Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 31, 2006, 297–314

A SELECTION THEORY FOR MULTIPLE-VALUED FUNCTIONS IN THE SENSE OF ALMGREN

Jordan Goblet

Université catholique de Louvain, Département de mathématique
Chemin du cyclotron 2, BE-1348 Louvain-La-Neuve, Belgium; goblet@math.ucl.ac.be

Abstract. A $\mathbf{Q}_k(\mathbf{R}^n)$-valued function is essentially a rule assigning k unordered and non necessarily distinct elements of $\mathbf{R}^n$ to each element of its domain set $A \subseteq \mathbf{R}^m$. For a $\mathbf{Q}_k(\mathbf{R})$-valued function f we construct a decomposition into k branches that naturally inherit the regularity properties of f. Next we prove that a measurable $\mathbf{Q}_k(\mathbf{R}^n)$-valued function admits a decomposition into k measurable branches. An example of a Lipschitzian $\mathbf{Q}_2(\mathbf{R}^2)$-valued function that does not admit a continuous decomposition is also provided and we state a selection result about multiple-valued functions defined on intervals. We finally give a new proof of Rademacher's theorem for multiple-valued functions. This proof is mainly based on the decomposition theory and it does not use Almgren's bi-Lipschitzian correspondence between $\mathbf{Q}_k(\mathbf{R}^n)$ and a cone $Q^* \subset \mathbf{R}^{P(n)k}$.

1. Introduction

In his big regularity paper [1], F. J. Almgren introduced the machinery of multiple-valued functions to study the partial regularity of area-minimizing integral currents. He proved that any m-dimensional mass-minimizing integral current is regular except on a set of Hausdorff dimension at most $m-2$. His regularity theory relies on a scheme of approximation of Dirichlet-minimizing multiple-valued functions. The success of Almgren's regularity theory raises the need of further studying multiple-valued functions and of making his work more accessible.

A multiple-valued function $f\colon A \subseteq \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ is essentially a rule assigning k unordered and non necessarily distinct elements of $\mathbf{R}^n$ to each element of its domain. Such maps are studied in complex analysis (see Appendix 5 in [8]). Indeed in complex function theory one often speaks of the "two-valued function $f(z) = z^{1/2}$". This is better considered as a function from $\mathbf{R}^2$ to $\mathbf{Q}_2(\mathbf{R}^2)$.

In Chapter 1 of [1], Almgren proved that the metric space $\mathbf{Q}_k(\mathbf{R}^n)$ is in explicit bi-Lipschitzian correspondence with a finite polyhedral cone Q^* included in a higher dimensional Euclidean space and his analysis is mainly based on this correspondence. In the present paper, we put this correspondence aside and we approach multiple-valued functions by selection arguments. For a $\mathbf{Q}_k(\mathbf{R}^n)$-valued function f, we construct a decomposition into k particular branches $f_1,\dots,f_k$: $\mathbf{R}^m \to \mathbf{R}^n$ and we study the properties of the branches such as continuity in accordance with those of f. For $n=1$, we prove in Proposition 4.1 that f admits

2000 Mathematics Subject Classification: Primary 54C65.

Lipschitzian (respectively Hölder continuous, continuous, measurable) branches if f is Lipschitzian (respectively Hölder continuous, continuous, measurable). For $n > 1$, we show in Proposition 5.1 that f can be split into measurable branches if f is measurable and we complete this result by a Lusin type theorem for multiple-valued functions. We also provide an example of a Lipschitzian $\mathbf{Q}_2(\mathbf{R}^2)$-valued function that does not admit a continuous decomposition and state a selection result already proved in [1] and [3] about multiple-valued functions defined on closed intervals.

We finally recall what is meant by "differentiability" in the context of multiple-valued functions. Essentially a multiple-valued function f is said to be affinely approximable at $a \in \mathbf{R}^m$ if it is possible to approach f near a by a multiple-valued function admitting an affine decomposition. We suggest an original proof of Rademacher's theorem based on the selection theory:

Theorem. *Let $f\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ be a Lipschitzian multiple-valued function. Then f is strongly affinely approximable at $\mathscr{L}^m$ almost all points of $\mathbf{R}^m$.*

We close our paper by using the 1-dimensional selection result to show that these affine approximations control the variation of f:

Theorem. *Let $f\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ be a Lipschitzian multiple-valued function and $[a,b] := \{a + t(b-a) \mid t \in [0,1]\}$ where $a, b \in \mathbf{R}^m$ such that $a \neq b$. If f is affinely approximable at $\mathscr{H}^1$ almost all points of $[a,b]$ then*

$$\mathscr{F}\big(f(a), f(b)\big) \leq \int_{[a,b]} \|Af(x)\| \, d\mathscr{H}^1(x).$$

2. Preliminaries

The scalar product of two vectors $x, y \in \mathbf{R}^n$ is denoted $(x|y)$, and the Euclidean norm of $x \in \mathbf{R}^n$ is denoted $|x|$. For a fixed space X with a metric d, let

$$\mathbf{B}(a,r) = \{x \in X \mid d(a,x) \leq r\}, \quad \mathbf{U}(a,r) = \{x \in X \mid d(a,x) < r\}$$

be the closed and open balls with center $a \in X$ and radius $r > 0$. If $L\colon \mathbf{R}^m \to \mathbf{R}^n$ is a linear mapping, we set $\|L\| = \sup\{|L(x)| \mid x \in \mathbf{R}^m \text{ with } |x| \leq 1\}$. For each point $x_i \in \mathbf{R}^n$, $[\![x_i]\!]$ denotes the Dirac measure at x_i. Denote $\mathbf{Q}_k(\mathbf{R}^n) = \big\{\sum_{i=1}^k [\![x_i]\!] \mid x_i \in \mathbf{R}^n\big\}$, where x_i and x_j are not necessarily distinct for $i \neq j$ and the integer k is already fixed. We can define a topology on $\mathbf{Q}_k(\mathbf{R}^n)$ by the equivalent metrics $\mathscr{G}$ and $\mathscr{F}$:

$$\mathscr{G}\bigg(\sum_{i=1}^k [\![x_i]\!], \sum_{j=1}^k [\![y_j]\!]\bigg) = \min\bigg\{\bigg(\sum_{i=1}^k |x_i - y_{\sigma(i)}|^2\bigg)^{1/2} \;\bigg|\; \sigma \text{ is a permutation of } \{1,\ldots,k\}\bigg\},$$

$$\mathscr{F}\left(\sum_{i=1}^{k}[\![x_i]\!],\sum_{j=1}^{k}[\![y_j]\!]\right)=\min\left\{\sum_{i=1}^{k}|x_i-y_{\sigma(i)}|\;\middle|\;\sigma\text{ is a permutation of }\{1,\dots,k\}\right\}.$$

We easily prove that $(\mathbf{Q}_k(\mathbf{R}^n),\mathscr{G})$ is a complete separable metric space. A map $f\colon A\subseteq\mathbf{R}^m\to\mathbf{Q}_k(\mathbf{R}^n)$ will be called a multiple-valued function or a $\mathbf{Q}_k(\mathbf{R}^n)$-valued function. Let us note that a $\mathbf{Q}_1(\mathbf{R}^n)$-valued function is nothing else than a classical function since $\mathbf{Q}_1(\mathbf{R}^n)$ is in bijection with $\mathbf{R}^n$.

To begin with, let us pay attention to a problem that will make it possible for us to reduce significantly the size of the coming proofs. Considering $f_1,\dots,f_k\colon A\subseteq\mathbf{R}^m\to\mathbf{R}^n$, we can define the following $\mathbf{Q}_k(\mathbf{R}^n)$-valued function

$$f:=\sum_{i=1}^{k}[\![f_i]\!] \tag{1}$$

and wonder if f inherits the regularity properties of $f_1,\dots,f_k$. We can then easily check Proposition 2.1.

Proposition 2.1. *Let $f_1,\dots,f_k\colon A\subseteq\mathbf{R}^m\to\mathbf{R}^n$ and f defined by* (1). *The following assertions are checked*:

(1) *If $f_1,\dots,f_k$ are Lipschitzian on $B\subseteq A$ then f is Lipschitzian on B and* $\operatorname{Lip}(f)\le\left(\sum_{i=1}^k\operatorname{Lip}^2(f_i)\right)^{1/2}$.

(2) *If $f_1,\dots,f_k$ are continuous on $B\subseteq A$ then f is continuous on B.*

To end this section, we will define what is meant by measurable $\mathbf{Q}_k(\mathbf{R}^n)$-valued function and prove a proposition which is similar to Proposition 2.1. As in [4] and [5], we call measure what is usually called outer measure.

Definition 2.1. Let μ be a measure on $\mathbf{R}^m$. A multiple-valued function $f\colon A\subseteq\mathbf{R}^m\to\mathbf{Q}_k(\mathbf{R}^n)$ is called μ-measurable if for all open $G\subseteq\mathbf{Q}_k(\mathbf{R}^n)$, $f^{-1}(G)$ is μ-measurable.

Proposition 2.2. *Let $f_1,\dots,f_k\colon A\subseteq\mathbf{R}^m\to\mathbf{R}^n$ and f defined by* (1). *If $f_1,\dots,f_k$ are μ-measurable then f is μ-measurable.*

Proof. Since $\mathbf{Q}_k(\mathbf{R}^n)$ is separable it is enough to show that $f^{-1}(U)$ is μ-measurable for all open balls $U\subset\mathbf{Q}_k(\mathbf{R}^n)$. Let $v=\sum_{i=1}^k[\![x_i]\!]\in\mathbf{Q}_k(\mathbf{R}^n)$ and $r>0$. We note that

$$\begin{aligned}
f^{-1}\big(\mathbf{U}(v,r)\big)&=\left\{y\in A\;\middle|\;\mathscr{G}\left(v,\sum_{i=1}^{k}[\![f_i(y)]\!]\right)<r\right\}\\
&=\bigcup_{\sigma}\left\{y\in A\;\middle|\;\sum_{i=1}^{k}|x_i-f_{\sigma(i)}(y)|^2<r^2\right\}\\
&=\bigcup_{\sigma}\;\bigcup_{\substack{r_1,\dots,r_k\\ \text{rational}\\ r_1^2+\cdots+r_k^2<r^2}}\;\bigcap_{i=1}^{k}f_{\sigma(i)}^{-1}\big(\mathbf{B}(x_i,r_i)\big)
\end{aligned}$$

so f is μ-measurable. $\square$

3. Canonical decomposition

The central idea in this section consists in decomposing multiple-valued functions in the most efficient possible way. If one has a $\mathbf{Q}_k(\mathbf{R}^n)$-valued function defined on $A \subseteq \mathbf{R}^m$, it is easy to find maps $f_1, \dots, f_k \colon A \to \mathbf{R}^n$ such that $f = \sum_{i=1}^k [\![f_i]\!]$ on A. However we will be concerned with the existence of branches which preserve some properties of f such as measurability, continuity .... We start by defining some necessary tools to this study.

For all $n, k \in \mathbf{N}_0$, we define the functions

$$\beta_{(n,k),1} \colon \mathbf{Q}_k(\mathbf{R}^n) \to \mathbf{R}^n : \ v \mapsto \min\bigl(\operatorname{spt}(v)\bigr)$$

where $\mathbf{R}^n$ is endowed with a total order. For $k \in \mathbf{N}_0$ and $v \in \mathbf{Q}_k(\mathbf{R}^n)$ fixed, we will define by induction with respect to $i \in \{1, \dots, k\}$ a sequence of pairs $\bigl(v_i, \beta_{(n,k),i}(v)\bigr)$ where $v_i \in \mathbf{Q}_{k-i+1}(\mathbf{R}^n)$ and $\beta_{(n,k),i}(v) \in \mathbf{R}^n$. Let us assume that $v_1 = v$ and that the sequence is defined until the index i, we then set successively

$$v_{i+1} := v_i - [\![\beta_{(n,k),i}(v)]\!] \quad \text{and} \quad \beta_{(n,k),i+1}(v) := \beta_{(n,k-i),1}(v_{i+1}).$$

By this process, we have just built the functions

$$\beta_{(n,k),i} \colon \mathbf{Q}_k(\mathbf{R}^n) \to \mathbf{R}^n \tag{2}$$

for $i = 1, \dots, k$ associated to a total order on $\mathbf{R}^n$ and such that

$$v = \sum_{i=1}^{k} [\![\beta_{(n,k),i}(v)]\!] \tag{3}$$

for all $v \in \mathbf{Q}_k(\mathbf{R}^n)$. Consequently, if we consider a multiple-valued function $f \colon A \subseteq \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$, an immediate consequence of (3) is that $f = \sum_{i=1}^k [\![f_i]\!]$ with

$$f_i := \beta_{(n,k),i} \circ f \tag{4}$$

for $i = 1, \dots, k$. Subsequently, we will study the properties of the selection defined by (4), assuming that $\mathbf{R}^n$ is endowed with the lexicographical order.

4. Decomposition properties for $n = 1$

Lemma 4.1. *For all $k \in \mathbf{N}_0$, $\operatorname{Lip}(\beta_{(1,k),i}) \le 1$ for $i = 1, \dots, k$.*

Proof. Let $k \in \mathbf{N}_0$ and $v = \sum_{j=1}^k [\![x_j]\!]$, $w = \sum_{j=1}^k [\![y_j]\!] \in \mathbf{Q}_k(\mathbf{R})$ such that $x_1 \le \cdots \le x_k$ and $y_1 \le \cdots \le y_k$. By the first part of the proof of Theorem 1.2 in [1], it is clear that

$$\mathscr{G}^2(v, w) = \sum_{j=1}^k |x_j - y_j|^2 \ge |x_i - y_i|^2 = |\beta_{(1,k),i}(v) - \beta_{(1,k),i}(w)|^2$$

for $i = 1, \dots, k$. □

We can then state Proposition 4.1 which immediately ensues from (4) and Lemma 4.1.

Proposition 4.1. *Let $f\colon A \subseteq \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R})$ be a multiple-valued function. Then there exist $f_1, \dots, f_k\colon A \subseteq \mathbf{R}^m \to \mathbf{R}$ such that $f = \sum_{i=1}^k [\![f_i]\!]$ on A and the following assertions hold*:

(1) *If f is continuous and $\omega_{f,x}$ denote the modulus of continuity of f at x i.e.*

$$\omega_{f,x}(\delta) = \sup\bigl\{\mathscr{G}(f(x), f(y)) \mid y \in A \text{ and } |x - y| \le \delta\bigr\}$$

then $f_1, \dots, f_k$ are continuous and $\omega_{f_i,x} \le \omega_{f,x}$ for $i = 1, \dots, k$ and for all $x \in A$ where $\omega_{f_i,x}$ denotes the modulus of continuity of f_i at x i.e.

$$\omega_{f_i,x}(\delta) = \sup\bigl\{|f_i(x) - f_i(y)| \mid y \in A \text{ and } |x - y| \le \delta\bigr\}.$$

(2) *If f is μ-measurable then $f_1, \dots, f_k$ are μ-measurable.*

This last proposition is useful as it can help us to easily prove a few theorems that are not obvious at first sight. Nevertheless, this proposition only settles when $n = 1$. Can we hope the same for $n > 1$? The next section will attempt to answer this question. In order to illustrate the interest of Proposition 4.1, we mention two extension theorems for $\mathbf{Q}_k(\mathbf{R})$-valued functions.

Theorem 4.1. *If $A \subset \mathbf{R}^m$ is closed and $f\colon A \to \mathbf{Q}_k(\mathbf{R})$ is continuous, then f has a continuous extension $\bar{f}\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R})$.*

Proof. This follows from Proposition 4.1, Tietze's extension theorem and Proposition 2.1. □

Theorem 4.2. *If $A \subset \mathbf{R}^m$ and $f\colon A \to \mathbf{Q}_k(\mathbf{R})$ is Lipschitzian, then f has a Lipschitzian extension $\bar{f}\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R})$ with $\operatorname{Lip}(\bar{f}) \le k^{1/2} \operatorname{Lip}(f)$.*

Proof. This follows from Proposition 4.1, Kirszbraun's extension theorem and Proposition 2.1. □

5. Decomposition properties for $n > 1$

As shown earlier, studying the hypothetical properties of the selection defined by (4) boils down to studying the properties of the functions (2). For $n = 1$, we showed that these maps are Lipschitzian. For $n > 1$, we will prove that they are merely Borelian. Let $e_1, \dots, e_n$ denote the standard basis vectors of $\mathbf{R}^n$. This notation will only be used in Lemma 5.1. Recall that $\mathbf{R}^n$ is endowed with the lexicographical order.

Lemma 5.1. *For all $k \in \mathbf{N}_0$ and $n \in \mathbf{N}_0 \backslash \{1\}$, $\beta_{(n,k),i}$ is Borelian for $i = 1, \dots, k$.*

Proof. Let $k \in \mathbf{N}_0$ and $n \in \mathbf{N}_0 \backslash \{1\}$. We will partition $\mathbf{Q}_k(\mathbf{R}^n)$ into a countable union of closed sets so that the restrictions of the functions $\beta_{(n,k),i}$ to these closed sets are continuous. It will then be clear that the functions $\beta_{(n,k),i}$ are Borelian.

It is obvious that

$$\mathbf{Q}_k(\mathbf{R}^n) = \bigcup_{a=1}^{\infty} K_a$$

where

$$K_a = \mathbf{Q}_k(\mathbf{R}^n) \cap \Bigg\{ \sum_{i=1}^{k} [\![x_i]\!] \;\Big|\text{if } (x_i|e_l) \neq (x_j|e_l) \text{ for } l \in \{1, \dots, n\} \\ \text{then } |(x_i|e_l) - (x_j|e_l)| \geq 1/a \Bigg\}.$$

Let us show that the sets K_a are closed and that the restrictions of the functions $\beta_{(n,k),i}$ to these sets are continuous.

Fix $a \in \mathbf{N}_0$. We take a sequence $(v_q = \sum_{i=1}^{k} [\![x_{i,q}]\!])_{q \in \mathbf{N}} \subset K_a$ such that $v_q \to v$ as $q \to \infty$ and we will show that $v \in K_a$. Let us consider any subsequence of $(v_q)_{q \in \mathbf{N}}$ still denoted by $(v_q)_{q \in \mathbf{N}}$. Since $(v_q)_{q \in \mathbf{N}}$ converges, it is bounded; hence there exists $M > 0$ such that $\mathscr{G}^2(v_q, k[\![0]\!]) = \sum_{i=1}^{k} |x_{i,q}|^2 < M$ for all $q \in \mathbf{N}$. Consequently the sequence $(x_{i,q})_{q \in \mathbf{N}}$ is bounded for $i = 1, \dots, k$. Therefore there exist $q_1 < q_2 < \cdots$ and $x_1, x_2, \dots, x_k \in \mathbf{R}^n$ such that for all $i \in \{1, \dots, k\}$, the subsequence $(x_{i,q_r})_{r \in \mathbf{N}}$ converges to x_i as $r \to \infty$. Then the subsequence $(v_{q_r})_{r \in \mathbf{N}}$ converges to $\sum_{i=1}^{k} [\![x_i]\!]$ as $r \to \infty$; hence $v = \sum_{i=1}^{k} [\![x_i]\!]$. If $(x_i|e_l) \neq (x_j|e_l)$ for $l \in \{1, \dots, n\}$ then there exists $p \in \mathbf{N}$ such that $(x_{i,q_r}|e_l) \neq (x_{j,q_r}|e_l)$ for all $r \geq p$; hence $|(x_{i,q_r}|e_l) - (x_{j,q_r}|e_l)| \geq 1/a$. By taking the limit, we deduce that $|(x_i|e_l) - (x_j|e_l)| \geq 1/a$; hence $v \in K_a$.

It only remains to prove that the functions $\beta_{(n,k),i}$ restricted to K_a are continuous. For $i, j \in \{1, \dots, k\}$ such that $i \neq j$, we state the following important remarks:

(1) If $(x_i|e_l) = (x_j|e_l)$ for $l \in \{1,\dots,n\}$ then there exists $p \in \mathbf{N}$ such that for all $r \ge p$, we have $(x_{i,q_r}|e_l) = (x_{j,q_r}|e_l)$. Otherwise, there exist $q_{r_1} < q_{r_2} < \cdots$ such that $(x_{i,q_{r_s}}|e_l) \ne (x_{j,q_{r_s}}|e_l)$ for all $s \in \mathbf{N}$; hence

$$|(x_{i,q_{r_s}}|e_l) - (x_{j,q_{r_s}}|e_l)| \ge 1/a$$

which on letting $s \to \infty$ implies the contradiction $|(x_i|e_l) - (x_j|e_l)| \ge 1/a$.

(2) If $(x_i|e_l) < (x_j|e_l)$ for $l \in \{1,\dots,n\}$ then there exists $p \in \mathbf{N}$ such that for all $r \ge p$, we have $(x_{i,q_r}|e_l) < (x_{j,q_r}|e_l)$. Otherwise, there exist $q_{r_1} < q_{r_2} < \cdots$ such that $(x_{i,q_{r_s}}|e_l) \ge (x_{j,q_{r_s}}|e_l)$ for all $s \in \mathbf{N}$ which on letting $s \to \infty$ implies the contradiction $(x_i|e_l) \ge (x_j|e_l)$.

(3) If $(x_i|e_l) > (x_j|e_l)$ for $l \in \{1,\dots,n\}$ then there exists $p \in \mathbf{N}$ such that for all $r \ge p$, we have $(x_{i,q_r}|e_l) > (x_{j,q_r}|e_l)$. Otherwise, there exist $q_{r_1} < q_{r_2} < \cdots$ such that $(x_{i,q_{r_s}}|e_l) \le (x_{j,q_{r_s}}|e_l)$ for all $s \in \mathbf{N}$ which on letting $s \to \infty$ implies the contradiction $(x_i|e_l) \le (x_j|e_l)$.

Consequently, if $x_1,\dots,x_k$ are ordered in a certain way then there exists $p \in \mathbf{N}$ such that $x_{1,q_r}, \dots, x_{k,q_r}$ are ordered in the same way for all $r \ge p$. Therefore $\beta_{(n,k),i}(v_{q_r}) \to \beta_{(n,k),i}(v)$ for $i = 1,\dots,k$. □

The following result is a consequence of (4) and Lemma 5.1.

Proposition 5.1. *Let μ be a measure on $\mathbf{R}^m$ and $f\colon A \subseteq \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ be a μ-measurable multiple-valued function. Then there exist μ-measurable functions $f_1,\dots,f_k\colon A \subseteq \mathbf{R}^m \to \mathbf{R}^n$ such that $f = \sum_{i=1}^k [\![f_i]\!]$ on A.*

The following theorem is nothing else than an adaptation of Lusin's theorem to multiple-valued functions.

Theorem 5.1. *Let μ be a Borel regular measure on $\mathbf{R}^m$ and $f\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ be a μ-measurable multiple-valued function. Assume $A \subset \mathbf{R}^m$ is μ-measurable and $\mu(A) < \infty$. Fix $\varepsilon > 0$. Then there exists a compact set $C \subset A$ and $f_1,\dots,f_k\colon \mathbf{R}^m \to \mathbf{R}^n$ such that*

(1) *$f = \sum_{i=1}^k [\![f_i]\!]$ on $\mathbf{R}^m$,*
(2) *$\mu(A \setminus C) < \varepsilon$,*
(3) *$f_i|_C$ is continuous for $i = 1,\dots,k$, and*
(4) *$f|_C$ is continuous.*

Proof. By Proposition 5.1, there exist μ-measurable functions $f_1,\dots,f_k\colon \mathbf{R}^m \to \mathbf{R}^n$ such that $f = \sum_{i=1}^k [\![f_i]\!]$ on $\mathbf{R}^m$. We then infer from Lusin's theorem that for $i = 1,\dots,k$, there exists a compact set $C_i \subset A$ such that $\mu(A \setminus C_i) < \varepsilon/k$ and $f_i|_{C_i}$ is continuous. We set $C := \bigcap_{i=1}^k C_i \subset A$ which is compact. Then it is clear that $\mu(A \setminus C) < \varepsilon$ and $f|_C$ is continuous by Proposition 2.1. □

To end this section, we show that a continuous $\mathbf{Q}_k(\mathbf{R}^n)$-valued function does no necessarily admit a continuous decomposition if $n > 1$. We consider the Lipschitzian $\mathbf{Q}_2(\mathbf{R}^2)$-valued function

$$\begin{aligned} &f\colon \mathbf{S}^1 \subset \mathbf{R}^2 \to \mathbf{Q}_2(\mathbf{S}^1)\colon \\ &\quad x = (\cos\theta, \sin\theta) \mapsto f(x) = \left[\!\!\left[\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right)\right]\!\!\right] + \left[\!\!\left[\left(-\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right)\right]\!\!\right] \end{aligned}$$

where $\mathbf{S}^1 := \{(\cos\theta, \sin\theta) \mid \theta \in [0, 2\pi)\}$ is the unit circle centered on the origin, and we study the following selection problem: can one select for all $x \in \mathbf{S}^1$, a point $g(x) \in \operatorname{spt}\big(f(x)\big)$ in such a way that g is continuous on $\mathbf{S}^1$? Assume that such a function exists. We introduce the mapping

$$h\colon \mathbf{S}^1 \to \mathbf{S}^1 : x = (\cos\theta, \sin\theta) \mapsto h(x) = (\cos 2\theta, \sin 2\theta)$$

and it is obvious that h is continuous and that its topological degree, written $\deg(h)$, is 2. On the other hand, $h \circ g = \mathrm{id}_{\mathbf{S}^1}$ so that $\deg(h \circ g) = 1$ and the topological degree of g is necessarily an integer which refutes the equality $\deg(h \circ g) = \deg(h)\deg(g)$.

Consequently, a continuous $\mathbf{Q}_k(\mathbf{R}^{n>1})$-valued function does not necessarily admit a continuous decomposition. This also proves that there is no order on $\mathbf{R}^{n>1}$ so that the functions defined by (2) are continuous.

Nevertheless, we can prove that every continuous multiple-valued function f defined on a closed interval can be split into single branches which inherit the modulus of continuity of f.

Proposition 5.2. *Let $f\colon [a,b] \subset \mathbf{R} \to \mathbf{Q}_k(\mathbf{R}^n)$ be a continuous multiple-valued function and ω_f denote the modulus of continuity of f. Then there exist continuous functions $f_1, \dots, f_k\colon [a,b] \subset \mathbf{R} \to \mathbf{R}^n$ such that $f = \sum_{i=1}^k [\![f_i]\!]$ on $[a,b]$ and*

$$\omega_{f_i} \le n^{1/2} k \omega_f$$

where ω_{f_i} denotes the modulus of continuity of f_i for $i = 1, \dots, k$.

Proof. We will only prove that if f is Lipschitzian then f admits a Lipschitzian selection $f_1, \dots, f_k$ such that $\operatorname{Lip}(f_i) \le \operatorname{Lip}(f)$ for $i = 1, \dots, k$ since it is the unique result we will need later. For a general argument, see the original Proposition 1.10 in [1] or Theorem 1.1 in [3].

We will construct multiple-valued functions $f^j\colon [a,b] \to \mathbf{Q}_k(\mathbf{R}^n)$ and functions $f_1^j, \dots, f_k^j : [a,b] \to \mathbf{R}^n$ with $f^j = \sum_{i=1}^k [\![f_i^j]\!]$ for each $j = 1, 2, \dots$ such that (passing to a subsequence of the j's) one obtains Lipschitzian functions $f_i = \lim_{j\to\infty} f_i^j$ with $f = \sum_{i=1}^k [\![f_i]\!]$.

For fixed $j \in \mathbf{N}_0$ and for each $l = 0, 1, \ldots, j$ we set $t_l = a + l\Delta t$ where $\Delta t = (b-a)/j$. Then we choose $f^j\colon [a,b] \to \mathbf{Q}_k(\mathbf{R}^n)$ subject to the requirement that for each $l = 1, \ldots, j$ and $t \in [t_{l-1}, t_l]$

$$f^j(t) = \sum_{i=1}^{k} \left[\!\!\left[\frac{t_l - t}{\Delta t} p_i + \frac{t - t_{l-1}}{\Delta t} q_i \right]\!\!\right]$$

corresponding to some choice of $p_1, \ldots, p_k, q_1, \ldots, q_k$ such that $f(t_{l-1}) = \sum_{i=1}^k [\![p_i]\!]$, $f(t_l) = \sum_{i=1}^k [\![q_i]\!]$ and $\mathscr{G}\big(f(t_{l-1}), f(t_l)\big) = \big(\sum_{i=1}^k |p_i - q_i|^2\big)^{1/2}$. One readily verifies the existence of piecewise affine continuous functions $f_1^j, \ldots, f_k^j\colon [a,b] \to \mathbf{R}^n$ such that $f^j = \sum_{i=1}^k [\![f_i^j]\!]$. We choose and fix such functions for each j. We can then easily prove that $f(x) = \lim_{j\to\infty} f^j(x)$ for all $x \in [a,b]$. Let us fix $j \in \mathbf{N}_0$ and $u \in \{1, \ldots, k\}$. We will now prove that f_u^j is in fact Lipschitzian with $\operatorname{Lip}(f_u^j) \le \operatorname{Lip}(f)$. Suppose $a \le c < d \le b$ and $r, s \in \{1, \ldots, j\}$ such that $c \in [t_{r-1}, t_r]$ and $d \in [t_{s-1}, t_s]$. We consider two cases.

(1) Suppose that $r = s$. We obtain that

$$\begin{aligned}
|f_u^j(c) - f_u^j(d)| &\le \left(\sum_{i=1}^k \left| \frac{t_r - d}{\Delta t} p_i + \frac{d - t_{r-1}}{\Delta t} q_i - \frac{t_r - c}{\Delta t} p_i - \frac{c - t_{r-1}}{\Delta t} q_i \right|^2 \right)^{1/2} \\
&= \left(\sum_{i=1}^k \left| \frac{c-d}{\Delta t} p_i - \frac{c-d}{\Delta t} q_i \right|^2 \right)^{1/2} = \frac{d-c}{\Delta t} \left(\sum_{i=1}^k |p_i - q_i|^2 \right)^{1/2} \\
&= \frac{d-c}{\Delta t} \mathscr{G}\big(f(t_{r-1}), f(t_r)\big) \le \operatorname{Lip}(f)(d-c).
\end{aligned}$$

(2) Suppose that $r < s$. We obtain that

$$\begin{aligned}
|f_u^j(c) - f_u^j(d)| &\le |f_u^j(c) - f_u^j(t_r)| + |f_u^j(t_r) - f_u^j(t_{r+1}) + \cdots \\
&\quad + |f_u^j(t_{s-2}) - f_u^j(t_{s-1})| + |f_u^j(t_{s-1}) - f_u^j(d)| \\
&\le \operatorname{Lip}(f)\big((t_r - c) + (t_{r+1} - t_r) + \cdots \\
&\quad + (t_{s-1} - t_{s-2}) + (d - t_{s-1})\big) \\
&= \operatorname{Lip}(f)(d-c).
\end{aligned}$$

We conclude that the sequences $\{f_1^j\}_{j\in\mathbf{N}}, \ldots, \{f_k^j\}_{j\in\mathbf{N}}$ belong to the following closed, bounded and equicontinuous family

$$\Omega = \Big\{ y \in C([a,b], \mathbf{R}^n) \;\Big|\; |y(x) - y(\bar{x})| \le \operatorname{Lip}(f)|x - \bar{x}| \text{ for all } x,\ \bar{x} \in [a,b] \text{ and } |y(a)| \le \max\big\{|v| \;\big|\; v \in \operatorname{spt}\big(f(a)\big)\big\} \Big\}.$$

By Arzela–Ascoli's theorem, there exist $j_1 < j_2 < \cdots$ and $f_1, \ldots, f_k \in \Omega$ such that $\{f_i^{j_l}\}_{l\in\mathbf{N}}$ converges uniformly to f_i as $l \to \infty$ for $i = 1, \ldots, k$. It is immediate that $f = \lim_{l\to\infty} f^{j_l} = \lim_{l\to\infty} \sum_{i=1}^k [\![f_i^{j_l}]\!] = \sum_{i=1}^k [\![f_i]\!]$. □

6. Rademacher's theorem for multiple-valued functions

Before presenting a theorem for multiple-valued functions identical to Rademacher's theorem, we will define the meaning of "differentiability" and "approximative differentiability" for multiple-valued functions in the way they were originally expressed by Almgren in [1].

Definition 6.1. A multiple-valued function $f\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ is said to be affine if and only if there exist affine maps $g_1, \dots, g_k\colon \mathbf{R}^m \to \mathbf{R}^n$ such that $f = \sum_{i=1}^k [\![g_i]\!]$.

One can check that if $f = \sum_{i=1}^k [\![g_i]\!]$ is such an affine function, then the functions $g_1, \dots, g_k$ are uniquely determined up to order.

Definition 6.2. Let $A \subseteq \mathbf{R}^m$ be an open set and $f\colon A \to \mathbf{Q}_k(\mathbf{R}^n)$. One says that f is affinely approximable at $a \in A$ if and only if there is an affine multiple-valued function $g\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathscr{G}\big(f(x), g(x)\big) \le \varepsilon |x-a| \quad \text{for all } x \in \mathbf{B}(a,\delta) \cap A.$$

Definition 6.3. Let $A \subseteq \mathbf{R}^m$ be an open set and $f\colon A \to \mathbf{Q}_k(\mathbf{R}^n)$. One says that f is approximately affinely approximable at $a \in A$ if and only if there is an affine multiple-valued function $g\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ satisfying

$$\operatorname{ap}\lim_{x \to a} \frac{\mathscr{G}\big(f(x), g(x)\big)}{|x-a|} = 0.$$

In other words, for any $\varepsilon > 0$,

$$\lim_{r \to 0} \frac{\mathscr{L}^m\big(\mathbf{B}(a,r) \cap \{x \mid \mathscr{G}\big(f(x), g(x)\big) > \varepsilon |x-a|\}\big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)} = 0.$$

In case f is affinely approximable (respectively approximately affinely approximable) at a, the affine multiple-valued function g is seen to be uniquely determined and is denoted $Af(a)$ (respectively $\operatorname{ap} Af(a)$). We set $\|Af(a)\| = \sum_{i=1}^k \|g_i - g_i(0)\|$, where $g_1, \dots, g_k$ are the unique affine functions such that $Af(a) = \sum_{i=1}^k [\![g_i]\!]$.

Definition 6.4. Let $A \subseteq \mathbf{R}^m$ be an open set and $f\colon A \to \mathbf{Q}_k(\mathbf{R}^n)$. One says that f is strongly affinely approximable (respectively strongly approximately affinely approximable) at $a \in A$ if and only if f is affinely approximable (respectively approximately affinely approximable) at a and if $g_1, \dots, g_k\colon \mathbf{R}^m \to \mathbf{R}^n$ are affine maps such that

$$Af(a) = \sum_{i=1}^k [\![g_i]\!] \quad (\text{respectively } \operatorname{ap} Af(a) = \sum_{i=1}^k [\![g_i]\!]),$$

then $g_i(a) = g_j(a)$ implies $g_i = g_j$ for all $i, j \in \{1, \dots, k\}$.

We now state two useful and analogous characterizations for the coming proofs.

Proposition 6.1. *Let $a \in A \subseteq \mathbf{R}^m$ and $f\colon A \to \mathbf{Q}_k(\mathbf{R}^n)$ such that $f = \sum_{i=1}^k [\![f_i]\!]$ where $f_i\colon A \to \mathbf{R}^n$ is differentiable at a for $i = 1, \dots, k$. Then f is affinely approximable at a.*

Proof. For $i = 1, \dots, k$ and for all $x \in \mathbf{R}^m$, we define $g_i(x) := f_i(a) + Df_i(a)(x-a)$ and we set $g := \sum_{i=1}^k [\![g_i]\!]$ which is obviously an affine multiple-valued function. Let $\varepsilon > 0$ and $i \in \{1, \dots, k\}$. Since f_i is differentiable at a, there exists $\delta_i > 0$ such that $x \in A$ and $|x - a| \le \delta_i$ implies $|f_i(x) - g_i(x)| \le (\varepsilon/k^{1/2})|x - a|$. We then choose $\delta := \min\{\delta_1, \dots, \delta_k\} > 0$ because $x \in A$ and $|x - a| \le \delta$ implies

$$\mathscr{G}\big(f(x), g(x)\big) \le \left(\sum_{i=1}^k |f_i(x) - g_i(x)|^2\right)^{1/2} \le \left(\sum_{i=1}^k \frac{\varepsilon^2}{k}|x-a|^2\right)^{1/2} = \varepsilon|x - a|. \ \square$$

Proposition 6.2. *Let $a \in A \subseteq \mathbf{R}^m$ and $f\colon A \to \mathbf{Q}_k(\mathbf{R}^n)$ such that $f = \sum_{i=1}^k [\![f_i]\!]$ where $f_i\colon A \to \mathbf{R}^n$ is approximately differentiable at a for $i = 1, \dots, k$. Then f is approximately affinely approximable at a.*

Proof. For $i = 1, \dots, k$ and for all $x \in \mathbf{R}^m$, we define

$$g_i(x) := f_i(a) + \mathrm{ap}Df_i(a)(x - a) \tag{5}$$

and we set

$$g := \sum_{i=1}^k [\![g_i]\!] \tag{6}$$

which is obviously an affine multiple-valued function. Let $\varepsilon > 0$ and $r > 0$. We easily check that

$$\begin{aligned}
&\frac{\mathscr{L}^m\big(\mathbf{B}(a,r) \cap \{x \in \mathbf{R}^m \mid \mathscr{G}\big(f(x), g(x)\big) > \varepsilon|x-a|\}\big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)} \\
&= \frac{\mathscr{L}^m\big(\mathbf{B}(a,r) \cap \{x \in \mathbf{R}^m \mid \mathscr{G}\big(f(x), g(x)\big) \le \varepsilon|x-a|\}^c\big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)} \\
&\le \frac{\mathscr{L}^m\big(\mathbf{B}(a,r) \cap \{x \in \mathbf{R}^m \mid \sum_{i=1}^k |f_i(x) - g_i(x)|^2 \le \varepsilon^2|x-a|^2\}^c\big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)} \\
&\le \frac{\mathscr{L}^m\left(\mathbf{B}(a,r) \cap \left(\bigcap_{i=1}^k \{x \in \mathbf{R}^m \mid |f_i(x) - g_i(x)|^2 \le \varepsilon^2 k^{-1}|x-a|^2\}\right)^c\right)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)}
\end{aligned}$$

$$= \frac{\mathscr{L}^m\left(\mathbf{B}(a,r) \cap \left(\bigcup_{i=1}^{k} \{x \in \mathbf{R}^m \mid |f_i(x) - g_i(x)|^2 > \varepsilon^2 k^{-1}|x-a|^2\}\right)\right)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)}$$

$$\leq \sum_{i=1}^{k} \frac{\mathscr{L}^m\big(\mathbf{B}(a,r) \cap \{x \in \mathbf{R}^m \mid |f_i(x) - g_i(x)| > \varepsilon k^{-1/2}|x-a|\}\big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)}$$

and the right-hand side converges to zero as $r \to 0$. □

Theorem 6.1. *Let $f\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R})$ be a Lipschitzian multiple-valued function. Then f is strongly affinely approximable at $\mathscr{L}^m$ almost all points of $\mathbf{R}^m$.*

Proof. According to Proposition 4.1, there exist Lipschitzian functions $f_1, \dots, f_k\colon \mathbf{R}^m \to \mathbf{R}$ such that $f = \sum_{i=1}^{k} [\![f_i]\!]$. We then deduce from the classical Rademacher's theorem that there exists $B \subset \mathbf{R}^m$ such that $\mathscr{L}^m(B) = 0$ and f_i is differentiable on B^c for $i = 1, \dots, k$. Proposition 6.1 implies that f is affinely approximable on B^c with $Af(a) = \sum_{i=1}^{k} [\![f_i(a) + Df_i(a)(\,\cdot - a)]\!]$ for all $a \in B^c$. For all $i, j \in \{1, \dots, k\}$, we define $Z_{ij} = \{a \in B^c \mid f_i(a) = f_j(a)\}$ and we deduce from Corollary 1 in Section 3.1 of [4] that $Df_i(a) = Df_j(a)$ for $\mathscr{L}^m$ almost all $a \in Z_{ij}$. □

The proof of Rademacher's theorem is no longer easy when $n > 1$ because a Lipschitzian multiple-valued function does not necessarily admit a Lipschitzian decomposition. The first proof was suggested by Almgren in [1] and is based on the existence of a bi-Lipschitzian correspondence between $\mathbf{Q}_k(\mathbf{R}^n)$ and a cone $Q^* \subset \mathbf{R}^{P(n)k}$ where $P(n)$ is an integer depending on n. The following proof does not use this correspondence.

Lemma 6.1. *Let $A \subseteq \mathbf{R}^m$ and $a \in A$. Suppose that $f_1, \dots, f_k\colon A \to \mathbf{R}^n$ are continuous at a. Then there exists $0 < r(a) \leq \infty$ such that*

$$\min_{\sigma}\left\{\sum_{i=1}^{k} |f_i(x) - f_{\sigma(i)}(a)|^2\right\} = \sum_{i=1}^{k} |f_i(x) - f_i(a)|^2 \quad \text{for all } x \in \mathbf{U}\big(a, r(a)\big) \cap A.$$

Proof. We define $\varepsilon := \min\{|f_i(a) - f_j(a)| \mid i, j \in \{1, \dots, k\} \text{ and } f_i(a) \neq f_j(a)\}$. Let $i \in \{1, \dots, k\}$. By the continuity of f_i at a, there exists $\delta_i > 0$ such that $x \in A$ and $|x - a| < \delta_i$ implies $|f_i(x) - f_i(a)| < \frac{1}{2}\varepsilon$. We choose $r(a) := \min\{\delta_1, \dots, \delta_k\} > 0$. Indeed for $i \in \{1, \dots, k\}$, σ a permutation of $\{1, \dots, k\}$ and $x \in \mathbf{U}\big(a, r(a)\big) \cap A$, we obtain the two following cases:

(1) If $f_{\sigma(i)}(a) = f_i(a)$ then $|f_i(x) - f_{\sigma(i)}(a)| = |f_i(x) - f_i(a)|$.

(2) If $f_{\sigma(i)}(a) \neq f_i(a)$ then it is clear that $f_{\sigma(i)}(a) \notin \mathbf{U}\big(f_i(a), \varepsilon\big)$. It ensues that $|f_i(x) - f_{\sigma(i)}(a)| \geq |f_i(x) - f_i(a)|$, otherwise we have the following contradiction

$$\varepsilon \leq |f_i(a) - f_{\sigma(i)}(a)| \leq |f_i(x) - f_{\sigma(i)}(a)| + |f_i(x) - f_i(a)| < 2|f_i(x) - f_i(a)| < \varepsilon. \; □$$

Theorem 6.2. *Let $f\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ be a Lipschitzian multiple-valued function and $A \subset \mathbf{R}^m$ be an $\mathscr{L}^m$-measurable set such that $\mathscr{L}^m(A) < \infty$. Fix $\varepsilon > 0$. Then there exists a compact set $C \subset A$ such that f is strongly affinely approximable at $\mathscr{L}^m$ almost all points of C and $\mathscr{L}^m(A \backslash C) < \varepsilon$.*

Proof. f is $\mathscr{L}^m$-measurable since f is Lipschitzian. By Theorem 5.1, there exist a compact set $C \subset A$ and $f_1, \dots, f_k\colon \mathbf{R}^m \to \mathbf{R}^n$ such that $f = \sum_{i=1}^k [\![f_i]\!]$, $\mathscr{L}^m(A\backslash C) < \varepsilon$ and $f_i|_C$ is continuous for $i = 1, \dots, k$.

Let $a \in C$. By Lemma 6.1, there exists $r(a) > 0$ such that $x \in \mathbf{U}\big(a, r(a)\big) \cap C$ implies

$$\sum_{i=1}^k |f_i(x) - f_i(a)|^2 = \mathscr{G}^2\big(f(x), f(a)\big) \le \operatorname{Lip}^2(f)|x-a|^2.$$

Consequently we get that

$$|f_i(x) - f_i(a)| \le \operatorname{Lip}(f)|x-a| \tag{7}$$

for $i = 1, \dots, k$ and for all $x \in \mathbf{U}\big(a, r(a)\big) \cap C$. Let

$$Z := \left\{ a \in C \;\middle|\; \lim_{r\to 0} \frac{\mathscr{L}^m\big(\mathbf{B}(a,r)\cap C\big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)} = 1 \right\},$$

so that $\mathscr{L}^m(C\backslash Z) = 0$ according to the Lebesgue density theorem. Notice also that

$$1 = \lim_{r\to 0} \frac{\mathscr{L}^m\big(\mathbf{B}(a,r)\cap C\big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)} = \lim_{r\to 0} \frac{\mathscr{L}^m\big(\mathbf{B}(a,r)\cap Z\big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)} + \underbrace{\lim_{r\to 0} \frac{\mathscr{L}^m\big(\mathbf{B}(a,r)\cap (C\backslash Z)\big)}{\mathscr{L}^m\big(\mathbf{B}(a,r)\big)}}_{=0}$$

for all $a \in Z$. Let $\bar{a} \in Z$ and $i \in \{1, \dots, k\}$. Let us set

$$h(x) := \frac{|f_i(x) - f_i(\bar{a})|}{|x - \bar{a}|}$$

for all $x \in \mathbf{R}^m$ and show that

$$\begin{aligned} \operatorname{ap}\limsup_{x\to\bar{a}} h(x) &:= \inf\left\{ t \in \mathbf{R} \mid \lim_{r\to 0} \frac{\mathscr{L}^m\big(\mathbf{B}(\bar{a},r)\cap \{x \in \mathbf{R}^m \mid h(x) > t\}\big)}{\mathscr{L}^m\big(\mathbf{B}(\bar{a},r)\big)} = 0 \right\} \\ &\le \operatorname{Lip}(f). \end{aligned} \tag{8}$$

Assume that $t > \operatorname{Lip}(f)$ and $r < r(\bar{a})$. Observe that (7) implies

$$\frac{\mathscr{L}^m\left(\left\{ x \in \mathbf{R}^m \;\middle|\; \dfrac{|f_i(x) - f_i(\bar{a})|}{|x-\bar{a}|} > t \right\} \cap \mathbf{B}(\bar{a},r)\right)}{\mathscr{L}^m\big(\mathbf{B}(\bar{a},r)\big)} \le \frac{\mathscr{L}^m\big(\mathbf{B}(\bar{a},r)\backslash C\big)}{\mathscr{L}^m\big(\mathbf{B}(\bar{a},r)\big)}.$$

Since $\bar{a} \in Z$, the right-hand side converges to zero as $r \to 0$ so that the statement (8) is proved. For $i = 1, \dots, k$, we deduce from Theorem 3.1.8[1] of [5] that f_i is approximately differentiable on $Z_i \subseteq Z$ with $\mathscr{L}^m(Z \backslash Z_i) = 0$. We fix $Z^* := \bigcap_{i=1}^k Z_i$ so that the functions f_i are approximately differentiable on Z^* and $\mathscr{L}^m(Z \backslash Z^*) = 0$. Let $a^* \in Z^*$. Proposition 6.2 implies that f is approximately affinely approximable at a^* by the affine multiple-valued function defined by (5) and (6). Let us show that f is in fact affinely approximable at a^* by the same affine multiple-valued function. Take note that the following argument is an adaptation of Lemma 3.1.5 of [5]. Let $\gamma > 0$. We choose $0 < \beta < 1$ such that

$$\beta\left(1 + \frac{\mathrm{Lip}(f) + \beta + \left(\sum_{i=1}^k \|\mathrm{ap}Df_i(a^*)\|^2\right)^{1/2}}{1-\beta}\right) < \gamma$$

and we define the set $W := \left\{x \in \mathbf{R}^m \mid \mathscr{G}\big(f(x), g(x)\big) \le \beta|x - a^*|\right\}$. Since f is approximately affinely approximable at a^*, there exists $\delta > 0$ such as, if $0 < r < \delta$, then

$$\frac{\mathscr{L}^m\big(\mathbf{B}(a^*, r) \cap W^c\big)}{\mathscr{L}^m\big(\mathbf{B}(a^*, r)\big)} < \beta^m.$$

Let us fix $x \in \mathbf{U}\big(a^*, \delta(1-\beta)\big)$ and regard $r := (|x - a^*|)/(1-\beta)$. It is clear that $r < \delta$ and $\mathbf{B}(x, \beta r) \subseteq \mathbf{B}(a^*, r)$. On the other hand, we remark that $\mathbf{B}(x, \beta r) \cap W \neq \emptyset$ for otherwise we obtain the following contradiction

$$\begin{aligned}(\beta r)^m \alpha(m) &= \mathscr{L}^m\big(\mathbf{B}(x, \beta r)\big) = \mathscr{L}^m\big(\mathbf{B}(x, \beta r) \cap W^c\big) \\ &\le \mathscr{L}^m\big(\mathbf{B}(a^*, r) \cap W^c\big) < \beta^m \mathscr{L}^m\big(\mathbf{B}(a^*, r)\big) = \beta^m r^m \alpha(m).\end{aligned}$$

where $\alpha(m)$ is the Lebesgue measure of the unit Euclidean ball in $\mathbf{R}^m$. Let $z \in \mathbf{B}(x, \beta r) \cap W$. We successively have

$$\begin{aligned}\mathscr{G}\big(f(x), g(x)\big) &\le \mathscr{G}\big(f(x), f(z)\big) + \mathscr{G}\big(f(z), g(z)\big) + \mathscr{G}\big(g(z), g(x)\big) \\ &\le \mathrm{Lip}(f)|x - z| + \beta|z - a^*| + \left(\sum_{i=1}^k |g_i(z) - g_i(x)|^2\right)^{1/2}\end{aligned}$$

[1] This theorem indicates that an approximate local growth condition on f suffices to guarantee that f is the union of a countable family of Lipschitzian functions.

$$
\begin{aligned}
&\le \mathrm{Lip}(f)\beta r + \beta(|z-x| + |x-a^*|) + \left(\sum_{i=1}^{k} |\mathrm{ap}Df_i(a^*)(z-x)|^2\right)^{1/2} \\
&\le \mathrm{Lip}(f)\beta r + \beta(\beta r + |x-a^*|) + |z-x|\left(\sum_{i=1}^{k} \|\mathrm{ap}Df_i(a^*)\|^2\right)^{1/2} \\
&\le \mathrm{Lip}(f)\beta r + \beta(\beta r + |x-a^*|) + \beta r\left(\sum_{i=1}^{k} \|\mathrm{ap}Df_i(a^*)\|^2\right)^{1/2} \\
&= \beta|x-a^*|\left(1 + \frac{\mathrm{Lip}(f) + \beta + \left(\sum_{i=1}^{k} \|\mathrm{ap}Df_i(a^*)\|^2\right)^{1/2}}{1-\beta}\right).
\end{aligned}
$$

We thus obtain that $\mathscr{G}\big(f(x), g(x)\big) < \gamma|x-a^*|$ so f is affinely approximable at a^*. Thus f is affinely approximable on Z^*. For all $i,j \in \{1,\ldots,k\}$, we define $Z^*_{ij} = \{a \in Z^* \mid f_i(a) = f_j(a)\}$ and we deduce from Theorem 3 in Section 6.1 of [4] that $\mathrm{ap}Df_i(a) = \mathrm{ap}Df_j(a)$ for $\mathscr{L}^m$ almost all $a \in Z^*_{ij}$. Consequently f is strongly affinely approximable at $\mathscr{L}^m$ almost all points of C. ▫

Theorem 6.3. *Let $f\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ be a Lipschitzian multiple-valued function. Then f is strongly affinely approximable at $\mathscr{L}^m$ almost all points of $\mathbf{R}^m$.*

Proof. Let us show that there exist disjoint, compact sets $\{C_i\}_{i=1}^{\infty} \subset \mathbf{R}^m$ such that

$$\mathscr{L}^m\left(\left(\bigcup_{i=1}^{\infty} C_i\right)^c\right) = 0$$

and for each $i = 1, 2, \ldots$

$$f \text{ is strongly affinely approximable at } \mathscr{L}^m \text{ almost all points of } C_i.$$

For each positive integer n, set $\mathbf{B}_n = \mathbf{B}(0,n)$. By Theorem 6.2, there exists a compact set $C_1 \subset \mathbf{B}_1$ such that $\mathscr{L}^m(\mathbf{B}_1 \backslash C_1) \le 1$ and f is strongly affinely approximable $\mathscr{L}^m$ almost everywhere on C_1. Assume now $C_1, \ldots, C_n$ have been constructed, there exists a compact set $C_{n+1} \subset \mathbf{B}_{n+1} \backslash \bigcup_{i=1}^{n} C_i$ such that $\mathscr{L}^m(\mathbf{B}_{n+1} \backslash \bigcup_{i=1}^{n+1} C_i) \le 1/(n+1)$ and f is strongly affinely approximable at $\mathscr{L}^m$ almost all points of C_{n+1}. Consequently the starting assertion is confirmed. For all $i \in \mathbf{N}_0$, we define the set $A_i := \{a \in C_i \mid f$ is strongly affinely approximable at $a\}$ and note that $\mathscr{L}^m(C_i \backslash A_i) = 0$. It is clear that $\mathscr{L}^m\big((\bigcup_{i=1}^{\infty} A_i)^c\big) = 0$ so f is strongly affinely approximable at $\mathscr{L}^m$ almost all points of $\mathbf{R}^m$. ▫

Theorem 6.4. *Let $f\colon \mathbf{R}^m \to \mathbf{Q}_k(\mathbf{R}^n)$ be a Lipschitzian multiple-valued function and $[a,b] := \{a + t(b-a) \mid t \in [0,1]\}$ where $a, b \in \mathbf{R}^m$ such that $a \ne b$.*

If f is affinely approximable at $\mathscr{H}^1$ almost all points of $[a,b]$ then

$$\mathscr{F}\big(f(a), f(b)\big) \leq \int_{[a,b]} \|Af(x)\| \, d\mathscr{H}^1(x).$$

Proof. By a standard argument applied to Proposition 5.2, there exist Lipschitzian functions $f_1, \dots, f_k \colon [a,b] \to \mathbf{R}^n$ such that $f|_{[a,b]} = \sum_{i=1}^k [\![f_i]\!]$. We introduce the map $\varphi \colon [0,1] \to [a,b] \colon t \mapsto a + t(b-a)$ and the direction $u = (b-a)/(|b-a|)$. We then obtain

$$\begin{aligned}
\mathscr{F}\big(f(b), f(a)\big) &\leq \sum_{i=1}^k |f_i(b) - f_i(a)| = \sum_{i=1}^k \big|f_i\big(\varphi(1)\big) - f_i\big(\varphi(0)\big)\big| \\
&= \sum_{i=1}^k \left| \int_0^1 (f_i \circ \varphi)'(t) \, dt \right| \leq \sum_{i=1}^k \int_0^1 \big|D_u f_i\big(\varphi(t)\big)\big| \, |\varphi'(t)| \, dt \\
&= \sum_{i=1}^k \int_{[a,b]} |D_u f_i(x)| \, d\mathscr{H}^1(x).
\end{aligned}$$

So it remains to prove the following assertion:

Claim. *Let $x \in]a,b[$ such that f is affinely approximable at x and $D_u f_i(x)$ exists for $i = 1, \dots, k$. Then $\sum_{i=1}^k |D_u f_i(x)| \leq \|Af(x)\|$.*

Since f is affinely approximable at x, it is clear that $Af(x)(x) = f(x) = \sum_{i=1}^k [\![f_i(x)]\!]$ so that

$$Af(x)(\,\cdot\,) = \sum_{i=1}^k [\![f_i(x) + L_i(x)(\,\cdot - x)]\!]$$

where $L_1(x), \dots, L_k(x) \colon \mathbf{R}^m \to \mathbf{R}^n$ are linear maps. Let

$$T = \{t \in \mathbf{R} \mid x + tu \in [a,b]\}.$$

For all $t \in T$, let σ_t be any permutation which attains the following minimum:

$$\min_\sigma \sum_{i=1}^k \big|f_{\sigma(i)}(x+tu) - f_i(x) - tL_i(x)(u)\big|.$$

We will now prove that there exists $\delta > 0$ such that $f_{\sigma_t(i)}(x) = f_i(x)$ for $i = 1, \dots, k$ and for all $t \in T$ satisfying $|t| < \delta$.

Fix $j \in \{1, \ldots, k\}$. By contradiction, let $(t_l)_{l\in\mathbf{N}} \subset T$ be a sequence converging to 0 such that $f_{\sigma_{t_l}(j)}(x) \neq f_j(x)$ for all $l \in \mathbf{N}$. Using the continuity of every f_q satisfying $f_q(x) \neq f_j(x)$, we have that there exists $\varepsilon > 0$ such that, for l large enough,

$$|f_j(x) - f_{\sigma_{t_l}(j)}(x + t_l u)| > \varepsilon.$$

Hence, for l large enough,

$$\begin{aligned}
&\frac{\min_\sigma \sum_{i=1}^k |f_{\sigma(i)}(x + t_l u) - f_i(x) - t_l L_i(x)(u)|}{|t_l|} \\
&\quad\geq \frac{|f_{\sigma_{t_l}(j)}(x + t_l u) - f_j(x) - t_l L_j(x)(u)|}{|t_l|} \\
&\quad\geq \frac{|f_{\sigma_{t_l}(j)}(x + t_l u) - f_j(x)|}{|t_l|} - \frac{|t_l L_j(x)(u)|}{|t_l|} \geq \frac{\varepsilon}{|t_l|} - \|L_j(x)\|.
\end{aligned}$$

Consequently,

$$\lim_{l\to\infty} \frac{\min_\sigma \sum_{i=1}^k \big|f_{\sigma(i)}(x + t_l u) - f_i(x) - t_l L_i(x)(u)\big|}{|t_l|} = \infty$$

which contradicts the fact that f is affinely approximable at x by $Af(x)$. Then, there exists $\delta_j > 0$ such that $f_{\sigma_t(j)}(x) = f_j(x)$ if $t \in T$ and $|t| < \delta_j$. If we set $\delta = \min\{\delta_j \mid j = 1, \ldots, k\}$ then $f_{\sigma_t(i)}(x) = f_i(x)$ for $i = 1, \ldots, k$ and for all $t \in T$ such that $|t| < \delta$.

Fix $\bar{\varepsilon} > 0$. Since f is affinely approximable at x by $Af(x)$, there exists $\gamma > 0$ such that if $t \in T$ and $0 < |t| < \gamma$ then

$$\sum_{i=1}^k \left| \frac{f_{\sigma_t(i)}(x + tu) - f_i(x)}{t} - L_i(x)(u) \right| < \bar{\varepsilon}.$$

On the other hand, if $t \in T$ and $0 < |t| < \min\{\delta, \gamma\}$ then

$$\sum_{i=1}^k \left| \frac{f_{\sigma_t(i)}(x + tu) - f_{\sigma_t(i)}(x)}{t} - L_i(x)(u) \right| < \bar{\varepsilon};$$

hence

$$\min_\sigma \sum_{i=1}^k \left| \frac{f_{\sigma(i)}(x + tu) - f_{\sigma(i)}(x)}{t} - L_i(x)(u) \right| < \bar{\varepsilon}.$$

Consequently,

$$\lim_{t\to 0} \mathscr{F}\left(\sum_{i=1}^k \left[\!\!\left[\frac{f_i(x + tu) - f_i(x)}{t} \right]\!\!\right], \sum_{i=1}^k [\![L_i(x)(u)]\!] \right) = 0.$$

Finally, by the triangular inequality, we obtain for all $t \neq 0$ that

$$\begin{aligned}\mathscr{F}\Big(\sum_{i=1}^{k}[\![D_u f_i(x)]\!], \sum_{i=1}^{k}[\![L_i(x)(u)]\!]\Big) \\ \leq \mathscr{F}\Big(\sum_{i=1}^{k}[\![D_u f_i(x)]\!], \sum_{i=1}^{k}[\![\big(f_i(x+tu)-f_i(x)\big)/t]\!]\Big) \\ + \mathscr{F}\Big(\sum_{i=1}^{k}[\![\big(f_i(x+tu)-f_i(x)\big)/t]\!], \sum_{i=1}^{k}[\![L_i(x)(u)]\!]\Big)\end{aligned}$$

where the right-hand side converges to zero as $t \to 0$. Therefore, there exists a permutation σ such that $D_u f_i(x) = L_{\sigma(i)}(x)(u)$ for $i = 1, \ldots, k$; hence

$$\sum_{i=1}^{k}|D_u f_i(x)| = \sum_{i=1}^{k}|L_i(x)(u)| \leq \sum_{i=1}^{k}\|L_i(x)\| = \|Af(x)\|. \qquad \square$$

Acknowledgement. The author would like to thank the referee for his helpful suggestions, and to acknowledge the fact that this type of questions was suggested to him by his thesis advisor Thierry De Pauw. The credit of the idea that the selection theory could imply a new proof of Rademacher's theorem goes to him, and this paper would not exist without the constantly challenging discussions we had.

References

[1] Almgren, F. J., Jr.: Almgren's big regularity paper. **Q**-valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension 2. - World Scientific Monograph Series in Mathematics, 1. World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[2] Almgren, F. J., Jr.: Dirichlet's problem for multiple-valued functions and the regularity of mass minimizing integral currents. In: Minimal Submanifolds and Geodesics (Proc. Japan–United States Sem., Tokyo, 1977), North-Holland, Amsterdam–New York, 1979, 1–6.

[3] De Lellis, C., C. R. Grisanti, and P. Tilli: Regular selections for multiple-valued functions. - Ann. Mat. Pura Appl. (4) 183, 2004, 79–95.

[4] Evans, L. C., and R. F. Gariepy: Measure Theory and Fine Properties of Functions. - CRC Press, Boca Raton, FL, 1992.

[5] Federer, H.: Geometric Measure Theory. - Springer-Verlag, New York, 1969.

[6] Morgan, F.: Geometric Measure Theory: A Beginners Guide. - Academic Press, Boston, 1988.

[7] Priestley, H. A.: Introduction to Complex Analysis. - Oxford University Press, Oxford, 1985.

[8] Whitney, H.: Complex Analytic Varieties. - Addison-Wesley, Reading, 1972.

Received 2 February 2005