|\varphi'(z)|^2}{\bigl(1-|\varphi(z)|^2\bigr)^{2\alpha}} g^s(z,a)\, dA(z) < \infty$;
(4) $\displaystyle \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \frac{|\varphi'(z)|^2}{\bigl(1-|\varphi(z)|^2\bigr)^{2\alpha}} \bigl(1-|\varphi_a(z)|^2\bigr)^s\, dA(z) < \infty.$

To be precise, the case $s=0$ of Theorem A does not appear in [17, Theorem 2.2.1(i)], but it has been included above since the same proof works also in this case.

A composition operator $C_\varphi\colon \mathscr{B}_\alpha^* \to Q_s^*$ is said to be *bounded* if there exists a positive constant C such that $\|C_\varphi(h)\|_{Q_s^*} \le C\|h\|_{\mathscr{B}_\alpha^*}$ for all $h \in \mathscr{B}_\alpha^*$. Hereafter it is agreed the same meaning for the boundedness of C_φ mapping from one hyperbolic class X^* into another hyperbolic class Y^*. This definition is of the same spirit as the definition in [9] of a bounded composition operator mapping from one meromorphic function class into another. However, it would be of interest to find metrics in $\mathscr{B}_\alpha^*$ and Q_s^* such that these classes would become complete metric spaces and the continuity of C_φ would be equivalent to the natural definition of a bounded composition operator given above.

The first result of this paper extends Theorem A to the corresponding hyperbolic classes.

Theorem 1.1. *Let $0<\alpha\le 1$, $0\le s\le 1$ and $\varphi\in B(\mathbf{D})$. Then the following statements are equivalent*:
(1) $C_\varphi\colon \mathscr{B}_\alpha \to Q_s$ *is bounded*;
(2) $C_\varphi : \mathscr{B}_\alpha^* \to Q_s^*$ *is bounded*;
(3) $C_\varphi : \mathscr{B}_{\alpha,0}^* \to Q_s^*$ *is bounded.*

Keeping the consideration mostly out of meromorphic function classes, it is settled to point out a somewhat surprising phenomenon which occurs here. Namely, by Theorem A, Theorem 1.1 and the theorem in [9], a composition operator C_φ mapping from $\mathscr{B}$ into Q_s is bounded, if and only if, it is bounded from $\mathscr{B}^*$ into Q_s^*, if and only if, it is bounded from $\mathscr{N}$ into $Q_s^\#$, where $\mathscr{N}$ denotes the class of normal functions and $Q_s^\#$ is the meromorphic Q_s-class. See [2] and [9] for necessary definitions. Since the functions in $\mathscr{B}_\alpha$ are bounded if $0<\alpha<1$, it is easy to see, by using functions with Hadamard gaps as in the proof of Theorem 1.1, that this result remains also valid when the domain space and classes are $\mathscr{B}_\alpha$, $\mathscr{B}_\alpha^*$ and $\mathscr{N}_\alpha^\#$, $0<\alpha<1$, respectively.

Two quantities a and b are said to be *comparable*, denoted by $a\simeq b$, if there exists a positive constant C such that $C^{-1}a\le b\le Ca$.

Example 1.2. For $0<\beta<1$, define $\phi_\beta(z)=1-(1-z)^\beta$. Then $\phi_\beta(z)$ is a conformal mapping which fixes the points zero and one, and maps $\mathbf{D}$ onto a lens-type region. Since $\phi_\beta^*(z) \simeq |1-z|^{-1}$ in $\mathbf{D}$, Theorem 1.1 and Theorem A imply that $C_{\phi_\beta}\colon \mathscr{B}^* \to Q_s^*$ is bounded if and only if s is positive. It is now shown that $C_{\phi_\beta}\colon Q_{s_1}^* \to Q_{s_2}^*$, $0 < s_1 \le 1$, is bounded if and only if s_2 is positive. Since $Q_{s_1}^* \subset \mathscr{B}^*$ with $\|h\|_{\mathscr{B}^*} \le C\|h\|_{Q_{s_1}^*}$, where C is a positive constant, it suffices to show that $C_{\phi_\beta}\colon Q_{s_1}^* \to \mathscr{D}$ is not bounded. But this easily follows by the fact $(\phi_\beta \circ \phi_\beta)^*(z) \simeq |1-z|^{-1}$ in $\mathbf{D}$, since Fatou's lemma and (3.2) below yield $\phi_\beta \in Q_{s_1}^*$ for $0 < s_1 \le 1$.

If $0 < \alpha < \infty$, $0 < s < \infty$ and $\varphi \in B(\mathbf{D})$, then $C_\varphi\colon Q_s \to \mathscr{B}_\alpha$ is bounded if and only if $\varphi \in \mathscr{B}_\alpha^*$ by [17, Theorem 2.2.1(iii)]. This result is extended for the corresponding hyperbolic classes in the following theorem.

Theorem 1.3. *Let $0 < \alpha \le 1$, $0 \le s \le 1$ and $\varphi \in B(\mathbf{D})$. Then the following statements are equivalent*:

(1) $C_\varphi\colon \mathscr{B} \to \mathscr{B}_\alpha$ *is bounded*;
(2) $C_\varphi\colon \mathscr{B}^* \to \mathscr{B}_\alpha^*$ *is bounded*;
(3) $C_\varphi\colon Q_s^* \to \mathscr{B}_\alpha^*$ *is bounded*;
(4) $\varphi \in \mathscr{B}_\alpha^*$.

Theorem 1.4 generalizes [16, Theorem 4.1(i)] since the term $\left|\varphi_a'\big(\varphi(z)\big)\right|$ in conditions (2) and (3) below can be replaced by $(1-|a|^2)^\tau/|1-\bar{a}\varphi(z)|^{1+\tau}$, $0 < \tau < \infty$, by Lemma B below (in Section 2).

Theorem 1.4. *Let $-1 < s_1 < \infty$, $0 < s_2 < \infty$ and $\varphi \in B(\mathbf{D})$. Then the following statements are equivalent*:

(1) $C_\varphi\colon \mathscr{D}_{s_1} \to Q_{s_2}$ *is bounded*;

(2) $\displaystyle\sup_{a,b\in\mathbf{D}} \int_{\mathbf{D}} \left|\varphi_a'\big(\varphi(z)\big)\right|^{2+s_1} |\varphi'(z)|^2 g^{s_2}(z,b)\, dA(z) < \infty;$

(3) $\displaystyle\sup_{a,b\in\mathbf{D}} \int_{\mathbf{D}} \left|\varphi_a'\big(\varphi(z)\big)\right|^{2+s_1} |\varphi'(z)|^2 \big(1-|\varphi_b(z)|^2\big)^{s_2}\, dA(z) < \infty.$

Moreover, if $s_1 \le 0$ and $s_2 \le 1$, then (1)–(3) *are equivalent to*

(4) $C_\varphi\colon \mathscr{D}_{s_1}^* \to Q_{s_2}^*$ *is bounded.*

By Theorem 1.3, $C_\varphi\colon \mathscr{D} \to \mathscr{B}$ is bounded for all $\varphi \in B(\mathbf{D})$. Moreover, since $Q_s = \mathscr{B}$ for $s > 1$, Theorem 1.4 implies that $C_\varphi\colon \mathscr{D} \to \mathscr{B}$ is bounded if and only if

$$\sup_{a,b\in\mathbf{D}} \int_{\mathbf{D}} |(\varphi_a \circ \varphi)'(z)|^2 g^s(z,b)\, dA(z) < \infty, \quad 1 < s < \infty. \tag{1.3}$$

However, (1.3) is equivalent to

$$\sup_{a,z\in\mathbf{D}} |(\varphi_a \circ \varphi)'(z)|(1-|z|^2) = \|\varphi\|_{\mathscr{B}^*} < \infty,$$

which is, of course, satisfied for all $\varphi \in B(\mathbf{D})$.

Example 1.5. For $0 \le p < \infty$, define $\psi_p(z) = (p+z)/(p+1)$. Then ψ_p is a conformal mapping which maps $\mathbf{D}$ onto the disc centered at $p/(p+1)$ with radius $1/(p+1)$. Clearly,

$$\|f \circ \varphi\|_{Q_{s_2}} \le C\|f \circ \varphi\|_{\mathscr{D}} \le C\|f\|_{\mathscr{D}} \tag{1.4}$$

for all $f \in \mathscr{D}$ and $\varphi \in B(\mathbf{D})$, thus, in particular, $C_{\psi_p}\colon \mathscr{D} \to Q_{s_2}$ is bounded. A similar reasoning shows that $C_{\psi_p}\colon \mathscr{D}^* \to Q^*_{s_2}$ is also bounded. However, a geometric argument or a straightforward calculation based on the identity

$$1 - \bar{a}\psi_p(z) = \frac{p(1-\bar{a}) + 1 - \bar{a}z}{p+1}$$

shows that $|1 - a\psi_p(z)| \le |1 - az|$ for all $z \in \mathbf{D}$ and $a \in (0,1)$, and therefore

$$\begin{aligned}
\sup_{a,b\in\mathbf{D}} \int_{\mathbf{D}} &\bigl|\varphi_a'\bigl(\psi_p(z)\bigr)\bigr|^{2+s_1}|\psi_p'(z)|^2\bigl(1-|\varphi_b(z)|^2\bigr)^{s_2}\,dA(z) \\
&\ge \lim_{a\to 1} \frac{(1-a^2)^{2+s_1+s_2}}{(p+1)^2} \int_{\mathbf{D}} \frac{(1-|z|^2)^{s_2}}{|1-a\psi_p(z)|^{2(2+s_1)}|1-az|^{2s_2}}\,dA(z) \\
&\ge \lim_{a\to 1} \frac{(1-a^2)^{2+s_1+s_2}}{(p+1)^2} \int_{\mathbf{D}} \frac{(1-|z|^2)^{s_2}}{|1-az|^{2(2+s_1+s_2)}}\,dA(z) \simeq \lim_{a\to 1}(1-a^2)^{-s_1},
\end{aligned}$$

from which it follows by Theorem 1.4 that $C_{\psi_p}\colon \mathscr{D}_{s_1} \to Q_{s_2}$ is not bounded if s_1 is positive.

Example 1.6. Let $0 < \beta < 1$, $-1 < s_1 < \infty$ and $0 \le s_2 < \infty$, and consider the map $\phi_\beta(z) = 1 - (1-z)^\beta$. It is proved that C_{ϕ_β} admits the same behavior as C_{ψ_p} does in the sense that $C_{\phi_\beta}\colon \mathscr{D}_{s_1} \to Q_{s_2}$ is bounded if and only if $-1 < s_1 \le 0$. In view of (1.4) it suffices to show that $C_{\phi_\beta}\colon \mathscr{D}_{s_1} \to Q_{s_2}$ is not bounded if $s_1 > 0$. To this end, choose $f_a(z) = (1-z)^{-a}$, $0 < a < \infty$. Now, by [5, Lemma on p. 65], there is a positive constant C_1 such that

$$\|f_a\|^2_{\mathscr{D}_{s_1}} = a^2 \int_{\mathbf{D}} \frac{(1-|z|^2)^{s_1}}{|1-z|^{2(1+a)}}\,d\sigma(z) \le C_1 a^2 \int_0^1 \frac{r\,dr}{(1-r)^{2a+1-s_1}},$$

and therefore $f_a \in \mathscr{D}_{s_1}$ for $0 < a < \frac{1}{2}s_1$. Moreover, denote $\omega_{a,\beta}(z) = (1-z)^{-a\beta} = f_a \circ \phi_\beta$. Then, by [2, Proposition 1], there is a positive constant C_2 such that

$$\|f_a \circ \phi_\beta\|_{Q_{s_2}} = \|\omega_{a,\beta}\|_{Q_{s_2}} \ge C_2\|\omega_{a,\beta}\|_{\mathscr{B}} = \infty,$$

and therefore $f \circ \phi_\beta \notin Q_{s_2}$. Thus $C_{\phi_\beta}\colon \mathscr{D}_{s_1} \to Q_{s_2}$ is not bounded if $s_1 > 0$.

The following result generalizes [16, Theorem 4.1(ii)].

Theorem 1.7. *Let $-1 < s_1 < \infty$, $0 \le s_2 < \infty$ and $\varphi \in B(\mathbf{D})$. Then the following statements are equivalent*:

(1) $C_\varphi\colon \mathscr{D}_{s_1} \to Q_{s_2}$ *is compact*;

(2) $\displaystyle\lim_{|a|\to 1} \sup_{b\in\mathbf{D}} \int_{\mathbf{D}} |\varphi_a'(\varphi(z))|^{2+s_1} |\varphi'(z)|^2 g^{s_2}(z,b)\, dA(z) = 0;$

(3) $\displaystyle\lim_{|a|\to 1} \sup_{b\in\mathbf{D}} \int_{\mathbf{D}} |\varphi_a'(\varphi(z))|^{2+s_1} |\varphi'(z)|^2 (1-|\varphi_b(z)|^2)^{s_2}\, dA(z) = 0.$

Since, by the general definition of a bounded (respectively compact) operator mapping from one Banach space into another, $C_\varphi\colon \mathscr{D}_{s_1} \to Q_{s_2,0}$ is bounded (respectively compact) if and only if $C_\varphi\colon \mathscr{D}_{s_1} \to Q_{s_2}$ is bounded (respectively compact) and $C_\varphi(\mathscr{D}_{s_1}) \subset Q_{s_2,0}$, the operator $C_\varphi\colon \mathscr{D}^*_{s_1} \to Q^*_{s_2,0}$ is said to be *bounded*, if $C_\varphi\colon \mathscr{D}^*_{s_1} \to Q^*_{s_2}$ is bounded and $C_\varphi(\mathscr{D}^*_{s_1}) \subset Q^*_{s_2,0}$.

Theorem 1.8. *Let $-1 < s_1 < \infty$, $0 < s_2 < \infty$ and $\varphi \in B(\mathbf{D})$. Then $C_\varphi\colon \mathscr{D}_{s_1} \to Q_{s_2,0}$ is bounded if and only if $C_\varphi\colon \mathscr{D}_{s_1} \to Q_{s_2}$ is bounded and the following two conditions are satisfied*:

(1) $\varphi \in Q_{s_2,0}$;

(2) $\displaystyle\lim_{|a|,|b|,t\to 1} \int_{|\varphi(z)|\ge t} \bigl|\varphi_a'\bigl(\varphi(z)\bigr)\bigr|^{2+s_1} |\varphi'(z)|^2 g^{s_2}(z,b)\, dA(z) = 0.$

Similarly, if $s_1 \le 0$ and $s_2 \le 1$, then $C_\varphi\colon \mathscr{D}^*_{s_1} \to Q^*_{s_2,0}$ is bounded if and only if $C_\varphi\colon \mathscr{D}^*_{s_1} \to Q^*_{s_2}$ is bounded, $\varphi \in Q^*_{s_2,0}$ and (2) *is satisfied.*

It is easy to show that the conditions (1) and (2) in Theorem 1.8 together are equivalent to

(3) $\displaystyle\lim_{|b|\to 1} \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \bigl|\varphi_a'\bigl(\varphi(z)\bigr)\bigr|^{2+s_1} |\varphi'(z)|^2 g^{s_2}(z,b)\, dA(z) = 0,$

and hence the first part of Theorem 1.8 implies the following result.

Theorem 1.9. *Let $-1 < s_1 < \infty$, $0 < s_2 < \infty$ and $\varphi \in B(\mathbf{D})$. Then $C_\varphi\colon \mathscr{D}_{s_1} \to Q_{s_2,0}$ is compact if and only if $C_\varphi\colon \mathscr{D}_{s_1} \to Q_{s_2}$ is compact and the condition* (3) *above is satisfied.*

The remaining part of the paper is organized as follows. In Section 2, some background material and auxiliary results needed later on are recalled, and Section 3 contains the proofs of the results presented in this section.

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2. Background material

A positive Borel measure μ on $\mathbf{D}$ is a *bounded s-Carleson measure*, if

$$\sup_I \frac{\mu\big(S(I)\big)}{|I|^s} < \infty, \quad 0 < s < \infty,$$

where $|I|$ denotes the arc length of a subarc I of the boundary of

$$S(I) = \left\{ z \in \mathbf{D} : \frac{z}{|z|} \in I,\ 1 - |I| \le |z| \right\}$$

is the Carleson box based on I, and the supremum is taken over all subarcs I such that $|I| \le 1$. Moreover, if

$$\lim_{|I|\to 0} \frac{\mu\big(S(I)\big)}{|I|^s} = 0, \quad 0 < s < \infty,$$

then μ is a *compact s-Carleson measure.* If $s = 1$, then a bounded (respectively compact) 1-Carleson measure is just a standard bounded (respectively compact) Carleson measure.

Some well-known and useful characterizations of bounded s-Carleson measures are gathered in the following lemma. For the proof, see [1, Theorem 13], [3, Lemma 2.1], [10, pp. 89–90], and [11, Proposition 2.1].

Lemma B. *Let μ be a positive Borel measure on $\mathbf{D}$, $1 < s < \infty$, $0 < r < 1$ and $0 < \tau < \infty$. Then the following statements are equivalent*:

(1) $K_1 = \sup_I \dfrac{\mu\big(S(I)\big)}{|I|^s} < \infty$;

(2) $K_2 = \sup_{z\in\mathbf{D}} \dfrac{\mu\big(D(z,r)\big)}{(1-|z|^2)^s} < \infty$;

(3) $K_3 = \sup_{z\in\mathbf{D}} \displaystyle\int_{\mathbf{D}} \left(\frac{(1-|z|^2)^\tau}{|1-\bar{z}w|^{1+\tau}} \right)^s d\mu(w) < \infty.$

Moreover, K_1, K_2 and K_3 are comparable.

Here $D(a,r) = \{z \in \mathbf{D} : |\varphi_a(z)| < r\}$ is the pseudo-hyperbolic disc of center $a \in \mathbf{D}$ and radius $0 < r < 1$. The pseudo-hyperbolic disc $D(a,r)$ is an Euclidean disc centered at $(1-r^2)a/(1-|a|^2r^2)$ with radius $(1-|a|^2)r/(1-|a|^2r^2)$, see [8, p. 3].

The following change of variables formula by C. S. Stanton, [6] and [15], was apparently first used by J. H. Shapiro [12] in the study of composition operators, and it also plays a key role in some of the proofs in this paper.

Lemma C. *Let g and u be positive measurable functions on $\mathbf{D}$, and let $\varphi \in B(\mathbf{D})$. Then*

$$\int_{\mathbf{D}} (g \circ \varphi)(z)|\varphi'(z)|^2 u(z)\, dA(z) = \int_{\mathbf{D}} g(w) U(\varphi, w)\, dA(w),$$

where

$$U(\varphi, w) = \sum_{z \in \varphi^{-1}\{w\}} u(z), \quad w \in \mathbf{D} \setminus \{\varphi(0)\}.$$

If $u(z) = (-\log|z|)^s$, then $U(\varphi, w)$ is the *generalized Nevanlinna counting function*

$$N_{\varphi,s}(w) = \sum_{z \in \varphi^{-1}\{w\}} \left(\log \frac{1}{|z|}\right)^s.$$

For the study of compactness, the following well-known result is needed. See [4, Proposition 3.11] for a similar result.

Lemma D. *Let $-1 < s_1 < \infty$, $0 \le s_2 < \infty$ and $\varphi \in B(\mathbf{D})$. Then $C_\varphi \colon \mathscr{D}_{s_1} \to Q_{s_2}$ is compact if and only if for any bounded sequence $\{f_n\}$ in $\mathscr{D}_{s_1}$ with $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$ as $n \to \infty$, $\|f_n \circ \varphi\|_{Q_{s_2}} \to 0$ as $n \to \infty$.*

3. Proofs

Proof of Theorem 1.1. It is enough to prove the implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) since (2) $\Rightarrow$ (3) is clearly true.

Suppose $C_\varphi \colon \mathscr{B}_\alpha \to Q_s$ is bounded, that is, (1) is satisfied. If $h \in \mathscr{B}_\alpha^*$, then

$$\int_{\mathbf{D}} \big((h \circ \varphi)^*(z)\big)^2 g^s(z,a)\, dA(z) \le \|h\|^2_{\mathscr{B}_\alpha^*} \int_{\mathbf{D}} \frac{|\varphi'(z)|^2}{\big(1 - |\varphi(z)|^2\big)^{2\alpha}} g^s(z,a)\, dA(z),$$

where the last integral is uniformly bounded for all $a \in \mathbf{D}$ by Theorem A, and therefore $C_\varphi \colon \mathscr{B}_\alpha^* \to Q_s^*$ is bounded. Thus (1) implies (2) is proved.

To prove (3) $\Rightarrow$ (1), let first $\alpha = 1$, and suppose that $C_\varphi \colon \mathscr{B}_0^* \to Q_s^*$ is bounded. If $h_b(z) = bz$, $b \in \mathbf{D}$, then $h^*(z) = |b|(1 - |bz|^2)^{-1}$ and $h_b \in \mathscr{B}_0^*$ for all $b \in \mathbf{D}$, and

$$\sup_{a \in \mathbf{D}} \int_{\mathbf{D}} \frac{|b|^2 |\varphi'(z)|^2}{\big(1 - |b\varphi(z)|^2\big)^2} g^s(z,a)\, dA(z) \le C\|h\|^2_{\mathscr{B}^*} \le C|b|^2$$

for some positive C by the assumption. Taking limit as $|b| \to 1$, $b \in \mathbf{D}$, Fatou's lemma with Theorem A implies that $C_\varphi \colon \mathscr{B} \to Q_s$ is bounded, that is, (1) with $\alpha = 1$ holds.

If $0<\alpha<1$, functions with Hadamard gaps may be used. Define

$$g_n(z)=\sum_{k=0}^{\infty}2^{k(\alpha-1)}(b_nz)^{2^k},$$

where $\{b_n\}\subset\mathbf{D}$ and $|b_n|\to 1$, as $n\to\infty$. Then $g_n\in\mathscr{B}_{\alpha,0}$ by [18, Theorem 1]. Since $|g_n(z)|\le\sum_{k=0}^{\infty}2^{k(\alpha-1)}$, there is a positive constant C, depending only on α, such that $h_n=C^{-1}g_n$ satisfies $|h_n(z)|\le\frac{1}{2}$ for all $z\in\mathbf{D}$ and $n\in\mathbf{N}$, and therefore $h_n^*(z)\simeq|h_n'(z)|$ in $\mathbf{D}$. Now one may argue as in [1, p. 133] and use Fatou's lemma to conclude that the condition (3) with $0<\alpha<1$ in Theorem A is satisfied, and hence (1) with $0<\alpha<1$ holds. ▫

Remark. To characterize bounded composition operators from $Q_{s_1}^*$ to $Q_{s_2}^*$ when $0<s_1,s_2<1$ appears to be more complicated. However, Example 1.2 shows that there is a function φ for which $C_\varphi\colon\mathscr{B}^*\to Q_{s_2}^*$ is bounded if and only $C_\varphi\colon Q_{s_1}^*\to Q_{s_2}^*$ is bounded, provided $0<s_1\le 1$.

Proof of Theorem 1.3. It suffices to prove the implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2) since (2) $\Rightarrow$ (3) is clearly true and (1) is equivalent to (4) by [17, Theorem 2.2.1(iii)].

Suppose $C_\varphi\colon Q_s^*\to\mathscr{B}_\alpha^*$ is bounded, that is, (3) holds, and let first $0<s\le 1$. If $\phi_{\beta,a}(z)=1-(1-\bar{a}z)^\beta$, where $0<\beta<1$ and $a\in\mathbf{D}$, then

$$\phi_{\beta,a}^*(z)=\frac{\beta|a|\,|1-\bar{a}z|}{1-|1-(1-\bar{a}z)^\beta|}\simeq\frac{\beta|a|}{|1-\bar{a}z|},\quad z\in\mathbf{D}. \tag{3.1}$$

By [7, Lemma 2.5], there is a positive constant C_1 such that

$$\begin{aligned}\int_{\mathbf{D}}\big(\phi_{\beta,a}^*(z)\big)^2\big(1-|\varphi_b(z)|^2\big)^s\,dA(z)&\simeq\beta^2|a|^2(1-|b|^2)^s\\&\quad\cdot\int_{\mathbf{D}}\frac{(1-|z|^2)^s}{|1-\bar{b}z|^{2s}|1-\bar{a}z|^2}\,dA(z)\\&\le C_1^2\beta^2|a|^2\frac{(1-|b|^2)^s}{|1-\bar{a}b|^s},\end{aligned}\tag{3.2}$$

and it follows that $\|\phi_{\beta,a}\|_{Q_s^*}\le C_1 2^{s/2}\beta|a|$ for all $a\in\mathbf{D}$, $0<\beta<1$ and $0<s\le 1$. By the assumption there exists a positive constant C_2 such that

$$\frac{\beta|a|}{2}\frac{|\varphi'(z)|}{|1-\bar{a}\varphi(z)|}(1-|z|^2)^\alpha\le(\phi_{\beta,a}\circ\varphi)^*(z)(1-|z|^2)^\alpha\le C_2\|\phi_{\beta,a}\|_{Q_s^*}\le C_2C_1 2^{s/2}\beta|a|,$$

and the assertion $\varphi\in\mathscr{B}_\alpha^*$ follows by choosing $a=\varphi(z)$. The case $s=0$ can be proved in a similar manner by choosing the test function $\varphi_a(z)/2$.

If $h\in\mathscr{B}^*=B(\mathbf{D})$, then

$$(h\circ\varphi)^*(z)(1-|z|^2)^\alpha\le\|h\|_{\mathscr{B}^*}\frac{|\varphi'(z)|}{1-|\varphi(z)|^2}(1-|z|^2)^\alpha\le\|h\|_{\mathscr{B}^*}\|\varphi\|_{\mathscr{B}_\alpha^*},$$

and (4) $\Rightarrow$ (2) follows. ▫

Proof of Theorem 1.4. To prove that (2) implies (1), the reasoning in the proof of [10, Theorem 2.2] is followed. By Lemma C and the subharmonicity of $|f'(z)|^2$, there is a positive constant C_1 such that

$$\begin{aligned}\int_{\mathbf{D}}|(f\circ\varphi)'(z)|^2 g^{s_2}(z,a)\,dA(z) &= \int_{\mathbf{D}}|f'(w)|^2 d\mu_{a,s_2}(w)\\ &\le C_1\int_{\mathbf{D}}\left(\frac{1}{(1-|w|^2)^2}\int_{D(w,1/2)}|f'(z)|^2 dA(z)\right)d\mu_{a,s_2}(w),\end{aligned}$$

where $d\mu_{a,s_2}(w)=N_{\varphi\circ\varphi_a,s_2}(w)\,dA(w)$. Then the symmetry

$$\chi_{D(z,r)}(w)=\chi_{D(w,r)}(z)$$

of the characteristic functions of pseudohyperbolic discs and Fubini's theorem yield

$$\int_{\mathbf{D}}|(f\circ\varphi)'(z)|^2 g^{s_2}(z,a)\,dA(z)\le C_1\int_{\mathbf{D}}|f'(z)|^2\left(\int_{D(z,1/2)}\frac{d\mu_{a,s_2}(w)}{(1-|w|^2)^2}\right)dA(z). \tag{3.3}$$

By Lemmas B and C, the assumption (2) is equivalent to

$$\sup_{a\in\mathbf{D}}\int_{D(z,1/2)} d\mu_{a,s_2}(w)\le C_2(1-|z|^2)^{2+s_1},\quad z\in\mathbf{D}, \tag{3.4}$$

for some positive constant C_2. Since $1-|w|\simeq 1-|z|$ for $w\in D(z,1/2)$, it follows by (3.3) and (3.4) that $C_\varphi\colon \mathscr{D}_{s_1}\to Q_{s_2}$ is bounded.

Suppose then that $C_\varphi\colon \mathscr{D}_{s_1}\to Q_{s_2}$ is bounded. For $a\in\mathbf{D}$, define $f_a(z)=\int_0^z\left(\varphi_a'(w)\right)^{1+s_1/2}dw$. Then, by Forelli–Rudin estimates [19, Lemma 4.2.2] there is a positive constant C_1, depending only on s_1, such that

$$\|f_a\|^2_{\mathscr{D}_{s_1}}=(1-|a|^2)^{1+s_1/2}\int_{\mathbf{D}}\frac{(1-|z|^2)^{s_1}}{|1-\bar{a}z|^{2+s_1}}\,d\sigma(z)\le C_1$$

for all $a\in\mathbf{D}$, and thus the family $\{f_a : a\in\mathbf{D}\}$ is norm bounded uniformly in $\mathscr{D}_{s_1}$. Since $C_\varphi\colon \mathscr{D}_{s_1}\to Q_{s_2}$ is bounded, there is a positive constant C_2 such that

$$\begin{aligned}\sup_{b\in\mathbf{D}}\int_{\mathbf{D}}|\varphi_a'(\varphi(z))|^{2+s_1}|\varphi'(z)|^2(1-|\varphi_b(z)|^2)^{s_2}\,d\sigma(z) &= \|f_a\circ\varphi\|^2_{Q_{s_2}}\\ &\le C_2\|f_a\|^2_{\mathscr{D}_{s_1}}\le C_1C_2\end{aligned}$$

for all $a\in\mathbf{D}$, and the condition (2) follows.

Since (2) and (3) are clearly equivalent, it is now proceeded to consider the hyperbolic case. Suppose that $s_1\le 0$ and $s_2\le 1$. If (2) is satisfied, then the same reasoning as in the first part of the proof shows that $C_\varphi\colon \mathscr{D}^*_{s_1} \to Q^*_{s_2}$ is bounded since also $(h^*)^2$ is a subharmonic function in $\mathbf{D}$.

Suppose then that $C_\varphi\colon \mathscr{D}^*_{s_1} \to Q^*_{s_2}$ is bounded. For $a \in \mathbf{D}$ and $\frac{1}{2} < \gamma \le 1$ let

$$(3.5)\quad f_{a,\gamma}(z) = \int_0^z \bigl(\varphi_a'(w)\bigr)^\gamma \, dw = \begin{cases} \dfrac{(1-|a|^2)^\gamma}{\bar a(1-2\gamma)}\bigl((1-\bar a z)^{1-2\gamma} - 1\bigr), & a \in \mathbf{D}\setminus\{0\},\\ z, & a = 0,\end{cases}$$

and

$$(3.6)\qquad h_{a,\gamma}(z) = \begin{cases} \dfrac{(2\gamma-1)\bar a}{6} f_{a,\gamma}(z), & a \in \mathbf{D}\setminus\{0\},\\ \dfrac{z}{2}, & a = 0.\end{cases}$$

Then $\|h_{a,\gamma}\|_\infty \le \frac{1}{2}$ for all $a \in \mathbf{D}$, and therefore $h^*_{a,\gamma}(z) \simeq |h'_{a,\gamma}(z)|$ in $\mathbf{D}$. The reasoning in the proof of the implication (1) $\Rightarrow$ (2) with the functions

$$h_{a,1+s_1/2}(z) = \frac{(1+s_1)\bar a}{6} \int_0^z \bigl(\varphi_a'(w)\bigr)^{1+s_1/2} \, dw$$

yields the assertion (2). □

Proof of Theorem 1.7. Suppose first that $C_\varphi\colon \mathscr{D}_{s_1} \to Q_{s_2}$ is compact and consider the functions $f_a(z) = \int_0^z (\varphi_a'(w))^{1+s_1/2}\, dw$. Since, by the proof of Theorem 1.4, there is a positive constant C such that $\|f_a\|_{\mathscr{D}_{s_1}} \le C$ for all $a \in \mathbf{D}$, and further $f_a \to 0$ uniformly on compact subsets as $|a| \to 1$, Lemma D gives (2).

Suppose now that (2) holds. Let $\{f_n\} \subset \mathscr{D}_{s_1}$ such that $\|f_n\|_{\mathscr{D}_{s_1}} \le C_1 < \infty$ for all $n \in \mathbf{N}$, and $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$. By Lemma D, it suffices to show that $\|f_n \circ \varphi\|_{Q_{s_2}} \to 0$ as $n \to \infty$. For $0 < \delta < 1$, let $\Delta(0,\delta)$ denote the Euclidean disc centered at the origin and of radius δ. A similar reasoning as in the proof of Theorem 1.4 with the fact $1-|w|^2 \simeq 1-|z|^2 \simeq |1-\bar z w|$, $w \in D(z,1/2)$, yields

$$\begin{aligned}
\|f_n \circ \varphi\|^2_{Q_{s_2}} &= \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f_n'(w)|^2 \, d\mu_{a,s_2}(w)\\
&\le C_2 \int_{\mathbf{D}} |f_n'(z)|^2 (1-|z|^2)^{s_1} \left(\sup_{a\in\mathbf{D}} \int_{D(z,1/2)} |\varphi_z'(w)|^{2+s_1} \, d\mu_{a,s_2}(w)\right) dA(z)\\
&\le C_2 \int_{\mathbf{D}\setminus\Delta(0,\delta)} |f_n'(z)|^2 (1-|z|^2)^{s_1} \left(\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |\varphi_z'(w)|^{2+s_1} \, d\mu_{a,s_2}(w)\right) dA(z)\\
&\quad + \frac{C_2 4^{1+s_1}}{(1-\delta)^2} \|\varphi\|^2_{Q_{s_2}} \int_{\Delta(0,\delta)} |f_n'(z)|^2 \, dA(z)\\
&= I_1(\delta) + I_2(\delta),
\end{aligned}$$

where C_2 is a positive constant. For a given $\varepsilon > 0$, by the assumption (2) and Lemma C, there exists a $\delta_0 \in (0,1)$ such that, for all $|z| > \delta_0$,

$$\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |\varphi_z'(w)|^{2+s_1}\, d\mu_{a,s_2}(w) \le \frac{\varepsilon^2}{2C_1^2C_2},$$

and it follows that $I_1(\delta_0) < \varepsilon^2/2$. In view of Theorem 1.4, the assumption (2) implies that $C_\varphi\colon \mathscr{D}_{s_1} \to Q_{s_2}$ is bounded, and hence $\varphi \in Q_{s_2}$. Since $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$, in particular, in $\Delta(0,\delta_0)$, there exists an $N \in \mathbf{N}$ such that, for $n \ge N$,

$$\int_{\Delta(0,\delta_0)} |f_n'(z)|^2\, dA(z) < \frac{\varepsilon^2(1-\delta_0)^2}{2C_2 4^{1+s_1}\|\varphi\|_{Q_{s_2}}^2},$$

and therefore $I_2(\delta_0) < \varepsilon^2/2$. Thus, as $n \ge N$, $\|f_n\circ\varphi\|_{Q_{s_2}} < \varepsilon$, and $C_\varphi\colon \mathscr{D}_{s_1} \to Q_{s_2}$ is compact by Lemma D.

Since (2) and (3) are clearly equivalent, the proof is complete. □

Proof of Theorem 1.8. Suppose first that $C_\varphi\colon \mathscr{D}_{s_1} \to Q_{s_2,0}$ is bounded, that is, $C_\varphi\colon \mathscr{D}_{s_1} \to Q_{s_2}$ is bounded and $C_\varphi(\mathscr{D}_{s_1}) \subset Q_{s_2,0}$. Then, by using the functions $f(z) = z \in \mathscr{D}_{s_1}$ and $f_a(z) = \int_0^z \big(\varphi_a'(w)\big)^{1+s_1/2}\, dw \in \mathscr{D}_{s_1}$, the inclusion $C_\varphi(\mathscr{D}_{s_1}) \subset Q_{s_2,0}$ implies that (1) and (2) are satisfied.

Suppose now that $C_\varphi\colon \mathscr{D}_{s_1} \to Q_{s_2}$ is bounded, and the conditions (1) and (2) are satisfied. It suffices to show that $f\circ\varphi \in Q_{s_2,0}$ for $f \in \mathscr{D}_{s_1}$. A similar reasoning as in the proof of Theorem 1.7 yields

$$\begin{aligned}\int_{\mathbf{D}} |(f\circ\varphi)'(z)|^2 g^{s_2}(z,a)\, dA(z) &\le C\int_{\mathbf{D}\setminus\Delta(0,t)} |f'(z)|^2(1-|z|^2)^{s_1}\\ &\quad\cdot\left(\int_{\mathbf{D}} |\varphi_z'(w)|^{2+s_1} d\mu_{a,s_2}(w)\right) dA(z)\\ &\quad + C\int_{\Delta(0,t)} |f'(z)|^2\left(\int_{D(z,1/2)} \frac{d\mu_{a,s_2}(w)}{(1-|z|^2)^2}\right) dA(z)\\ &= I_1(t) + I_2(t),\end{aligned} \tag{3.7}$$

where C is a positive constant and $0 < t < 1$. To deal with $I_1(t)$, write

$$\begin{aligned}\int_{\mathbf{D}} |\varphi_z'(w)|^{2+s_1}\, d\mu_{a,s_2}(w) &= \int_{\Delta(0,t)} |\varphi_z'(w)|^{2+s_1}\, d\mu_{a,s_2}(w)\\ &\quad + \int_{\mathbf{D}\setminus\Delta(0,t)} |\varphi_z'(w)|^{2+s_1}\, d\mu_{a,s_2}(w)\\ &\le \frac{2^{2+s_1}}{(1-t)^{2+s_1}}\int_{\mathbf{D}} |\varphi'(u)|^2 g^{s_2}(u,a)\, d\sigma(u)\\ &\quad + \int_{\mathbf{D}\setminus\Delta(0,t)} |\varphi_z'(w)|^{2+s_1}\, d\mu_{a,s_2}(w) = I_3(t) + I_4(t).\end{aligned}$$

By the assumption (2), for a given $\varepsilon > 0$, there exists a $\delta_1 \in (0,1)$ such that

$$I_4(t) < \frac{\varepsilon^2}{3C\|f\|^2_{\mathscr{D}_{s_1}}} \tag{3.8}$$

for all $|a|, |z|, t \geq \delta_1$. Let $t \geq \delta_1$ be fixed. Since $\varphi \in Q_{s_2,0}$ by the assumption (1), there exists a $\delta_2 \in [\delta_1, 1)$ such that

$$I_3(t) < \frac{\varepsilon^2(1-t)^{2+s_1}}{3\cdot 2^{2+s_1}C\|f\|^2_{\mathscr{D}_{s_1}}} \tag{3.9}$$

for all $|a| \geq \delta_2$. Since $|z| \geq t$ in the term $I_1(t)$, it follows by combining (3.7), (3.8) and (3.9) that $I_1(t) \leq 2\varepsilon^2/3$ for $|a| \geq \delta_2$. Further, since $\varphi \in Q_{s_2,0}$ by the assumption (1), there exists a $\delta_3 \in [\delta_2, 1)$ such that

$$I_2(t) \leq \frac{C\pi t^2}{(1-t^2)^2} \sup_{|z|=t} |f'(z)|^2 \int_{\mathbf{D}} |\varphi'(z)|^2 g^{s_2}(z,a)\, dA(z) < \frac{\varepsilon^2}{3}$$

for all $|a| \geq \delta_3$. Therefore one finally concludes

$$\int_{\mathbf{D}} |(f\circ\varphi)'(z)|^2 g^{s_2}(z,a)\, dA(z) \leq I_1(t) + I_2(t) < \frac{2\varepsilon^2}{3} + \frac{\varepsilon^2}{3} = \varepsilon^2$$

for all $|a| \geq \delta_3$, that is, $f\circ\varphi \in Q_{s_2,0}$ as one wished to prove. □

Proof of Theorem 1.9. As it was pointed out in Section 1, Theorem 1.9 follows by Theorem 1.8. □

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